A NOTE ON LÁRUSSON-SIGURDSSON’S PAPER

ARMEN EDIGARIAN*

1. Introduction

Let $X$ be a complex manifold. We denote $\mathcal{A}_X$ the family of all mappings $f : \overline{D} \to X$ which are holomorphic in a neighborhood of the closure $\overline{D}$ of the unit disc $D$. A disc functional on $X$ is a function $H : \mathcal{A}_X \to \mathbb{R} \cup \{-\infty\}$. The envelope of $H$ is the function $E_H : X \to \mathbb{R} \cup \{-\infty\}$ defined by the formula

$$E_H(x) := \inf \{ H(f) : f \in \mathcal{A}_X, \quad f(0) = x \}, \quad x \in X.$$

E. Poletsky [5], [6], [7], has shown that certain disc functionals on domains in $\mathbb{C}^n$ have plurisubharmonic envelopes. Later, for three classes of disc functionals plurisubharmonicity of envelopes on a class of complex manifolds were shown by F. Lárusson and R. Sigurdsson [4]. The paper is motivated by results from [4].

Let us consider the following two functionals.

Let $\phi : X \to \mathbb{R} \cup \{-\infty\}$ be an upper semi-continuous function. Define the functional $H_1 = H_1^\phi$ by the formula

$$H_1(f) = \frac{1}{2\pi} \int_0^{2\pi} \phi(f(e^{i\theta})) d\theta, \quad f \in \mathcal{A}_X.$$ 

In [4] this functional is called the Poisson functional.

Let $\nu$ be a plurisubharmonic function on $X$. We define the functional $H_2 = H_2^\nu$ as follows. If $f \in \mathcal{A}_X$ and $\nu \circ f$ is not identically $-\infty$, then

$$H_2(f) = \frac{1}{2\pi} \int_0^{2\pi} \log(1 + \Delta u)(\nu \circ f),$$

where $\Delta u$ is the generalized Laplacian of a subharmonic function $u$. If $f \in \mathcal{A}_X$ and $\nu \circ f = -\infty$, then we put $H_2(f) = 0$. In [4] the functional $H_2$ is called the Riesz functional.

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Following [4], we define \( \mathcal{P} \) as the class of complex manifolds \( X \) for which there exists a finite sequence of complex manifolds and holomorphic mappings

\[
X_0 \xrightarrow{h_1} X_1 \xrightarrow{h_2} \ldots \xrightarrow{h_m} X_m = X, \quad m \geq 0,
\]

where \( X_0 \) is a domain in a Stein manifold and each \( h_i, i = 1, \ldots, m \), is either a covering or a finite branched covering. More on the class \( \mathcal{P} \) could be found in [4].

For a complex manifold \( X \) we denote \( \text{PSH}(X) \) the set of all plurisubharmonic functions on \( X \). We assume that the constant function \( -\infty \) is plurisubharmonic.

Recall the following result from [4]

**Theorem 1.1.** Let \( X \) be a manifold in \( \mathcal{P} \). If \( \phi \) is an upper semi-continuous function on \( X \), then \( E_{H_1^\phi} \) is plurisubharmonic, and

\[
E_{H_1^\phi} = \sup \{ u \in \text{PSH}(X) : u \leq \phi \}.
\]

If \( v \) is a continuous plurisubharmonic function on \( X \), then \( E_{H_2^\phi} \) is plurisubharmonic, and

\[
E_{H_2^\phi} = \sup \{ u \in \text{PSH}(X) : u \leq 0, \mathcal{L}(u) \geq \mathcal{L}(v) \},
\]

where \( \mathcal{L}(u) \) is the Levi form \( i\theta \partial u \) of \( u \).

In Theorem 1.1 the plurisubharmonicity of \( H_2 \) is obtained as a corollary from the plurisubharmonicity of \( H_1 \) (see [4]). Actually, this is the reason why in Theorem 1.1 the authors assumed the continuity of \( v \). The main purpose of this note is to show the plurisubharmonicity of \( H_2 \) for any plurisubharmonic function \( v \).

