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SEQUENCES FOR COMPLEXES II

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1. Introduction and Notation

This short paper elaborates on an example given in [4] to illustrate an application of sequences for complexes:

> Let R be a local ring with a dualizing complex D, and let M be a finitely generated R-module; then a sequence $x_1, \dots, x_n$ is part of a system of parameters for M if and only if it is a $\mathbf{R}\mathrm{Hom}_R(M, D)$-sequence [4, 5.10].

The final Theorem 3.9 of this paper generalizes the result above in two directions: the dualizing complex is replaced by a Cohen-Macaulay semidualizing complex (see [3, Sec. 2] or 3.8 below for definitions), and the finite module is replaced by a complex with finite homology.

Before we can even state, let alone prove, this generalization of [4, 5.10] we have to introduce and study *parameters* for complexes. For a finite R-module M every M-sequence is part of a system of parameters for M, so, loosely speaking, regular elements are just special parameters. For a complex X, however, parameters and regular elements are two different things, and kinship between them implies strong relations between two measures of the size of X: the *amplitude* and the *Cohen-Macaulay defect* (both defined below). This is described in 3.5, 3.6, and 3.7.

The definition of parameters for complexes is based on a notion of *anchor prime ideals*. These do for complexes what minimal prime ideals do for modules, and the quantitative relations between dimension and depth under dagger duality—studied in [3]—have a qualitative description in terms of anchor and associated prime ideals.

Throughout R denotes a commutative, Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. We use the same notation as in [4], but for convenience we recall a few basic facts.

The homological position and size of a complex X is captured by the *su-*

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premum, *infimum*, and *amplitude*:

$$\sup X = \sup\{\, \ell \in \mathsf{Z} \mid \mathrm{H}_\ell(X) \neq 0 \,\},$$
$$\inf X = \inf\{\, \ell \in \mathsf{Z} \mid \mathrm{H}_\ell(X) \neq 0 \,\}, \qquad \text{and}$$
$$\operatorname{amp} X = \sup X - \inf X.$$

By convention, $\sup X = -\infty$ and $\inf X = \infty$ if $\mathrm{H}(X) = 0$.

The *support* of a complex X is the set

$$\operatorname{Supp}_R X = \{\, \mathfrak{p} \in \operatorname{Spec} R \mid X_\mathfrak{p} \not\simeq 0 \,\} = \bigcup_\ell \operatorname{Supp}_R \mathrm{H}_\ell(X).$$

As usual $\operatorname{Min}_R X$ is the subset of minimal elements in the support.

The *depth* and the (*Krull*) *dimension* of an R-complex X are defined as follows:

$$\operatorname{depth}_R X = -\sup(\mathbf{R}\mathrm{Hom}_R(k, X)), \quad \text{for } X \in \mathscr{D}_-(R), \qquad \text{and}$$
$$\dim_R X = \sup\{\, \dim R/\mathfrak{p} - \inf X_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R X \,\},$$

cf. [6, Sec. 3]. For modules these notions agree with the usual ones. It follows from the definition that

$$\dim_R X \geq \dim_{R_\mathfrak{p}} X_\mathfrak{p} + \dim R/\mathfrak{p} \tag{1.1}$$

for $X \in \mathscr{D}(R)$ and $\mathfrak{p} \in \operatorname{Spec} R$; and there are always inequalities:

$$-\inf X \leq \dim_R X \qquad \text{for} \quad X \in \mathscr{D}_+(R); \qquad \text{and} \tag{1.2}$$

$$-\sup X \leq \operatorname{depth}_R X \quad \text{for} \quad X \in \mathscr{D}_-(R). \tag{1.3}$$

A complex $X \in \mathscr{D}^\mathrm{f}_\mathrm{b}(R)$ is *Cohen-Macaulay* if and only if $\dim_R X = \operatorname{depth}_R X$, that is, if an only if the *Cohen-Macaulay defect*,

$$\operatorname{cmd}_R X = \dim_R X - \operatorname{depth}_R X,$$

is zero. For complexes in $\mathscr{D}^\mathrm{f}_\mathrm{b}(R)$ the Cohen-Macaulay defect is always non-negative, cf. [6, Cor. 3.9].

2. Anchor Prime Ideals

In [4] we introduced associated prime ideals for complexes. The analysis of the support of a complex is continued in this section, and the aim is now to identify the prime ideals that do for complexes what the minimal ones do for modules.

DEFINITIONS 2.1. Let $X \in \mathscr{D}_+(R)$; we say that $\mathfrak{p} \in \operatorname{Spec} R$ is an *anchor prime ideal* for X if and only if $\dim_{R_\mathfrak{p}} X_\mathfrak{p} = -\inf X_\mathfrak{p} > -\infty$. The set of anchor prime ideals for X is denoted by $\operatorname{Anc}_R X$; that is,

$$\operatorname{Anc}_R X = \{\, \mathfrak{p} \in \operatorname{Supp}_R X \mid \dim_{R_\mathfrak{p}} X_\mathfrak{p} + \inf X_\mathfrak{p} = 0 \,\}.$$

For $n \in \mathsf{N}_0$ we set

$$\mathrm{W}_n(X) = \{\, \mathfrak{p} \in \operatorname{Supp}_R X \mid \dim_R X - \dim R/\mathfrak{p} + \inf X_\mathfrak{p} \leq n \,\}.$$

