1. Introduction and Notation

This short paper elaborates on an example given in [4] to illustrate an application of sequences for complexes:

Let $R$ be a local ring with a dualizing complex $D$, and let $M$ be a finitely generated $R$-module; then a sequence $x_1, \ldots, x_n$ is part of a system of parameters for $M$ if and only if it is a $\text{RHom}_R(M, D)$-sequence [4, 5.10].

The final Theorem 3.9 of this paper generalizes the result above in two directions: the dualizing complex is replaced by a Cohen-Macaulay semi-dualizing complex (see [3, Sec. 2] or 3.8 below for definitions), and the finite module is replaced by a complex with finite homology.

Before we can even state, let alone prove, this generalization of [4, 5.10] we have to introduce and study parameters for complexes. For a finite $R$-module $M$ every $M$-sequence is part of a system of parameters for $M$, so, loosely speaking, regular elements are just special parameters. For a complex $X$, however, parameters and regular elements are two different things, and kinship between them implies strong relations between two measures of the size of $X$: the amplitude and the Cohen-Macaulay defect (both defined below). This is described in 3.5, 3.6, and 3.7.

The definition of parameters for complexes is based on a notion of anchor prime ideals. These do for complexes what minimal prime ideals do for modules, and the quantitative relations between dimension and depth under dagger duality—studied in [3]—have a qualitative description in terms of anchor and associated prime ideals.

Throughout $R$ denotes a commutative, Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. We use the same notation as in [4], but for convenience we recall a few basic facts.

The homological position and size of a complex $X$ is captured by the su-
**premum, infimum, and amplitude:**

\[
\text{sup } X = \sup \{ \ell \in \mathbb{Z} \mid H_\ell(X) \neq 0 \},
\]

\[
\text{inf } X = \inf \{ \ell \in \mathbb{Z} \mid H_\ell(X) \neq 0 \},
\]

and

\[
\text{amp } X = \text{sup } X - \text{inf } X.
\]

By convention, \( \text{sup } X = -\infty \) and \( \text{inf } X = \infty \) if \( H(X) = 0 \).

The *support* of a complex \( X \) is the set

\[
\text{Supp}_R X = \{ \mathfrak{p} \in \text{Spec } R \mid X_\mathfrak{p} \not\simeq 0 \} = \bigcup \text{Supp}_R H_\ell(X).
\]

As usual \( \text{Min}_R X \) is the subset of minimal elements in the support.

The *depth* and the *(Krull)* *dimension* of an \( R \)-complex \( X \) are defined as follows:

\[
\text{depth}_R X = -\sup(\mathcal{R}\text{Hom}_R(k, X)), \quad \text{for } X \in \mathcal{D}^-(R), \quad \text{and}
\]

\[
\text{dim}_R X = \sup \{ \text{dim } R/\mathfrak{p} - \inf X_\mathfrak{p} \mid \mathfrak{p} \in \text{Supp}_R X \},
\]

cf. [6, Sec. 3]. For modules these notions agree with the usual ones. It follows from the definition that

(1.1) \[
\text{dim}_R X \geq \text{dim}_{R_\mathfrak{p}} X_\mathfrak{p} + \text{dim } R/\mathfrak{p}
\]

for \( X \in \mathcal{D}(R) \) and \( \mathfrak{p} \in \text{Spec } R \); and there are always inequalities:

(1.2) \[
-\inf X \leq \text{dim}_R X \quad \text{for } X \in \mathcal{D}^+(R); \quad \text{and}
\]

(1.3) \[
-\sup X \leq \text{depth}_R X \quad \text{for } X \in \mathcal{D}^-(R).
\]

A complex \( X \in \mathcal{D}^b_+(R) \) is *Cohen-Macaulay* if and only if \( \text{dim}_R X = \text{depth}_R X \), that is, if an only if the *Cohen-Macaulay defect*,

\[
\text{cmd}_R X = \text{dim}_R X - \text{depth}_R X,
\]

is zero. For complexes in \( \mathcal{D}^b_+(R) \) the Cohen-Macaulay defect is always non-negative, cf. [6, Cor. 3.9].

