MATH. SCAND. 91 (2002), 214–220

DISCRETENESS OF SUBGROUPS OF SL(2, $\mathbf{C}$) CONTAINING ELLIPTIC ELEMENTS

PEKKA TUKIA and XIANTAO WANG

Abstract

The following result is the main result of the paper. Let $G \subset \mathrm{SL}(2, \mathbf{C})$ be non-elementary. If G contains an elliptic element of order at least 3, then G is discrete if and only if each non-elementary subgroup generated by two elliptic elements of G is discrete.

1. Introduction

One of the consequences of Jørgensen's inequality [7] is that a non-elementary subgroup of $\mathrm{SL}(2, \mathbf{C})$ is discrete if every two-generator subgroup is discrete (Jørgensen [7], [8]); if $G \subset \mathrm{SL}(2, \mathbf{R})$, the discreteness follows as soon as every cyclic subgroup is discrete (Jørgensen [8]). These results were extended by Wang and Yang who showed that G is discrete if every subgroup generated by two loxodromic elements is discrete [11]; if G contains parabolic elements, then G is discrete as soon as every subgroup generated by two parabolic elements is discrete [12]. The main result of this paper is Theorem 3.1 showing that G is discrete as soon as every non-elementary subgroup generated by two elliptic elements is discrete; here we must assume that G contains an elliptic element of order at least 3. We will also show (Theorem 4.1) that if every subgroup generated by an elliptic and a loxodromic element is discrete, then G is discrete, provided that there are elliptic elements of order at least 3. See the references [1], [2], [3], [5], [6], [9], [10] for further discussions of these theorems.

Finally, we complement these results for groups containing parabolic elements. We will prove that if G is non-elementary and contains parabolic elements, then G is discrete if every non-elementary subgroup generated by a parabolic and a loxodromic element is discrete (Theorem 5.1). The missing result would be that if non-elementary G contains parabolic and elliptic elements, then G is discrete if every subgroup generated by a parabolic and an elliptic element is discrete but this question is left open.

Received March 15, 2000.

The reason for the assumption that there are elliptic elements of order at least 3 is that two elliptic elements f and g of order 2 always generate an elementary group G. This follows for algebraic reasons since the cyclic group generated by fg is of index 1 or 2 in G.

ACKNOWLEDGEMENTS. This work was completed during the second author's visit to the University of Helsinki. He wishes to express his thanks to the Department of Mathematics of the University of Helsinki for hospitality. Research of the second author was partly supported by FNS of China, grant number 19801011.

2. Notations and preliminary results

We denote by $H^3 = \{(x, y, z) \in \mathbf{R}^3 : z > 0\}$ the hyperbolic 3-space and the hyperbolic space with boundary is $\bar{H}^3 = H^3 \cup \overline{\mathbf{C}}$. If $f \in \mathrm{SL}(2, \mathbf{C})$, we regard f as a Möbius transformation of $\overline{\mathbf{C}}$ and denote by $\tilde{f}$ the Poincaré extension of f to $\bar{H}^3$ and write

$$\mathrm{fix}(f) = \{x \in \overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\} : f(x) = x\},$$

$$\mathrm{ord}(f) = \text{the order of } f \text{ when } f \text{ is regarded as a Möbius transformation,}$$

$$A_f = \{z \in \bar{H}^3 : \tilde{f}(z) = z\};$$

the notation A_f is used only if f is elliptic and A_f is the the *axis* of f.

The letter G always denotes a subgroup of $\mathrm{SL}(2, \mathbf{C})$ unless otherwise stated. The group G is called *elementary* if there is $z_0 \in \bar{H}^3$ such that the orbit

$$Gz_0 = \{\tilde{f}(z_0) : f \in G\}$$

is finite. In fact (cf. [1, p. 84]), if G is elementary, then either there is a common fixed point in H^3 of the elements of G (if G is elliptic) or a one- or two-point orbit in $\overline{\mathbf{C}}$ and this is the fixed point set of parabolic or loxodromic elements in the group. We call G is *non-elementary* if G is not elementary.