Let \( \phi \) be a plurisuperharmonic function on a complex manifold \( X \), \( \phi \neq +\infty \). We put \( H^\phi(f) = \frac{1}{2\pi} \int_0^{2\pi} \phi(f(e^{i\theta})) \, d\theta \) for \( f \in \mathcal{A}_X \) such that \( \phi \circ f \neq +\infty \) and \( H^\phi(f) = +\infty \) for \( f \in \mathcal{A}_X \) such that \( \phi \circ f \equiv +\infty \). Note that if \( \phi \circ f \neq +\infty \), then \( \phi \circ f \in L^1(\mathbb{T}) \), where \( \mathbb{T} \) is the unit circle. According to our definition \( H^\phi \) is not a disc functional, because it may take the value \( +\infty \). Nevertheless, we may consider the envelope \( E_{H^\phi} \) of \( H^\phi \). It is not difficult to see that \( E_{H^\phi} < +\infty \). We have even more. Namely, we have the following results.

**Theorem 1.2.** Let \( X \) be a complex manifold and let \( \phi \) be a plurisuperharmonic function on \( X \), \( \phi \neq +\infty \). Then \( E_{H^\phi} < +\infty \) and \( E_{H^\phi} \) is an upper semicontinuous function on \( X \).
Theorem 1.3. Let $X$ be a manifold in $\mathcal{P}$ and let $\phi$ be a plurisuperharmonic function on $X$, $\phi \not\equiv +\infty$. Then $E_{H^\phi}$ is a plurisubharmonic function and

\[(1) \quad E_{H^\phi} = \sup\{u \in \text{PSH}(X) : u \leq \phi\} \quad \text{on} \quad X.\]

By the Riesz representation, for a plurisubharmonic function $v$ on a complex manifold $X$ and a holomorphic mapping $f \in \mathcal{A}_X$ such that $v \circ f \not\equiv -\infty$ we have

\[H_v^2(f) = v(f(0)) - \frac{1}{2\pi} \int_0^{2\pi} v(f(e^{i\theta})) \, d\theta.\]

So,

\[(2) \quad H_v^2(f) = v(f(0)) + H_i^{-v}(f) \quad \text{and} \quad E_{H_i^2} = v + E_{H_i^{-v}}.\]

As a simple corollary of Theorem 1.2 and equation (2) we have immediately the following.

Corollary 1.4. Let $X$ be a complex manifold and let $v$ be a plurisubharmonic function on $X$. Then $E_{H_i^2}$ is an upper semicontinuous function in $X$.

Using results from [4], Theorem 1.3, and equation (2) we have the following.

Corollary 1.5. Let $X$ be a manifold in $\mathcal{P}$ and let $v$ be a plurisubharmonic function on $X$. Then $E_{H_i^2}$ is a plurisubharmonic function and

\[E_{H_i^2} = \sup\{u \in \text{PSH}(X) : u \leq 0, \mathcal{L}(u) \geq \mathcal{L}(v)\} \quad \text{on} \quad X.\]

2. Proof of Theorem 1.2

The following two simple results (Lemma 2.1 and Lemma 2.2) play a crucial role in our considerations.

Lemma 2.1. Let $\Omega$ be a domain in $\mathbb{C}^n$ and let $\phi$ be a plurisuperharmonic function on $\Omega$. Then for any $y_0 \in \Omega$ and any $\epsilon > 0$ there exists $r_0 > 0$ such that for any $y_1 \in B(y_0, r), r \in (0, r_0)$, we have

\[\phi(y_0) \geq \frac{1}{b_n r^{2n}} \int_{B_n(y_1, r)} \phi(y) \, d\lambda_n(y) - \epsilon,\]

where $B_n(y_0, r) := \{y \in \mathbb{C}^n : \|y - y_0\| < r\}, B_n := B_n(0, 1), b_n := \lambda_n(B_n)$, and $\lambda_n$ is the Lebesgue measure in $\mathbb{C}^n$.