### 2. Anchor Prime Ideals

In [4] we introduced associated prime ideals for complexes. The analysis of the support of a complex is continued in this section, and the aim is now to identify the prime ideals that do for complexes what the minimal ones do for modules.
DEFINITIONS 2.1. Let $X \in \mathcal{D}_+(R)$; we say that $\mathfrak{p} \in \text{Spec } R$ is an anchor prime ideal for $X$ if and only if $\dim R_\mathfrak{p} X_\mathfrak{p} = -\inf X_\mathfrak{p} > -\infty$. The set of anchor prime ideals for $X$ is denoted by $\text{Anc}_R X$; that is,

$$\text{Anc}_R X = \{ \mathfrak{p} \in \text{Supp } R X | \dim R_\mathfrak{p} X_\mathfrak{p} + \inf X_\mathfrak{p} = 0 \}.$$ 

For $n \in \mathbb{N}_0$ we set

$$W_n(X) = \{ \mathfrak{p} \in \text{Supp } R X | \dim R X - \dim R/\mathfrak{p} + \inf X_\mathfrak{p} \leq n \}.$$ 

OBSERVATION 2.2. Let $S$ be a multiplicative system in $R$, and let $\mathfrak{p} \in \text{Spec } R$. If $\mathfrak{p} \cap S = \emptyset$ then $S^{-1}\mathfrak{p}$ is a prime ideal in $S^{-1}R$, and for $X \in \mathcal{D}_+(R)$ there is an isomorphism $S^{-1}X_{S^{-1}\mathfrak{p}} \simeq X_\mathfrak{p}$ in $\mathcal{D}(R_\mathfrak{p})$. In particular, $\inf S^{-1}X_{S^{-1}\mathfrak{p}} = \inf X_\mathfrak{p}$ and $\dim S^{-1}R_{S^{-1}\mathfrak{p}} S^{-1}X_{S^{-1}\mathfrak{p}} = \dim R_\mathfrak{p} X_\mathfrak{p}$. Thus, the next biconditional holds for $X \in \mathcal{D}_+(R)$ and $\mathfrak{p} \in \text{Spec } R$ with $\mathfrak{p} \cap S = \emptyset$.

(2.1) $\mathfrak{p} \in \text{Anc}_R X \iff S^{-1}\mathfrak{p} \in \text{Anc}_{S^{-1}R} S^{-1}X.$

THEOREM 2.3. For $X \in \mathcal{D}_+(R)$ there are inclusions:

(a) $\text{Min}_R X \subseteq \text{Anc}_R X$; and

(b) $W_0(X) \subseteq \text{Anc}_R X$.

Furthermore, if $\text{amp } X = 0$, that is, if $X$ is equivalent to a module up to a shift, then

(c) $\text{Anc}_R X = \text{Min}_R X \subseteq \text{Ass}_R X$;

and if $X$ is Cohen-Macaulay, that is, $X \in \mathcal{D}_b(R)$ and $\dim R X = \text{depth}_R X$, then

(d) $\text{Ass}_R X \subseteq \text{Anc}_R X = W_0(X)$.

PROOF. In the following $X$ belongs to $\mathcal{D}_+(R)$.

(a): If $\mathfrak{p}$ belongs to $\text{Min}_R X$ then $\text{Supp } R_\mathfrak{p} X_\mathfrak{p} = \{ \mathfrak{p} \}$, so $\dim R_\mathfrak{p} X_\mathfrak{p} = -\inf X_\mathfrak{p}$, that is, $\mathfrak{p} \in \text{Anc}_{R_\mathfrak{p}} X_\mathfrak{p}$ and hence $\mathfrak{p} \in \text{Anc}_R X$ by (2.1).