OBSERVATION 2.2. Let S be a multiplicative system in R, and let $\mathfrak{p} \in \operatorname{Spec} R$. If $\mathfrak{p} \cap S = \emptyset$ then $S^{-1}\mathfrak{p}$ is a prime ideal in $S^{-1}R$, and for $X \in \mathscr{D}(R)$ there is an isomorphism $S^{-1}X_{S^{-1}\mathfrak{p}} \simeq X_\mathfrak{p}$ in $\mathscr{D}(R_\mathfrak{p})$. In particular, $\inf S^{-1}X_{S^{-1}\mathfrak{p}} = \inf X_\mathfrak{p}$ and $\dim_{S^{-1}R_{S^{-1}\mathfrak{p}}} S^{-1}X_{S^{-1}\mathfrak{p}} = \dim_{R_\mathfrak{p}} X_\mathfrak{p}$. Thus, the next biconditional holds for $X \in \mathscr{D}_+(R)$ and $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathfrak{p} \cap S = \emptyset$.

$$\mathfrak{p} \in \operatorname{Anc}_R X \iff S^{-1}\mathfrak{p} \in \operatorname{Anc}_{S^{-1}R} S^{-1}X. \tag{2.1}$$

THEOREM 2.3. *For $X \in \mathscr{D}_+(R)$ there are inclusions:*

(a) $$\operatorname{Min}_R X \subseteq \operatorname{Anc}_R X; \qquad \textit{and}$$

(b) $$\mathrm{W}_0(X) \subseteq \operatorname{Anc}_R X.$$

Furthermore, if $\operatorname{amp} X = 0$, that is, if X is equivalent to a module up to a shift, then

(c) $$\operatorname{Anc}_R X = \operatorname{Min}_R X \subseteq \operatorname{Ass}_R X;$$

and if X is Cohen-Macaulay, that is, $X \in \mathscr{D}_\mathrm{b}^\mathrm{f}(R)$ and $\dim_R X = \operatorname{depth}_R X$, then

(d) $$\operatorname{Ass}_R X \subseteq \operatorname{Anc}_R X = \mathrm{W}_0(X).$$

PROOF. In the following X belongs to $\mathscr{D}_+(R)$.

(a): If $\mathfrak{p}$ belongs to $\operatorname{Min}_R X$ then $\operatorname{Supp}_{R_\mathfrak{p}} X_\mathfrak{p} = \{\mathfrak{p}_\mathfrak{p}\}$, so $\dim_{R_\mathfrak{p}} X_\mathfrak{p} = -\inf X_\mathfrak{p}$, that is, $\mathfrak{p}_\mathfrak{p} \in \operatorname{Anc}_{R_\mathfrak{p}} X_\mathfrak{p}$ and hence $\mathfrak{p} \in \operatorname{Anc}_R X$ by (2.1).

(b): Assume that $\mathfrak{p}$ belongs to $\mathrm{W}_0(X)$, then $\dim_R X = \dim R/\mathfrak{p} - \inf X_\mathfrak{p}$, and since $\dim_R X \geq \dim_{R_\mathfrak{p}} X_\mathfrak{p} + \dim R/\mathfrak{p}$ and $\dim_{R_\mathfrak{p}} X_\mathfrak{p} \geq -\inf X_\mathfrak{p}$, cf. (1.1) and (1.2), it follows that $\dim_{R_\mathfrak{p}} X_\mathfrak{p} = -\inf X_\mathfrak{p}$, as desired.

(c): For $M \in \mathscr{D}_0(R)$ we have

$$\operatorname{Anc}_R M = \{\, \mathfrak{p} \in \operatorname{Supp}_R M \mid \dim_{R_\mathfrak{p}} M_\mathfrak{p} = 0 \,\} = \operatorname{Min}_R M,$$

and the inclusion $\mathrm{Min}_R\, M \subseteq \mathrm{Ass}_R\, M$ is well-known.

(d): Assume that $X \in \mathscr{D}_\mathrm{b}^\mathrm{f}(R)$ and $\dim_R X = \operatorname{depth}_R X$, then $\dim_{R_\mathfrak{p}} X_\mathfrak{p} = \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Supp}_R\, X$, cf. [5, (16.17)]. If $\mathfrak{p} \in \mathrm{Ass}_R\, X$ we have

$$\dim_{R_\mathfrak{p}} X_\mathfrak{p} = \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} = -\sup X_\mathfrak{p} \leq -\inf X_\mathfrak{p},$$

cf. [4, Def. 2.3], and it follows by (1.2) that equality must hold, so $\mathfrak{p}$ belongs to $\mathrm{Anc}_R\, X$.

For each $\mathfrak{p} \in \mathrm{Supp}_R\, X$ there is an equality

$$\dim_R X = \dim_{R_\mathfrak{p}} X_\mathfrak{p} + \dim R/\mathfrak{p},$$

cf. [5, (17.4)(b)], so $\dim_R X - \dim R/\mathfrak{p} + \inf X_\mathfrak{p} = 0$ for $\mathfrak{p}$ with $\dim_{R_\mathfrak{p}} X_\mathfrak{p} = -\inf X_\mathfrak{p}$. This proves the inclusion $\mathrm{Anc}_R\, X \subseteq \mathrm{W}_0(X)$.