Proof. Fix $y_0 \in \Omega$ and $\epsilon > 0$. We may assume that $\phi(y_0) \neq +\infty$. Put $\epsilon_1 := \frac{\epsilon}{\alpha_n - 1}$. Since $\phi$ is a lower semicontinuous function, there exists $r_0 > 0$ such that

\[\phi(y) + \epsilon_1 \geq \phi(y_0), \quad y \in B_n(y_0, 2r) \subset \subset \Omega.\]
Fix $r \in (0, r_0)$ and $y_1 \in B_n(y_0, r)$. We have

$$
\phi(y_0) \geq \frac{1}{b_n(2r)^{2n}} \int_{B_n(y_0, 2r)} \phi(y) d\lambda_n(y)
$$

$$
\geq \frac{1}{b_n(2r)^{2n}} \int_{B_n(y_1, r)} \phi(y) d\lambda_n(y)
$$

$$
+ \frac{1}{b_n(2r)^{2n}} \int_{B_n(y_0, 2r) \setminus B_n(y_1, r)} \phi(y) d\lambda_n(y)
$$

$$
\geq \frac{1}{b_n(2r)^{2n}} \int_{B_n(y_1, r)} \phi(y) d\lambda_n(y)
$$

$$
+ \frac{1}{b_n(2r)^{2n}} (\phi(y_0) - \epsilon) (b_n(2r)^{2n} - b_n r^{2n})
$$

$$
\geq \frac{1}{b_n(2r)^{2n}} \int_{B_n(y_1, r)} \phi(y) d\lambda_n(y) + (\phi(y_0) - \epsilon) \left( 1 - \frac{1}{2^{2n}} \right)
$$

$$
= \frac{1}{b_n(2r)^{2n}} \int_{B_n(y_1, r)} \phi(y) d\lambda_n(y) + \phi(y_0) - \epsilon \left( 1 - \frac{1}{2^{2n}} \right).
$$

So,

$$
\phi(y_0) + \epsilon \geq \frac{1}{b_n r^{2n}} \int_{B_n(y_1, r)} \phi(y) d\lambda_n(y).
$$

**Lemma 2.2.** Let $\phi : T \times B_n \to [-\infty, +\infty)$ be an integrable function. Then

$$
\frac{1}{2\pi b_n} \int_0^{2\pi} \int_{B_n} \phi(e^{i\theta}, y) d\theta d\lambda_n(y) = \frac{1}{2\pi b_n} \int_0^{2\pi} \int_{B_n} \phi(e^{i\theta}, e^{i\theta} y) d\theta d\lambda_n(y).
$$

Therefore, there exists $y_0 \in B_n$ such that

$$
\frac{1}{2\pi b_n} \int_0^{2\pi} \int_{B_n} \phi(e^{i\theta}, y) d\theta d\lambda_n(y) \geq \frac{1}{2\pi} \int_0^{2\pi} \phi(e^{i\theta}, e^{i\theta} y_0) d\theta.
$$

**Proof.** Easily follows from measure theory.

Recall also the following result (see Lemma 2.3 in [4]).

**Theorem 2.3.** Let $X$ be a complex manifold and let $f_0 \in \mathcal{A}_X$. Then there exist $r > 1$, an open neighborhood $V$ of $x_0 = f_0(0)$, and $f \in \mathcal{O}(D_r \times V, X)$ such that
(i) \( f(z, x_0) = f_0(z) \) for all \( z \in D_r \),
(ii) \( f(0, x) = x \) for all \( x \in V \),

where \( D_r := \{ z \in \mathbb{C} : |z| < r \} \).