(b): Assume that $\mathfrak{p}$ belongs to $W_0(X)$, then $\dim R X = \dim R/\mathfrak{p} - \inf X_\mathfrak{p}$, and since $\dim R X \geq \dim R_\mathfrak{p} X_\mathfrak{p} + \dim R/\mathfrak{p}$ and $\dim R_\mathfrak{p} X_\mathfrak{p} \geq -\inf X_\mathfrak{p}$, cf. (1.1) and (1.2), it follows that $\dim R_\mathfrak{p} X_\mathfrak{p} = -\inf X_\mathfrak{p}$, as desired.

(c): For $M \in \mathcal{D}_0(R)$ we have

$$\text{Anc}_R M = \{ \mathfrak{p} \in \text{Supp } R M | \dim R_\mathfrak{p} M_\mathfrak{p} = 0 \} = \text{Min}_R M,$$
and the inclusion $\text{Min}_R M \subseteq \text{Ass}_R M$ is well-known.

(d): Assume that $X \in \mathcal{D}_+(R)$ and $\dim_R X = \text{depth}_R X$, then $\dim_{R_\wp} X_\wp = \text{depth}_{R_\wp} X_\wp$ for all $\wp \in \text{Supp}_R X$, cf. [5, (16.17)]. If $\wp \in \text{Ass}_R X$ we have

$$\dim_{R_\wp} X_\wp = \text{depth}_{R_\wp} X_\wp = - \sup X_\wp \leq - \inf X_\wp,$$

cf. [4, Def. 2.3], and it follows by (1.2) that equality must hold, so $\wp$ belongs to $\text{Anc}_R X$.

For each $\wp \in \text{Supp}_R X$ there is an equality

$$\dim_R X = \dim_{R_\wp} X_\wp + \dim R/\wp,$$

cf. [5, (17.4)(b)], so $\dim_R X - \dim R/\wp + \inf X_\wp = 0$ for $\wp$ with $\dim_{R_\wp} X_\wp = - \inf X_\wp$. This proves the inclusion $\text{Anc}_R X \subseteq W_0(X)$.

**Corollary 2.4.** For $X \in \mathcal{D}_0(R)$ there is an inclusion:

(a) $\text{Min}_R X \subseteq \text{Ass}_R X \cap \text{Anc}_R X$;

and for $\wp \in \text{Ass}_R X \cap \text{Anc}_R X$ there is an equality:

(b) $\text{cmd}_{R_\wp} X_\wp = \text{amp} X_\wp$.

**Proof.** Part (a) follows by 2.3 (a) and [4, Prop. 2.6]; part (b) is immediate by the definitions of associated and anchor prime ideals, cf. [4, Def. 2.3].

**Corollary 2.5.** If $X \in \mathcal{D}_+(R)$, then

$$\dim_R X = \sup \{ \dim R/\wp + \dim_{R_\wp} X_\wp \mid \wp \in \text{Anc}_R X \}.$$

**Proof.** It is immediate by the definitions that

$$\dim_R X = \sup \{ \dim R/\wp - \inf X_\wp \mid \wp \in \text{Supp}_R X \} \geq \sup \{ \dim R/\wp - \inf X_\wp \mid \wp \in \text{Anc}_R X \} = \sup \{ \dim R/\wp + \dim_{R_\wp} X_\wp \mid \wp \in \text{Anc}_R X \};$$

and the opposite inequality follows by 2.3 (b).

**Proposition 2.6.** The following hold:

(a) If $X \in \mathcal{D}_+(R)$ and $\wp$ belongs to $\text{Anc}_R X$, then $\dim_{R_\wp}(H_{\inf X_\wp}(X_\wp)) = 0$.

(b) If $X \in \mathcal{D}_0(R)$, then $\text{Anc}_R X$ is a finite set.