COROLLARY 2.4. *For $X \in \mathscr{D}_\mathrm{b}(R)$ there is an inclusion:*

$$\text{(a)} \qquad \mathrm{Min}_R\, X \subseteq \mathrm{Ass}_R\, X \cap \mathrm{Anc}_R\, X;$$

and for $\mathfrak{p} \in \mathrm{Ass}_R\, X \cap \mathrm{Anc}_R\, X$ there is an equality:

$$\text{(b)} \qquad \operatorname{cmd}_{R_\mathfrak{p}} X_\mathfrak{p} = \operatorname{amp} X_\mathfrak{p}.$$

PROOF. Part (a) follows by 2.3 (a) and [4, Prop. 2.6]; part (b) is immediate by the definitions of associated and anchor prime ideals, cf. [4, Def. 2.3].

COROLLARY 2.5. *If $X \in \mathscr{D}_+^\mathrm{f}(R)$, then*

$$\dim_R X = \sup\{\dim R/\mathfrak{p} + \dim_{R_\mathfrak{p}} X_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Anc}_R\, X\}.$$

PROOF. It is immediate by the definitions that

$$\begin{aligned}
\dim_R X &= \sup\{\dim R/\mathfrak{p} - \inf X_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}_R\, X\}\\
&\geq \sup\{\dim R/\mathfrak{p} - \inf X_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Anc}_R\, X\}\\
&= \sup\{\dim R/\mathfrak{p} + \dim_{R_\mathfrak{p}} X_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Anc}_R\, X\};
\end{aligned}$$

and the opposite inequality follows by 2.3 (b).

PROPOSITION 2.6. *The following hold:*

(a) *If $X \in \mathscr{D}_+(R)$ and $\mathfrak{p}$ belongs to* $\mathrm{Anc}_R\, X$, *then* $\dim_{R_\mathfrak{p}}(\mathrm{H}_{\inf X_\mathfrak{p}}(X_\mathfrak{p})) = 0$.

(b) *If $X \in \mathscr{D}_\mathrm{b}^\mathrm{f}(R)$, then* $\mathrm{Anc}_R\, X$ *is a finite set.*

PROOF. (a): Assume that $\mathfrak{p} \in \mathrm{Anc}_R\, X$; by [6, Prop. 3.5] we have

$$-\inf X_\mathfrak{p} = \dim_{R_\mathfrak{p}} X_\mathfrak{p} \geq \dim_{R_\mathfrak{p}}(\mathrm{H}_{\inf X_\mathfrak{p}}(X_\mathfrak{p})) - \inf X_\mathfrak{p},$$

and hence $\dim_{R_\mathfrak{p}}(\mathrm{H}_{\inf X_\mathfrak{p}}(X_\mathfrak{p})) = 0$.

(b): By (a) every anchor prime ideal for X is minimal for one of the homology modules of X, and when $X \in \mathscr{D}_\mathrm{b}^\mathrm{f}(R)$ each of the finitely many homology modules has a finite number of minimal prime ideals.

OBSERVATION 2.7. By Nakayama's lemma it follows that

$$\inf \mathrm{K}(x_1, \ldots, x_n; Y) = \inf Y,$$

for $Y \in \mathscr{D}_+^\mathrm{f}(R)$ and elements $x_1, \ldots, x_n \in \mathfrak{m}$.

PROPOSITION 2.8 (Dimension of Koszul Complexes). *The following hold for a complex $Y \in \mathscr{D}_+^\mathrm{f}(R)$ and elements $x_1, \ldots, x_n \in \mathfrak{m}$:*

(a) $\dim_R \mathrm{K}(x_1, \ldots, x_n; Y)$
$= \sup\{ \dim R/\mathfrak{p} - \inf Y_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}_R\, Y \cap \mathrm{V}(x_1, \ldots, x_n) \}$; *and*

(b) $\dim_R Y - n \leq \dim_R \mathrm{K}(x_1, \ldots, x_n; Y) \leq \dim_R Y$.

Furthermore:

(c) *The elements $x_1, \ldots, x_n$ are contained in a prime ideal $\mathfrak{p} \in \mathrm{W}_n(Y)$; and*

(d) $\dim_R \mathrm{K}(x_1, \ldots, x_n; Y) = \dim_R Y$ *if and only if $x_1, \ldots, x_n \in \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{W}_0(Y)$.*

PROOF. Since $\mathrm{Supp}_R \mathrm{K}(x_1, \ldots, x_n; Y) = \mathrm{Supp}_R\, Y \cap \mathrm{V}(x_1, \ldots, x_n)$ (see [6, p. 157] and [4, 3.2]) (a) follows by the definition of Krull dimension and 2.7. In (b) the second inequality follows from (a); the first one is established through four steps:

1° $Y = R$: The second equality below follows from the definition of Krull dimension as $\mathrm{Supp}_R \mathrm{K}(x_1, \ldots, x_n) = \mathrm{Supp}_R \mathrm{H}_0(\mathrm{K}(x_1, \ldots, x_n)) = \mathrm{V}(x_1, \ldots, x_n)$, cf. [4, 3.2]; the inequality is a consequence of Krull's Principal Ideal Theorem, see for example [8, Thm. 13.6].

$$\begin{aligned}
\dim_R \mathrm{K}(x_1, \ldots, x_n; Y) &= \dim_R \mathrm{K}(x_1, \ldots, x_n) \\
&= \sup\{ \dim R/\mathfrak{p} \mid \mathfrak{p} \in \mathrm{V}(x_1, \ldots, x_n) \} \\
&= \dim R/(x_1, \ldots, x_n) \\
&\geq \dim R - n \\
&= \dim_R Y - n.
\end{aligned}$$