**Lemma 2.4.** Let \( x_0 \in X, \beta \in \mathbb{R} \), and assume that \( E_H(x_0) < \beta \). Then there exist a neighborhood \( V \) of \( x_0 \) in \( X \), \( r > 1 \), and \( f \in \mathcal{O}(D_r \times B_n(r) \times V, X) \), such that \( f(0, 0, x) = f(0, y, x) = x \), \( y \in B_n(r) \), and

\[
\frac{1}{b_n} \int_{B_n} H(f(\cdot, y, x)) d\lambda_n(y) < \beta \quad \text{for all} \quad x \in V.
\]

**Proof of Lemma 2.4.** By definition there exists \( f_0 \in \mathcal{A}_X \) such that \( f_0(0) = x_0 \) and \( H(f_0) < \beta \). According to Theorem 2.3 there exist \( \tilde{r} > 1 \), an open neighborhood \( \tilde{V} \) of \( x_0 \), and \( \tilde{f} \in \mathcal{O}(D_{\tilde{r}} \times \tilde{V}, X) \) such that \( \tilde{f}(z, x_0) = f_0(z) \) for all \( z \in D_{\tilde{r}} \) and \( \tilde{f}(0, x) = x \) for all \( x \in \tilde{V} \).

Let \( (U, \zeta) \) be a local coordinate centered at \( x_0 \). We may assume that \( U \subset \tilde{V} \) and \( \zeta : U \rightarrow \mathbb{B}_n \) and \( \zeta(x_0) = 0 \). Consider the function

\[
F(w) = \frac{1}{2\pi} \int_0^{2\pi} \phi(\tilde{f}(e^{i\theta}, \zeta^{-1}(w))) d\theta, \quad w \in \mathbb{B}_n.
\]

Note that \( F \) is a plurisuperharmonic function in \( \mathbb{B}_n \). Fix an \( \epsilon > 0 \) such that \( H(f_0) < \beta - \epsilon \). Then there exists \( r > 0 \) such that

\[
\frac{1}{b_n} \int_{B_n} F(y_1 + ry) d\lambda_n(y) \leq F(0) + \epsilon,
\]

for any \( y_1 \in B_n(r) \). Put \( f(z, y, x) := \tilde{f}(z, \zeta^{-1}(\zeta(x) + rzy)) \) (use here (3)) and \( V := \zeta^{-1}(B_n(r)) \).

**Proof of Theorem 1.2.** Let \( x_0 \in X \) be fixed. Let us show that \( E_{H^\phi}(x_0) < +\infty \). Assume that \( (U, \zeta) \) is a local coordinate centered at \( x_0 \), i.e. \( \zeta(x_0) = 0 \). We may assume that \( \zeta : U \rightarrow \zeta(U) = B_n(2) \). Take an \( x_1 \in U \) such that \( \phi(x_1) < +\infty \). Consider the superharmonic function \( u := \phi \circ f \), where \( f(z) := \zeta^{-1}\left(z \frac{\zeta(x)}{||\zeta(x)||}\right) \). Note that \( f(0) = x_0 \) and \( u(||\zeta(x)||) = \phi(x_1) < +\infty \). Hence, \( H(f) < +\infty \).

Now, let \( \beta > E_H(x_0) \) be fixed. According to Lemma 2.4 there exist a neighborhood \( V \) of \( x_0 \) in \( X \), \( r > 1 \), and \( f \in \mathcal{O}(D_r \times B_n(r) \times V, X) \), such that \( f(0, 0, x) = x \) and

\[
\frac{1}{b_n} \int_{B_n} H(f(\cdot, w, x)) d\lambda_n(y) < \beta \quad \text{for all} \quad x \in V.
\]
Fix $x \in V$. By Lemma 2.2 there exists $y_0 \in B_n$ such that
\[
\frac{1}{b_n} \int_{B_n} H(f(\cdot, y, x)) d\lambda_n(y) \geq H(g),
\]
where $g(z) = f(z, zy_0, x)$. It suffices to note that $g(0) = x$.