**Proof.** (a): Assume that $\wp \in \text{Anc}_R X$; by [6, Prop. 3.5] we have

$$- \inf X_\wp = \dim_{R_\wp} X_\wp \geq \dim_{R_\wp}(H_{\inf X_\wp}(X_\wp)) - \inf X_\wp,$$
and hence \( \dim_{R_0}(H_{\inf X}(X_p)) = 0 \).

(b): By (a) every anchor prime ideal for \( X \) is minimal for one of the homology modules of \( X \), and when \( X \in D_f^i(R) \) each of the finitely many homology modules has a finite number of minimal prime ideals.

**Observation 2.7.** By Nakayama’s lemma it follows that

\[
\inf K(x_1, \ldots, x_n; Y) = \inf Y,
\]

for \( Y \in D_f^i(R) \) and elements \( x_1, \ldots, x_n \in \mathfrak{m} \).

**Proposition 2.8 (Dimension of Koszul Complexes).** The following hold for a complex \( Y \in D_f^i(R) \) and elements \( x_1, \ldots, x_n \in \mathfrak{m} \):

(a) \[
\dim R K(x_1, \ldots, x_n; Y) = \sup \{ \dim R / \mathfrak{p} - \inf Y_{\mathfrak{p}} \mid \mathfrak{p} \in \text{Supp} R Y \cap \mathcal{V}(x_1, \ldots, x_n) \}; \text{ and}
\]

(b) \[
\dim R Y - n \leq \dim R K(x_1, \ldots, x_n; Y) \leq \dim R Y.
\]

Furthermore:

(c) The elements \( x_1, \ldots, x_n \) are contained in a prime ideal

\[
\mathfrak{p} \in W_0(Y);
\]

and

(d) \[
\dim R K(x_1, \ldots, x_n; Y) = \dim R Y \text{ if and only if } x_1, \ldots, x_n \in \mathfrak{p}
\]

for some \( \mathfrak{p} \in W_0(Y) \).

**Proof.** Since \( \text{Supp} R K(x_1, \ldots, x_n; Y) = \text{Supp} R Y \cap \mathcal{V}(x_1, \ldots, x_n) \) (see [6, p. 157] and [4, 3.2]) (a) follows by the definition of Krull dimension and 2.7. In (b) the second inequality follows from (a); the first one is established through four steps:

1\( ^{\circ} \) \( Y = R \): The second equality below follows from the definition of Krull dimension as \( \text{Supp} R K(x_1, \ldots, x_n) = \text{Supp} R H_0(K(x_1, \ldots, x_n)) = \mathcal{V}(x_1, \ldots, x_n) \), cf. [4, 3.2]; the inequality is a consequence of Krull’s Principal Ideal Theorem, see for example [8, Thm. 13.6].

\[
\dim R K(x_1, \ldots, x_n; Y) = \dim R K(x_1, \ldots, x_n)
\]

\[
= \sup \{ \dim R / \mathfrak{p} \mid \mathfrak{p} \in \mathcal{V}(x_1, \ldots, x_n) \}
\]

\[
= \dim R / (x_1, \ldots, x_n)
\]

\[
\geq \dim R - n
\]

\[
= \dim R Y - n.
\]
2° \( Y = B \), a cyclic module: By \( \bar{x}_1, \ldots, \bar{x}_n \) we denote the residue classes in \( B \) of the elements \( x_1, \ldots, x_n \); the inequality below is by 1°.

\[
\dim_R K(x_1, \ldots, x_n; Y) = \dim_R K(\bar{x}_1, \ldots, \bar{x}_n) \\
= \dim_R K(\bar{x}_1, \ldots, \bar{x}_n) \\
\geq \dim B - n \\
= \dim_R Y - n.
\]

3° \( Y = H \in \mathcal{D}^0_0(R) \): We set \( B = R/\text{Ann}_R H \); the first equality below follows by [6, Prop. 3.11] and the inequality by 2°.