The following characterization of non-elementariness of a group generated by two elliptic elements is crucial for us. It follows easily from the facts that an elementary group has either a fixpoint in $\bar{H}^3$ or there is a two-point orbit, cf. [1, p. 84]. We state this in

LEMMA 2.1. *Two elliptic elements $f, g \in \mathrm{SL}(2, \mathbf{C})$, whose orders are not both equal to 2, generate a non-elementary group if and only if their Poincaré extensions to $\bar{H}^3$ have no common fixed points.*

The same conclusion is true if f is elliptic of order at least 3 and g is loxodromic.

LEMMA 2.2. *Let G be non-elementary. If $g \in G$ is elliptic, then there are infinitely many elements g_i of G such that each g_i is conjugate to g in G and no two $\tilde{g}_i$'s have common fixpoints (i.e. their axes are disjoint).*

PROOF. Suppose $g \in G$ is elliptic. Since G is non-elementary, G contains a loxodromic element f (see [1, Theorem 5.1.3]) such that

$$\mathrm{fix}(f) \cap \mathrm{fix}(g) = \phi.$$

Thus A_g is disjoint from $\mathrm{fix}(f)$ and hence $f^k(A_f)$ tend toward the attracting fixpoint of f as $k \to \infty$. It easily follows that there is a sequence $k_1 < k_2 \dots$ of integers such that the sets

$$f^{k_i}(A_g) = A_{f^{k_i} g f^{-k_i}}$$

are pairwise disjoint. This proves the lemma.

Lemmas 2.1 and 2.2 have the

COROLLARY 2.3. *Let G be non-elementary. Then G contains a non-elementary subgroup generated by two elliptic elements if and only if G contains an elliptic element of order at least 3.*

3. A discreteness criterion of subgroups of SL(2, C) with elliptic elements

In this section, we will prove the following main result of this paper.

THEOREM 3.1. *Let G be non-elementary. If G contain an elliptic element of order at least* 3, *then G is discrete if and only if each non-elementary subgroup generated by two elliptic elements of G is discrete.*

REMARK. Two elliptic elements of order 2 always generate an elementary discrete group, cf. the Introduction. This is the reason for the assumption that there are elliptic elements of order at least 3.

We start with

LEMMA 3.2. *Let G be non-elementary. If G contains elliptic elements and each non.elementary subgroup generated by two elliptic elements of G is discrete, then G contains no purely elliptic sequence $\{f_n\}$ such that $f_n \to I$ as $n \to \infty$. Here I is the identity element.*

PROOF. Suppose G contains a purely elliptic sequence $\{f_n\}$ such that $f_n \to I$ as $n \to \infty$. We can obtain by passing to a subsequence (denoted in the same manner) that $\mathrm{fix}(f_n)$ tends in the Hausdorff metric toward a one- or two-point set Y. Since G is non-elementary, there is a loxodromic element $h \in G$ (cf. [1, Theorem 5.1.3]) such that

$$\mathrm{fix}(h) \cap Y = \phi.$$

Thus, since $\mathrm{fix}(f_n) \to Y$, there are a neighborhood V of the attracting fixpoint of h and a neighborhood W of the repelling fixpoint of h such that for large n

$$A_{f_n} \cap V = \phi \qquad \text{and} \qquad A_{f_n} \cap W = \phi.$$

The latter of these formulae implies that there are p and n_0 such that $h^p(A_{f_n}) \subset V$ if $n > n_0$. We fix such p and n_0. Now, $h^p(A_{f_n})$ is the axis of $h^p f_n h^{-p}$. Hence the axes of f_n and $g_n = h^p f_n h^{-p}$ do not intersect and hence $\langle f_n, g_n \rangle$ is non-elementary by Lemma 2.1. However, both $f_n \to I$ and $g_n \to I$ and hence the commutator $[f_n, g_n] \to I$ and so Jørgensen's inequality

$$|\,\mathrm{tr}^2(f_n) - 4| + |\,\mathrm{tr}[f_n, g_n] - 2| \geq 1$$

is violated for large n. This contradiction proves the lemma.