2° $Y = B$, a cyclic module: By $\bar{x}_1, \dots, \bar{x}_n$ we denote the residue classes in B of the elements $x_1, \dots, x_n$; the inequality below is by 1°.

$$\begin{aligned}\dim_R \mathrm{K}(x_1, \dots, x_n; Y) &= \dim_R \mathrm{K}\,(\bar{x}_1, \dots, \bar{x}_n)\\ &= \dim_B \mathrm{K}\,(\bar{x}_1, \dots, \bar{x}_n)\\ &\geq \dim B - n\\ &= \dim_R Y - n.\end{aligned}$$

3° $Y = H \in \mathscr{D}_0^{\mathrm{f}}(R)$: We set $B = R/\operatorname{Ann}_R H$; the first equality below follows by [6, Prop. 3.11] and the inequality by 2°.

$$\begin{aligned}\dim_R \mathrm{K}(x_1, \dots, x_n; Y) &= \dim_R \mathrm{K}(x_1, \dots, x_n; B)\\ &\geq \dim_R B - n\\ &= \dim_R Y - n.\end{aligned}$$

4° $Y \in \mathscr{D}_{\mathrm{b}}^{\mathrm{f}}(R)$: The first equality below follows by [6, Prop. 3.12] and the last by [6, Prop. 3.5]; the inequality is by 3°.

$$\begin{aligned}\dim_R \mathrm{K}(x_1, \dots, x_n; Y) &= \sup\{\dim_R \mathrm{K}(x_1, \dots, x_n; \mathrm{H}_\ell(Y)) - \ell \mid \ell \in \mathsf{Z}\}\\ &\geq \sup\{\dim_R \mathrm{H}_\ell(Y) - n - \ell \mid \ell \in \mathsf{Z}\}\\ &= \dim_R Y - n.\end{aligned}$$

This proves (b).

In view of (a) it now follows that

$$\dim_R Y - n \leq \dim R/\mathfrak{p} - \inf Y_\mathfrak{p}$$

for some $\mathfrak{p} \in \operatorname{Supp}_R Y \cap \mathrm{V}(x_1, \dots, x_n)$. That is, the elements $x_1, \dots, x_n$ are contained in a prime ideal $\mathfrak{p} \in \operatorname{Supp}_R Y$ with

$$\dim_R Y - \dim R/\mathfrak{p} + \inf Y_\mathfrak{p} \leq n,$$

and this proves (c).

Finally, it is immediate by the definitions that

$$\dim_R Y = \sup\{\dim R/\mathfrak{p} - \inf Y_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R Y \cap \mathrm{V}(x_1, \dots, x_n)\}$$

if and only if $\mathrm{W}_0(Y) \cap \mathrm{V}(x_1, \dots, x_n) \neq \emptyset$. This proves (d).

THEOREM 2.9. *If $Y \in \mathscr{D}_{\mathrm{b}}^{\mathrm{f}}(R)$, then the next two numbers are equal.*

$$\mathrm{d}(Y) = \dim_R Y + \inf Y; \qquad \textit{and}$$

$$\mathrm{s}(Y) = \inf\{\, s \in \mathsf{N}_0 \mid \exists\, x_1, \dots, x_s : \mathfrak{m} \in \mathrm{Anc}_R \, \mathrm{K}(x_1, \dots, x_s; Y)\,\}.$$

PROOF. There are two inequalities to prove.

$\mathrm{d}(Y) \le \mathrm{s}(Y)$: Let $x_1, \dots, x_s \in \mathfrak{m}$ be such that $\mathfrak{m} \in \mathrm{Anc}_R \, \mathrm{K}(x_1, \dots, x_s; Y)$; by 2.8 (b) and 2.7 we then have

$$\dim_R Y - s \le \dim_R \mathrm{K}(x_1, \dots, x_s; Y) = -\inf \mathrm{K}(x_1, \dots, x_s; Y) = -\inf Y,$$

so $\mathrm{d}(Y) \le s$, and the desired inequality follows.

$\mathrm{s}(Y) \le \mathrm{d}(Y)$: We proceed by induction on d(Y). If $\mathrm{d}(Y) = 0$ then $\mathfrak{m} \in \mathrm{Anc}_R Y$ so $\mathrm{s}(Y) = 0$. If $\mathrm{d}(Y) > 0$ then $\mathfrak{m} \notin \mathrm{Anc}_R Y$, and since $\mathrm{Anc}_R Y$ is a finite set, by 2.6(b), we can choose an element $x \in \mathfrak{m} - \cup_{\mathfrak{p} \in \mathrm{Anc}_R Y} \mathfrak{p}$. We set $K = \mathrm{K}(x; Y)$; it is cleat that $\mathrm{s}(Y) \le \mathrm{s}(K) + 1$. Furthermore, it follows by 2.8 (a) and 2.3 (b) that $\dim_R K < \dim_R Y$ and thereby $\mathrm{d}(K) < \mathrm{d}(Y)$, cf. 2.7. Thus, by the induction hypothesis we have

$$\mathrm{s}(Y) \le \mathrm{s}(K) + 1 \le \mathrm{d}(K) + 1 \le \mathrm{d}(Y);$$

as desired.