3. Proof of Theorem 1.3

From [4] it follows that it suffices to prove Theorem 1.3 for domains in $\mathbb{C}^n$. So, in this section we assume that $X$ is a domain in $\mathbb{C}^n$ and $\phi$ is a plurisuperharmonic function on $X$, $\phi \not\equiv +\infty$. Moreover, the equality (1) follows from the plurisubharmonicity of $E_{H\phi}$ (see also [5], [6]).

For the proof of Theorem 1.3 it suffices to show that
\[
E_{H}(h(0)) \leq \frac{1}{2\pi} \int_0^{2\pi} E_{H}(h(e^{i\theta})) d\theta
\]
for every $h \in \mathcal{A}_X$ such that $\phi \circ h \not\equiv +\infty$ (since we know that $E_H$ is upper semi-continuous).

The idea of the proof of (6) goes back to E. Poletsky ([5], [6]) and proceeds as follows. It suffices to show that for every $\epsilon > 0$ and $v \in C(X, \mathbb{R})$ with $v \geq E_H$ there exists $g \in \mathcal{A}_X$ such that $g(0) = h(0)$ and
\[
H(g) \leq \frac{1}{2\pi} \int_0^{2\pi} v(h(e^{i\theta})) d\theta + \epsilon.
\]
For the construction of $g$, first we show that there exists $r > 1$ and $F \in C^\infty(D_r \times \mathbb{T}, X)$ such that $F(\cdot, w) \in \mathcal{A}_X$, $F(0, w) = h(w)$ for all $w \in \mathbb{T}$, and
\[
\frac{1}{2\pi} \int_0^{2\pi} H(F(\cdot, e^{i\theta})) d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} v(h(e^{i\theta})) d\theta + \epsilon.
\]
Next we show that there exist $s \in (1, r)$ and $G \in \mathcal{O}(D_s \times D_s, X)$ such that $G(0, w) = h(w)$ for all $w \in D_s$ and
\[
\frac{1}{2\pi} \int_0^{2\pi} H(G(\cdot, e^{i\theta})) d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} H(F(\cdot, e^{i\theta})) d\theta + \epsilon.
\]
Finally, we show that there exists $\theta_0 \in [0, 2\pi)$ such that if $g$ is defined by the formula $g(z) = G(e^{i\theta_0}z, z)$ then
\[
H(g) \leq \frac{1}{2\pi} \int_0^{2\pi} H(G(\cdot, e^{i\theta})) d\theta.
\]
As we see, main steps of the proof completely coincide with the proof of plurisubharmonicity of $E_{H^q}$ for an upper semi-continuous function $\phi$ (see the discussion before Lemma 2.3 in [4]). But the proofs of these steps turn out to be very technical and complicated.

Let us start with the following simple result, which follows from the measure theory.

**Lemma 3.1.** Let $h \in \mathcal{A}_X$ be such that $\phi \circ h \not\equiv +\infty$ and, therefore, $\phi \circ h \in L^1(T)$. Then for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int_I \phi \circ h(w) \, d\sigma(w) < \epsilon$$

for any measurable set $I \subset T$ with $\sigma(I) < \delta$, where $\sigma$ is the arc length measure on $T$.

**Lemma 3.2** (cf. Lemma 5.5 in [5], Lemma 2.5 in [4]). Let $h \in \mathcal{A}_X$ be such that $\phi \circ h \not\equiv +\infty$, $\epsilon > 0$, and $v \in C(X, \mathbb{R})$ with $v \geq EH$. Then there exist $r > 1$ and $F \in C^\infty(D_r \times X)$ such that $F(\cdot, w) \in \mathcal{O}(D_r, X)$, $F(0, w) = h(w)$ for all $w \in T$, and

$$\frac{1}{2\pi} \int_0^{2\pi} H(F(\cdot, e^{i\theta})) \, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} v(e^{i\theta}) \, d\theta + \epsilon. \tag{7}$$