Proof of Theorem 3.1. Suppose that the non-elementary G contains an elliptic element of order at least 3 and that every non-elementary subgroup generated by two elliptic elements of G is discrete but G is not discrete. Then G contains a sequence $\{f_n\}$ such that $f_n \to I$ as $n \to \infty$ and where each $f_n \neq I$. We can assume that $\mathrm{fix}(f_n)$ tends in the Hausdorff metric toward a one- or two-point set Y. We can find an elliptic element $g \in G$ of order at least 3 whose fixpoint set is disjoint from Y, cf. Lemma 2.2. Thus we can assume that $\mathrm{fix}(f_n)$, $\mathrm{fix}(g)$ and $\mathrm{fix}(f_n g f_n^{-1})$ are disjoint for large n.

Let $h_n = f_n g f_n^{-1}$ and set $G_n = \langle g, h_n \rangle$. If G_n is non-elementary, then G_n is discrete by the assumptions of the theorem and hence Jørgensen's inequality

$$|\,\mathrm{tr}^2(g h_n^{-1}) - 4| + |\,\mathrm{tr}[g h_n^{-1}, h_n] - 2| \geq 1$$

is true. However, $h_n \to g$ and hence the left hand side of the above inequality tends to 0. This is a contradiction and so G_n is elementary for large n. This is possible only if the axes of g and h_n intersect (Lemma 2.1) and since they do not have common fixpoints in $\overline{\mathbf{C}}$, they must have a common fixpoint p_n in H^3.

It follows that $h_n^{-1} g$ also has the fixpoint $p_n \in H^3$ and hence $\{h_n^{-1} g\}$ is a purely elliptic sequence tending to I and this is impossible by Lemma 3.2.

4. The elliptic-loxodromic case

As an application of theorem 3.1, we will prove a theorem which is midway between our theorem and the theorem of Wang and Yang in [11] where it was shown that if G is non-elementary and every non-elementary subgroup generated by two loxodromic elements is discrete, then G is discrete.

THEOREM 4.1. *Let G be non-elementary. If G contains an elliptic element of order at least 3, then G is discrete if and only if each non-elementary subgroup $\langle f, g\rangle$ of G is discrete where f is elliptic and g is loxodromic.*

Theorem 4.1 follows from theorem 3.1 and the following lemma.

LEMMA 4.2. *Let G be non-elementary such that G contains an elliptic element of order at least 3. Suppose that every non-elementary subgroup of G generated by one elliptic and one loxodromic element of G is discrete. Then every non-elementary subgroup of G generated by two elliptic elements is discrete.*

PROOF. Let f_1 and g_1 be elliptic elements of G such that $H = \langle f_1, g_1\rangle$ is non-elementary. We show that H is discrete. If $f_1 g_1$ is loxodromic, then $H = \langle f_1, f_1 g_1\rangle$ is discrete by assumption. If $f_1 g_1$ is non-loxodromic, then the proof of case (3) of Lemma 3 of [11] shows that H is conjugate to a non-elementary subgroup of $\mathrm{SL}(2, \mathsf{R})$, cf. also Lemma 5.23 of Gehring and Martin [4].

Thus we can assume that $H \subset \mathrm{SL}(2, \mathsf{R})$. However, a non-discrete and non-elementary subgroup of $\mathrm{SL}(2, \mathsf{R})$ contains a sequence $\{h_n\}$ of hyperbolic elements such that $h_n \to I$ as $n \to \infty$, cf. [3, Corollary p. 199]. Pass first to a subsequence so that the sets $\mathrm{fix}(h_n)$ have the Hausdorff limit X which is a one- or two-point set. Since H is non-elementary, at least one of the elliptic elements f_1 or g_1 is of order at least 3, say $\mathrm{ord}(f_1) \geq 3$. Again, like in the proof of Lemma 3.2, the non-elementariness of H implies that we can conjugate f_1 in H to obtain $f \in H$ so that $\mathrm{fix}(f)$ is disjoint from X. Thus we can assume that for large n

$$\mathrm{fix}(f) \cap \mathrm{fix}(h_n) = \phi.$$

It follows (Lemma 2.1) that $H_n = \langle f, h_n\rangle$ is a non-elementary subgroup of H for large n. Hence H_n is discrete as a subgroup of H. However, this is a contradiction since now Jørgensen's inequality

$$|\,\mathrm{tr}^2(h_n) - 4| + |\,\mathrm{tr}[h_n, f] - 2| \geq 1$$

is violated for large n since $h_n \to I$ as $n \to \infty$.