3. Parameters

By 2.9 the next definitions extend the classical notions of systems and sequences of parameters for finite modules (e.g., see [8, § 14] and the appendix in [2]).

DEFINITIONS 3.1. Let Y belong to $\mathscr{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ and set $d = \dim_R Y + \inf Y$. A set of elements $x_1, \dots, x_d \in \mathfrak{m}$ are said to be a *system of parameters* for Y if and only if $\mathfrak{m} \in \mathrm{Anc}_R \, \mathrm{K}(x_1, \dots, x_d; Y)$.

A sequence $\boldsymbol{x} = x_1, \dots, x_n$ is said to be a *Y-parameter sequence* if and only if it is part of a system of parameters for Y.

LEMMA 3.2. *Let Y belong to $\mathscr{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ and set $d = \dim_R Y + \inf Y$. The next two conditions are equivalent for elements $x_1, \dots, x_d \in \mathfrak{m}$.*

(i) *$x_1, \dots, x_d$ is a system of parameters for Y.*

(ii) *For every $j \in \{0, \dots, d\}$ there is an equality:*

$$\dim_R \mathrm{K}(x_1, \dots, x_j; Y) = \dim_R Y - j;$$

and $x_{j+1}, \dots, x_d$ is a system of parameters for $\mathrm{K}(x_1, \dots, x_j; Y)$.

PROOF. (i) $\Rightarrow$ (ii): Assume that $x_1, \dots, x_d$ is a system of parameters for Y, then

$$\begin{aligned}
-\inf \mathrm{K}(x_1, \dots, x_d; Y) &= \dim_R \mathrm{K}(x_1, \dots, x_d; Y) \\
&= \dim_R \mathrm{K}(x_{j+1}, \dots, x_d; \mathrm{K}(x_1, \dots, x_j; Y)) \\
&\geq \dim_R \mathrm{K}(x_1, \dots, x_j; Y) - (d-j) \quad \text{by 2.8 (b)} \\
&\geq \dim_R Y - j - (d-j) \quad \text{by 2.8 (b)} \\
&= \dim_R Y - d \\
&= -\inf Y.
\end{aligned}$$

By 2.7 it now follows that $-\inf Y = \dim_R \mathrm{K}(x_1, \dots, x_j; Y) - (d-j)$, so

$$\dim_R \mathrm{K}(x_1, \dots, x_j; Y) = d - j - \inf Y = \dim_R Y - j,$$

as desired. It also follows that $\mathrm{d}(\mathrm{K}(x_1, \dots, x_j; Y)) = d - j$, and since

$$\mathfrak{m} \in \mathrm{Anc}_R \mathrm{K}(x_1, \dots, x_d; Y) = \mathrm{Anc}_R \mathrm{K}(x_{j+1}, \dots, x_d; \mathrm{K}(x_1, \dots, x_j; Y)),$$

we conclude that $x_{j+1}, \dots, x_d$ is a system of parameters for $\mathrm{K}(x_1, \dots, x_j; Y)$.

(ii) $\Rightarrow$ (i): If $\dim_R \mathrm{K}(x_1, \dots, x_j; Y) = \dim_R Y - j$ then $\mathrm{d}(\mathrm{K}(x_1, \dots, x_j; Y)) = d - j$; and if $x_{j+1}, \dots, x_d$ is a system of parameters for $\mathrm{K}(x_1, \dots, x_j; Y)$ then $\mathfrak{m}$ belongs to

$$\mathrm{Anc}_R \mathrm{K}(x_{j+1}, \dots, x_d; \mathrm{K}(x_1, \dots, x_j; Y)) = \mathrm{Anc}_R \mathrm{K}(x_1, \dots, x_d; Y),$$

so $x_1, \dots, x_d$ must be a system of parameters for Y.

PROPOSITION 3.3. *Let $Y \in \mathscr{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. The following conditions are equivalent for a sequence $\boldsymbol{x} = x_1, \dots, x_n$ in $\mathfrak{m}$.*

(i) *$\boldsymbol{x}$ is a Y-parameter sequence.*

(ii) *For each $j \in \{0, \dots, n\}$ there is an equality:*

$$\dim_R \mathrm{K}(x_1, \dots, x_j; Y) = \dim_R Y - j;$$

and $x_{j+1}, \dots, x_n$ is a $\mathrm{K}(x_1, \dots, x_j; Y)$-parameter sequence.

(iii) *There is an equality:*

$$\dim_R \mathrm{K}(x_1, \dots, x_n; Y) = \dim_R Y - n.$$

PROOF. It follows by 3.2 that (i) implies (ii), and (iii) follows from (ii). Now, set $K = \mathrm{K}(\boldsymbol{x}; Y)$ and assume that $\dim_R K = \dim_R Y - n$. Choose, by 2.9,

$s = \mathrm{s}(K) = \dim_R K + \inf K$ elements $w_1, \dots, w_s$ in $\mathfrak{m}$ such that $\mathfrak{m}$ belongs to $\operatorname{Anc}_R \mathrm{K}(w_1, \dots, w_s; K) = \operatorname{Anc}_R \mathrm{K}(x_1, \dots, x_n, w_1, \dots, w_s; Y)$. Then, by 2.7, we have

$$n + s = (\dim_R Y - \dim_R K) + (\dim_R K + \inf K) = \dim_R Y + \inf Y = d,$$

so $x_1, \dots, x_n, w_1, \dots, w_s$ is a system of parameters for Y, whence $x_1, \dots, x_n$ is a Y-parameter sequence.