**Proof of Lemma 3.2.** Let $w_0 \in T$. Put $x_0 = h(w_0)$. From Lemma 2.4 it follows that there exist $r_0 > 1$, $f_0 \in \mathcal{O}(D_{r_0} \times B_n(r_0) \times V_0, X)$ such that $f_0(0, 0, x) = x$, $x \in V_0$, and

$$\frac{1}{b_n} \int_{B_n} H(f_0(\cdot, y, x)) \, d\lambda_n(y) < v(x_0) \quad \text{for all} \quad x \in V_0.$$

By replacing $V_0$ by a smaller neighborhood of $x_0$ we get

$$\frac{1}{b_n} \int_{B_n} H(f_0(\cdot, y, x)) \, d\lambda_n(y) \leq v(x) + \frac{\epsilon}{4}, \quad x \in V_0.$$

We can take an open arc $I_0 \subset T$ containing $w_0$ such that $h(w) \in V_0$ for all $w \in I_0$. Define $F_0 : D_{r_0} \times B_n(r_0) \times I_0 \to X$ by $F_0(z, y, w) = f(z, y, h(w))$. By replacing $r_0$ by a smaller number in $(1, \infty)$ and $I_0$ by a smaller open arc containing $w_0$, we may assume that $F_0(D_{r_0} \times B_n(r_0) \times I_0)$ is relatively compact in $X$.

Using compactness argument, we see that there exist a covering $\{I_v\}_{v=1}^V$ of $T$ by open arcs, $r_v > 1$, $F_v \in C^\infty(D_{r_v} \times B_n(r_v) \times I_v, X)$ such that

a) $F_v(\cdot, \cdot, w) \in \mathcal{O}(D_{r_v} \times B_n(r_v), X), \quad$
b) $F_\nu(0, 0, w) = h(w)$,

c) $F_\nu(D_{r_\nu} \times B_n(r_\nu) \times I_\nu)$ is relatively compact in $X$,

d) $\frac{1}{b_\nu} \int_0^{2\pi} H(F_\nu(\cdot, y, w)) d\lambda_n(y) < v(h(w)) + \frac{\epsilon}{4}$,

for $w \in I_\nu$, $\nu = 1, \ldots, N$.

Put $r := \min_\nu r_\nu$. Let $M \subset X$ be a compact set such that $\bigcup_{\nu=1}^N F_\nu(D_{r_\nu} \times B_n(r_\nu) \times I_\nu) \subset M$ and let $C > \sup_M |v|$. By Lemma 3.1 there exists a $\delta > 0$ such that for any measurable set $I \subset T$ with $\sigma(I) < \delta$ we have

$$\int_I \phi \circ h d\sigma < \frac{\epsilon}{4}. $$

There exist a subset $A \subset \{1, \ldots, N\}$ and disjoint closed arcs $J_\nu \subset I_\nu$, $\nu \in A$, such that $\sigma(T \setminus \bigcup J_\nu) < \min\{\delta, \frac{\epsilon}{2\pi}\}$. By possibly removing some arc $I_\nu$ from the covering of $T$, we may assume that $A = \{1, \ldots, N\}$. We take disjoint open arcs $K_\nu$ such that $J_\nu \subset K_\nu \subset I_\nu$. Now, we take a function $\rho \in C^\infty(T)$ such that

- $0 \leq \rho \leq 1$,
- $\rho(w) = 1$ for $w \in \bigcup J_\nu$,
- $\rho(w) = 0$ for $w \in T \setminus \bigcup K_\nu$,

Note that

$$\int_{J_\nu} \frac{1}{b_\nu} \int_{B_n} H(F_\nu(\cdot, y, w)) d\sigma(w) d\lambda_n(y) \leq \int_{J_\nu} v(h(w)) d\sigma(w) + \frac{\epsilon}{4} \sigma(J_\nu).$$

Hence, there exists $y_\nu \in B_n$ such that

$$\int_{J_\nu} H(F_\nu(\cdot, y_\nu, w)) d\sigma(w) d\lambda_n(y) \leq \int_{J_\nu} v(h(w)) d\sigma(w) + \frac{\epsilon}{4} \sigma(J_\nu).$$