5. The parabolic-loxodromic case

We complement our results and prove

THEOREM 5.1. *Let G be a non-elementary group containing parabolic elements. Then G is discrete if and only if every non-elementary subgroup generated by a parabolic and a loxodromic element of G is discrete.*

PROOF. Suppose that G is not discrete although every subgroup generated by a parabolic and a loxodromic element is discrete. We derive a contradiction as follow. Thus there is an infinite sequence f_i of G, $f_i \neq I$, such that $f_i \to I$ as $i \to \infty$. Pass to a subsequence so that $\mathrm{fix}(f_i)$ have the Hausdorff limit X which is a one- or two-point set. Since G is non-elementary and contains parabolic elements, we can find a parabolic element g whose fixpoint is not a point of X. Thus, for large i, the fixpoint sets of g and f_i are disjoint. Using this fact, a simple calculation shows that for large i there is n_i such that $h_i = f_i g^{n_i}$ is loxodromic. Since f_i and g do not have common fixpoints, neither have h_i and g and so $H_i = \langle g, h_i \rangle = \langle g, f_i \rangle$ is non-elementary and hence discrete by assumption. However, for large i,

$$|\operatorname{tr}^2(f_i) - 4| + |\operatorname{tr}[f_i, g_j] - 2| < 1$$

and so H_i would have to be elementary by Jørgensen's inequality. This contradiction proves the theorem.

REFERENCES

1. Beardon, A. F., *The geometry of discrete groups*, Springer-Verlag, 1983.
2. Beardon, A. F., *Some remarks on nondiscrete Möbius groups*, Ann. Acad. Sci. Fenn Ser. A I Math. 21 (1996), 69–79.
3. Doyle, C. and James, D., *Discreteness criteria and higher order generations for subgroups of* SL(2, R), Illinois J. Math. 25 (1981), 191–200.
4. Gehring, F. W. and Martin, G. J., *Inequalities for Möbius transformations*, J. Reine Angew. Math. 418 (1991), 31–76.
5. Gilman, J., *Inequalities in discrete subgroups of* PSL(2, R), Canad. J. Math. 40 (1988), 114–130.
6. Isachenko, N. A., *Systems of generators of subgroups of* PSL(2, C), Siberian Math. J. 31 (1990), 162–165.
7. Jørgensen, T., *On discrete groups of Möbius transformations*, Amer. J. Math. 98 (1976), 739–749.
8. Jørgensen, T., *A note on subgroups of* SL(2, C), Quart. J. Math. Oxford 28 (1977), 209–212.
9. Rosenberger, G., *Some remarks on a paper of C. Doyle and D. James on subgroups of* SL(2, R), Illinois J. Math. 28 (1984), 348–351.
10. Rosenberger, G., *Minimal generating systems of a subgroup of* SL(2, C), Proc. Edinburgh Math. Soc. 31 (1988), 261–265.
11. Wang, X. and Yang, W., *Discreteness criterion for subgroups in* SL(2, C), Math. Proc. Cambridge Philos. Soc. 124 (1998), 51–55.

12. Wang, X. and Yang, W., *Dense subgroups and discrete subgroups in* $\mathrm{SL}(2, C)$, Quart. J. Math. Oxford 50 (1999), 517–521.

PEKKA TUKIA
DEPARTMENT OF MATHEMATICS
P. O. BOX 4 (YLIOPISTONKATU 5)
FIN-00014 UNIVERSITY OF HELSINKI
FINLAND
E-mail: pekka.tukia@helsinki.fi

XIANTAO WANG
DEPARTMENT OF MATHEMATICS
HUNAN UNIVERSITY
CHANGSHA, HUNAN 410082
P. R. OF CHINA
E-mail: xtwang@mail.hunu.edu.cn