We now recover a classical result (e.g., see [2, Prop. A.4]):

COROLLARY 3.4. *Let M be an R-module. The following conditions are equivalent for a sequence $\boldsymbol{x} = x_1, \dots, x_n$ in $\mathfrak{m}$.*

(i) *$\boldsymbol{x}$ is an M-parameter sequence.*

(ii) *For each $j \in \{0, \dots, n\}$ there is an equality:*

$$\dim_R M/(x_1, \dots, x_j)M = \dim_R M - j;$$

and $x_{j+1}, \dots, x_n$ is an $M/(x_1, \dots, x_j)M$-parameter sequence.

(iii) *There is an equality:*

$$\dim_R M/(x_1, \dots, x_n)M = \dim_R M - n.$$

PROOF. By [6, Prop. 3.12] and [5, (16.22)] we have

$$\begin{aligned}
&\dim_R \mathrm{K}(x_1, \dots, x_j; M) \\
&\quad = \sup\{\dim_R(M \otimes^{\mathbf{L}}_R \mathrm{H}_\ell(\mathrm{K}\,(x_1, \dots, x_j))) - \ell \mid \ell \in \mathsf{Z}\} \\
&\quad = \sup\{\dim_R(M \otimes_R \mathrm{H}_\ell(\mathrm{K}\,(x_1, \dots, x_j))) - \ell \mid \ell \in \mathsf{Z}\} \\
&\quad = \dim_R(M \otimes_R R/(x_1, \dots, x_j)).
\end{aligned}$$

THEOREM 3.5. *Let $Y \in \mathscr{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. The following hold for a sequence $\boldsymbol{x} = x_1, \dots, x_n$ in $\mathfrak{m}$.*

(a) *There is an inequality:*

$$\operatorname{amp} \mathrm{K}(\boldsymbol{x}; Y) \geq \operatorname{amp} Y;$$

and equality holds if and only if $\boldsymbol{x}$ is a Y-sequence.

(b) *There is an inequality:*

$$\operatorname{cmd}_R \mathrm{K}(\boldsymbol{x}; Y) \geq \operatorname{cmd}_R Y;$$

and equality holds if and only if $\boldsymbol{x}$ is a Y-parameter sequence.

(c) *If $\boldsymbol{x}$ is a maximal Y-sequence, then*

$$\operatorname{amp} Y \leq \operatorname{cmd}_R \mathrm{K}(\boldsymbol{x}; Y).$$

(d) *If $\boldsymbol{x}$ is a system of parameters for Y, then*

$$\operatorname{cmd}_R Y \leq \operatorname{amp} \mathrm{K}(\boldsymbol{x}; Y).$$

PROOF. In the following K denotes the Koszul complex $\mathrm{K}(\boldsymbol{x}; Y)$.

(a): Immediate by 2.7 and [4, Prop. 5.1].

(b): By [4, Thm. 4.7 (a)] and 2.8 (b) we have

$$\operatorname{cmd}_R K = \dim_R K - \operatorname{depth}_R K = \dim_R K + n - \operatorname{depth}_R Y \geq \operatorname{cmd}_R Y,$$

and by 3.3 equality holds if and only if $\boldsymbol{x}$ is a Y-parameter sequence.

(c): Suppose $\boldsymbol{x}$ is a maximal Y-sequence, then

$$\begin{aligned} \operatorname{amp} Y &= \sup Y - \inf K && \text{by 2.7} \\ &= -\operatorname{depth}_R K - \inf K && \text{by [4, Thm. 5.4]} \\ &\leq \operatorname{cmd}_R K && \text{by (1.2).} \end{aligned}$$

(d): Suppose $\boldsymbol{x}$ is system of parameters for Y, then

$$\begin{aligned} \operatorname{amp} K &= \sup K + \dim_R K \\ &\geq \dim_R K - \operatorname{depth}_R K && \text{by (1.3)} \\ &= \operatorname{cmd}_R Y && \text{by (b).} \end{aligned}$$

THEOREM 3.6. *The following hold for $Y \in \mathscr{D}_{\mathrm{b}}^{\mathrm{f}}(R)$.*

(a) *The next four conditions are equivalent.*

  (i) *There is a maximal Y-sequence which is also a Y-parameter sequence.*

  (ii) $\operatorname{depth}_R Y + \sup Y \leq \dim_R Y + \inf Y$.

  (ii') $\operatorname{amp} Y \leq \operatorname{cmd}_R Y$.

  (iii) *There is a maximal strong Y-sequence which is also a Y-parameter sequence.*

(b) *The next four conditions are equivalent.*

  (i) *There is a system of parameters for Y which is also a Y-sequence.*

  (ii) $\dim_R Y + \inf Y \leq \operatorname{depth}_R Y + \sup Y$.

  (ii') $\operatorname{cmd}_R Y \leq \operatorname{amp} Y$.

(iii) *There is a system of parameters for Y which is also a strong Y-sequence.*

(c) *The next four conditions are equivalent.*

(i) *There is a system of parameters for Y which is also a maximal Y-sequence.*

(ii) $\dim_R Y + \inf Y = \operatorname{depth}_R Y + \sup Y$.

(ii') $\operatorname{cmd}_R Y = \operatorname{amp} Y$.