We define $F : D_r \times T \to X$ by

$$F(z, w) = \begin{cases} F_\nu(\rho(w)z, y_\nu, w), & z \in D_r, w \in K_\nu, \\ h(w), & z \in D_r, w \in T \setminus \bigcup K_\nu. \end{cases}$$

The choice of $\rho$ ensures that $F \in C^\infty(D_r \times T, X)$, $F(\cdot, w) \in C(D_r, X)$, and $F(0, w) = h(w)$, $w \in T$. Since $\phi$ is a plurisuperharmonic function,

$$(8) \quad H(F(\cdot, w)) \leq \phi(F(0, w) = \phi(h(w)),$$ \quad w \in T. $$
If we combine the inequalities we already have, then we get
\[
\frac{1}{2\pi} \int_{0}^{2\pi} H(F(\cdot, e^{i\theta})) \, d\theta \leq \sum_{\nu} \frac{1}{2\pi} \int_{J_{\nu}} H(F_{\nu}(\cdot, y_{\nu}, w)) \, d\sigma(w) + \frac{\epsilon}{4} \\
\leq \sum_{\nu} \frac{1}{2\pi} \int_{J_{\nu}} v \circ h \, d\sigma + \frac{\epsilon}{2} \leq \frac{1}{2\pi} \int_{T} v \circ h \, d\sigma + \epsilon,
\]
and we have proved (7).

Recall the following result (see Lemma 2.6 in [4], cf. Lemma 5.6 in [5] and Lemma 6 in [1]).

**Lemma 3.3.** Let \( r > 1, h \in \mathcal{O}(D_{r}, X) \), and \( F \in C^\infty(D_{r} \times T, X) \), such that \( F(\cdot, w) \in \mathcal{O}(D_{r}, X) \), and \( F(0, w) = h(w) \) for all \( w \in T \). Then there exist \( s \in (1, r) \), a natural number \( j_{0} \), and a sequence \( F_{j} \in \mathcal{O}(D_{s} \times A_{j}, X), j \geq j_{0} \), where \( A_{j} \) is an open annulus containing \( T \), such that:

(i) \( F_{j} \to F \) uniformly on \( D_{s} \times T \) as \( j \to \infty \),

(ii) there is an integer \( \ell_{j} \geq j \) such that the map \( (z, w) \mapsto F_{j}(zw^{\ell_{j}}, w) \) can be extended to a map \( G_{j} \in \mathcal{O}(D_{s}^{2}, X) \), where \( s_{j} \in (1, s) \), and

(iii) \( G_{j}(0, w) = h(w) \) for all \( w \in D_{s_{j}} \).

**Lemma 3.4.** Let \( h \) and \( F \) satisfy the conditions of Lemma 3.2. Then for every \( \epsilon > 0 \) there exist \( s \in (1, r) \) and \( G \in \mathcal{O}(D_{s} \times D_{s}, X) \) such that \( G(0, w) = h(w) \) for all \( w \in D_{s} \), and
\[
\frac{1}{2\pi} \int_{0}^{2\pi} H(G(\cdot, e^{i\theta})) \, d\theta \leq \frac{1}{2\pi} \int_{0}^{2\pi} H(F(\cdot, e^{i\theta})) \, d\theta + \epsilon.
\]