(iii) *There is a system of parameters for Y which is also a maximal strong Y-sequence.*

Proof. Let $Y \in \mathscr{D}_{\mathrm{b}}^{\mathrm{f}}(R)$, set $\mathrm{n}(Y) = \operatorname{depth}_R Y + \sup Y$ and $\mathrm{d}(Y) = \dim_R Y + \inf Y$.

(a): A maximal Y-sequence is of length $\mathrm{n}(Y)$, cf. [4, Cor. 5.5], and the length of a Y-parameter sequence is at most $\mathrm{d}(Y)$. Thus, (i) implies (ii) which in turn is equivalent to (ii'). Furthermore, a maximal strong Y-sequence is, in particular, a maximal Y-sequence, cf. [4, Cor. 5.7], so (iii) is stronger than (i). It is now sufficient to prove the implication (ii) $\Rightarrow$ (iii): We proceed by induction. If $\mathrm{n}(Y) = 0$ then the empty sequence is a maximal strong Y-sequence and a Y-parameter sequence. Let $\mathrm{n}(Y) > 0$; the two sets $\operatorname{Ass}_R Y$ and $\mathrm{W}_0(Y)$ are both finite, and since $0 < \mathrm{n}(Y) \leq \mathrm{d}(Y)$ none of them contain $\mathfrak{m}$. We can, therefore, choose an element $x \in \mathfrak{m} - \bigcup_{\operatorname{Ass}_R Y \cup \mathrm{W}_0(Y)} \mathfrak{p}$, and x is then a strong Y-sequence, cf. [4, Def. 3.3], and a Y-parameter sequence, cf. 3.3 and 2.8. Set $K = \mathrm{K}(x; Y)$, by [4, Thm. 4.7 and Prop. 5.1], respectively, 2.8 and 2.7 we have

$$\operatorname{depth}_R K + \sup K = \mathrm{n}(Y) - 1 \leq \mathrm{d}(Y) - 1 = \dim_R K + \inf K.$$

By the induction hypothesis there exists a maximal strong K-sequence $w_1, \dots, w_{n-1}$ which is also a K-parameter sequence, and it follows by [4, 3.5] and 3.3 that $x, w_1, \dots, w_{n-1}$ is a strong Y-sequence and a Y-parameter sequence, as wanted.

The proof of (b) i similar to the proof of (a), and (c) follows immediately by (a) and (b).

Theorem 3.7. *The following hold for $Y \in \mathscr{D}_{\mathrm{b}}^{\mathrm{f}}(R)$:*

(a) *If* $\operatorname{amp} Y = 0$, *then any Y-sequence is a Y-parameter sequence.*

(b) *If* $\operatorname{cmd}_R Y = 0$, *then any Y-parameter sequence is a strong Y-sequence.*

Proof. The empty sequence is a Y-parameter sequence as well as a strong Y-sequence, this founds the base for a proof by induction on the length n of the sequence $\boldsymbol{x} = x_1, \dots, x_n$. Let $n > 0$ and set $K = \mathrm{K}(x_1, \dots, x_{n-1}; Y)$; by 2.8 (a) we have

(∗)
$$\begin{aligned}\dim_R \mathrm{K}(x_1, \dots, x_n; Y) &= \dim_R \mathrm{K}(x_n; K)\\ &= \sup\{\dim R/\mathfrak{p} - \inf K_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Supp}_R K \cap \mathrm{V}(x_n)\}.\end{aligned}$$

Assume that $\mathrm{amp}\, Y = 0$. If $\boldsymbol{x}$ is a Y-sequence, then $\mathrm{amp}\, K = 0$ by 3.5 (a) and $x_n \notin \mathrm{z}_R K$, cf. [4, Def. 3.3]. As $\mathrm{z}_R K = \cup_{\mathfrak{p} \in \mathrm{Ass}_R K} \mathfrak{p}$, cf. [4, 2.5], it follows by (b) and (c) in 2.3 that x_n is not contained in any prime ideal $\mathfrak{p} \in \mathrm{W}_0(K)$; so from (∗) we conclude that $\dim_R \mathrm{K}(x_n; K) < \dim_R K$, and it follows by 2.8 (b) that $\dim_R \mathrm{K}(x_n, K) = \dim_R K - 1$. By the induction hypothesis $\dim_R K = \dim_R Y - (n-1)$, so $\dim_R \mathrm{K}(x_1, \dots, x_n; Y) = \dim_R Y - n$ and it follows by 3.3 that $\boldsymbol{x}$ is a Y-parameter sequence. This proves (a).

We now assume that $\mathrm{cmd}_R Y = 0$. If $\boldsymbol{x}$ is a Y-parameter sequence then, by the induction hypothesis, $x_1, \dots, x_{n-1}$ is a strong Y-sequence, so it is sufficient to prove that $x_n \notin \mathrm{Z}_R K$, cf. [4, 3.5]. By 3.3 it follows that x_n is a K-parameter sequence, so $\dim_R \mathrm{K}(x_n; K) = \dim_R K - 1$ and we conclude from (∗) that $x_n \notin \cup_{\mathfrak{p} \in \mathrm{W}_0(K)} \mathfrak{p}$. Now, by 3.5 (b) we have $\mathrm{cmd}_R K = 0$, so it follows from 2.3 (d) that $x_n \notin \cup_{\mathfrak{p} \in \mathrm{Ass}_R K} \mathfrak{p} = \mathrm{Z}_R K$. This proves (b).

Semi-dualizing Complexes 3.8. We recall two basic definitions from [3]:

A complex $C \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ is said to be *semi-dualizing* for R if and only if the homothety morphism $\chi_C^R \colon R \to \mathbf{R}\mathrm{Hom}_R(C, C)$ is an isomorphism [3, (2.1)].

Let C be a semi-dualizing complex for R. A complex $Y \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ is said to be *C-reflexive* if and only if the *dagger dual* $Y^{\dagger_C} = \mathbf{R}\mathrm{Hom}_R(Y, C)$ belongs to $\mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ and the biduality morphism $\delta_Y^C \colon Y \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(Y, C), C)$ is invertible in $\mathcal{D}(R)$ [3, (2.7)].

Relations between dimension and depth for C-reflexive complexes are studied in [3, sec. 3], and the next result is an immediate consequence of [3, (3.1) and (2.10)].

Let C be a semi-dualizing complex for R and let Z be a C-reflexive complex. The following holds for $\mathfrak{p} \in \mathrm{Spec}\, R$: If $\mathfrak{p} \in \mathrm{Anc}_R Z$ then $\mathfrak{p} \in \mathrm{Ass}_R Z^{\dagger_C}$, and the converse holds in C is Cohen-Macaulay.

A *dualizing complex*, cf. [7], is a semi-dualizing complex of finite injective dimension, in particular, it is Cohen-Macaulay, cf. [3, (3.5)]. If D is a dualizing complex for R, then, by [7, Prop. V.2.1], all complexes $Y \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ are D-reflexive; in particular, all finite R-modules are D-reflexive and, therefore, [4, 5.10] is a special case of the following:

THEOREM 3.9. *Let C be a Cohen-Macaulay semi-dualizing complex for R, and let $\boldsymbol{x} = x_1, \ldots, x_n$ be a sequence in $\mathfrak{m}$. If Y is C-reflexive, then $\boldsymbol{x}$ is a Y-parameter sequence if and only if it is a $\mathbf{R}\mathrm{Hom}_R(Y, C)$-sequence; that is*

$$\boldsymbol{x} \textit{ is a } Y\textit{-parameter sequence} \iff \boldsymbol{x} \textit{ is a } \mathbf{R}\mathrm{Hom}_R(Y, C)\textit{-sequence.}$$

PROOF. We assume that C is a Cohen-Macaulay semi-dualizing complex for R and that Y is C-reflexive, cf. 3.8. The desired biconditional follows by the next chain, and each step is explained below (we use the notation $-^{\dagger_C}$ introduced in 3.8).

$$\begin{aligned}
\boldsymbol{x} \text{ is a } Y\text{-parameter sequence} &\iff \mathrm{cmd}_R\, \mathrm{K}(\boldsymbol{x}; Y) = \mathrm{cmd}_R\, Y \\
&\iff \mathrm{amp}\, \mathrm{K}(\boldsymbol{x}; Y)^{\dagger_C} = \mathrm{amp}\, Y^{\dagger_C} \\
&\iff \mathrm{amp}\, \mathrm{K}(\boldsymbol{x}; Y^{\dagger_C}) = \mathrm{amp}\, Y^{\dagger_C} \\
&\iff \boldsymbol{x} \text{ is a } Y^{\dagger_C}\text{-sequence.}
\end{aligned}$$

The first biconditional follows by 3.5 (b) and the last by 3.5 (a). Since $\mathrm{K}(\boldsymbol{x})$ is a bounded complex of free modules (hence of finite projective dimension), it follows from [3, Thm. (3.17)] that also $\mathrm{K}(\boldsymbol{x}; Y)$ is C-reflexive, and the second biconditional is then immediate by the CMD-formula [3, Cor. (3.8)]. The third one is established as follows:

$$\begin{aligned}
\mathrm{K}(\boldsymbol{x}; Y)^{\dagger_C} &\simeq \mathbf{R}\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}) \otimes^{\mathbf{L}}_R Y, C) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}), Y^{\dagger_C}) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}), R \otimes^{\mathbf{L}}_R Y^{\dagger_C}) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}), R) \otimes^{\mathbf{L}}_R Y^{\dagger_C} \\
&\sim \mathrm{K}(\boldsymbol{x}) \otimes^{\mathbf{L}}_R Y^{\dagger_C} \\
&\simeq \mathrm{K}(\boldsymbol{x}; Y^{\dagger_C}),
\end{aligned}$$

where the second isomorphism is by adjointness and the fourth by, so-called, tensor-evaluation, cf. [1, (1.4.2)]. It is straightforward to check that $\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}), R)$ is isomorphic to the Koszul complex $\mathrm{K}(\boldsymbol{x})$ shifted n degrees to the right, and the symbol $\sim$ denotes isomorphism up to shift.

If C is a semi-dualizing complex for R, then both C and R are C-reflexive complexes, cf. [3, (2.8)], so we have an immediate corollary to the theorem:

COROLLARY 3.10. *If C is a Cohen-Macaulay semi-dualizing complex for R, then the following hold for a sequence $\boldsymbol{x} = x_1, \ldots, x_n$ in $\mathfrak{m}$.*

(a) *$\boldsymbol{x}$ is a C-parameter sequence if and only if it is an R-sequence.*

(b) *$\boldsymbol{x}$ is an R-parameter sequence if and only if it is a C-sequence.*

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