**Proof of Lemma 3.4.** For any fixed \( z, w \in T \) there exists \( r(z, w) > 0 \) such that
\[
\frac{1}{b_{n}} \int_{B_{n}} \phi(y_{1} + ry) \, d\lambda_{n}(y) \leq \phi(F(z, w)) + \frac{\epsilon}{2}
\]
for \( y_{1} \in B(F(z, w), r), r \in (0, r(z, w)) \). Hence, for any fixed \( z, w \in T \) we have
\[
\limsup_{m \to \infty} \limsup_{k \to \infty} \frac{1}{b_{n}} \int_{B_{n}} \phi\left(F_{k}(z, w) + \frac{1}{m}y\right) \, d\lambda_{n}(y) \leq \phi(F(z, w)) + \frac{\epsilon}{2}.
\]
By Fatou’s theorem, we have
\[
\limsup_{m \to \infty} \limsup_{k \to \infty} \frac{1}{4\pi^{2}} \int_{0}^{2\pi} \int_{0}^{2\pi} \left[ \frac{1}{b_{n}} \int_{B_{n}} \phi\left(F_{k}(e^{i\theta}, e^{i\tau}) + \frac{1}{m}y\right) \, d\lambda_{n}(y) \right] \, d\theta \, d\tau
\]
\[
\leq \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \lim \sup_{m \to \infty} \lim \sup_{k \to \infty} \frac{1}{b_n} \int_{B_n} \phi \left( F_k(e^{i\theta}, e^{i\tau}) + \frac{1}{m} y \right) d\lambda_n(y) \, d\theta \, d\tau
\]
\[
\leq \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \phi(F(e^{i\theta}, e^{i\tau})) \, d\theta \, d\tau + \frac{\epsilon}{2}.
\]

Hence, there exist \(m_0\) and \(k_0\) such that
\[
\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \left[ \frac{1}{b_n} \int_{B_n} \phi \left( F_{k_0}(e^{i\theta}, e^{i\tau}) + \frac{1}{m_0} y \right) d\lambda_n(y) \right] \, d\theta \, d\tau
\]
\[
\leq \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \phi(F(e^{i\theta}, e^{i\tau})) \, d\theta \, d\tau + \epsilon.
\]

So, there exists \(y_0 \in B_n\) such that
\[
\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \phi \left( F_{k_0}(e^{i\theta}, e^{i\tau}) + \frac{1}{m_0} e^{i\theta} y_0 \right) \, d\theta \, d\tau
\]
\[
\leq \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \phi(F(e^{i\theta}, e^{i\tau})) \, d\theta \, d\tau + \epsilon.
\]

Put \(G(z, w) = G_{k_0}(z, w) + \frac{1}{m_0} z w^\ell_0 y_0\), where \(G_{k_0}\) is given by Lemma 3.3 (iii).

**Lemma 3.5.** Let \(s > 1\) and \(G \in \mathcal{O}(\mathbb{D}_s \times \mathbb{D}_s, X)\). Then there exists \(g \in \mathcal{O}(\mathbb{D}_s, X)\) such that \(g(0) = G(0, 0)\) and
\[
H(g) \leq \frac{1}{2\pi} \int_0^{2\pi} H(G(\cdot, e^{i\theta})) \, d\theta.
\]

**Proof of Lemma 3.5.** Note that
\[
\frac{1}{2\pi} \int_0^{2\pi} H(G(\cdot, e^{i\theta})) \, d\theta = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \phi(G(e^{i\tau}, e^{i\theta})) \, d\tau \, d\theta
\]
\[
= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \phi(G(e^{i\tau}, e^{i\theta+i\tau})) \, d\tau \, d\theta.
\]

So, there exists \(\theta_0 \in [0, 2\pi)\) such that
\[
\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \phi(G(e^{i\tau}, e^{i\theta+i\tau})) \, d\tau \, d\theta \geq \frac{1}{2\pi} \int_0^{2\pi} \phi(G(e^{i\tau}, e^{i\theta_0} e^{i\tau})) \, d\tau.
\]

Put \(g(z) = G(z, e^{i\theta_0} z)\).
Remark 3.6. In a forthcoming paper [2], the author will continue the study of plurisubharmonicity of the Poisson functional.

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REFERENCES

INSTITUTE OF MATHEMATICS
JAGIELLONIAN UNIVERSITY
REYMONTA 4-526
30-059 KRAKÓW
POLAND
E-mail: edigaria@im.uj.edu.pl