\[
\dim_R K(x_1, \ldots, x_n; Y) = \dim_R K(x_1, \ldots, x_n; B) \\
\geq \dim B - n \\
= \dim_R Y - n.
\]

4° \( Y \in \mathcal{D}^1_0(R) \): The first equality below follows by [6, Prop. 3.12] and the last by [6, Prop. 3.5]; the inequality is by 3°.

\[
\dim_R K(x_1, \ldots, x_n; Y) = \sup \{ \dim_R K(x_1, \ldots, x_n; H_\ell(Y)) - \ell \mid \ell \in \mathbb{Z} \} \\
\geq \sup \{ \dim_R H_\ell(Y) - n - \ell \mid \ell \in \mathbb{Z} \} \\
= \dim_R Y - n.
\]

This proves (b).

In view of (a) it now follows that

\[
\dim_R Y - n \leq \dim R/\mathfrak{p} - \inf Y_\mathfrak{p}
\]

for some \( \mathfrak{p} \in \text{Supp}_R Y \cap V(x_1, \ldots, x_n) \). That is, the elements \( x_1, \ldots, x_n \) are contained in a prime ideal \( \mathfrak{p} \in \text{Supp}_R Y \) with

\[
\dim_R Y - \dim R/\mathfrak{p} + \inf Y_\mathfrak{p} \leq n,
\]

and this proves (c).

Finally, it is immediate by the definitions that

\[
\dim_R Y = \sup \{ \dim R/\mathfrak{p} - \inf Y_\mathfrak{p} \mid \mathfrak{p} \in \text{Supp}_R Y \cap V(x_1, \ldots, x_n) \}
\]

if and only if \( W_0(Y) \cap V(x_1, \ldots, x_n) \neq \emptyset \). This proves (d).
Theorem 2.9. If $Y \in \mathcal{D}_b^f(R)$, then the next two numbers are equal.

$$d(Y) = \dim_R Y + \inf Y; \quad \text{and}$$

$$s(Y) = \inf\{ s \in \mathbb{N}_0 \mid \exists x_1, \ldots, x_j : \mathfrak{m} \in \text{Anc}_R K(x_1, \ldots, x_j; Y) \}.$$ 

Proof. There are two inequalities to prove.

$d(Y) \leq s(Y)$: Let $x_1, \ldots, x_s \in \mathfrak{m}$ be such that $\mathfrak{m} \in \text{Anc}_R K(x_1, \ldots, x_j; Y)$; by 2.8 (b) and 2.7 we then have

$$\dim_R Y - s \leq \dim_R K(x_1, \ldots, x_s; Y) = -\inf K(x_1, \ldots, x_s; Y) = -\inf Y,$$

so $d(Y) \leq s$, and the desired inequality follows.

$s(Y) \leq d(Y)$: We proceed by induction on $d(Y)$. If $d(Y) = 0$ then $\mathfrak{m} \in \text{Anc}_R Y$ so $s(Y) = 0$. If $d(Y) > 0$ then $\mathfrak{m} \notin \text{Anc}_R Y$, and since $\text{Anc}_R Y$ is a finite set, by 2.6(b), we can choose an element $x \in \mathfrak{m} - \cup_{p \in \text{Anc}_R Y}$. We set $K = K(x; Y)$; it is clear that $s(Y) \leq s(K) + 1$. Furthermore, it follows by 2.8 (a) and 2.3 (b) that $\dim_R K < \dim_R Y$ and thereby $d(K) < d(Y)$, cf. 2.7. Thus, by the induction hypothesis we have

$$s(Y) \leq s(K) + 1 \leq d(K) + 1 \leq d(Y);$$

as desired.

3. Parameters

By 2.9 the next definitions extend the classical notions of systems and sequences of parameters for finite modules (e.g., see [8, § 14] and the appendix in [2]).

Definitions 3.1. Let $Y$ belong to $\mathcal{D}_b(R)$ and set $d = \dim_R Y + \inf Y$. A set of elements $x_1, \ldots, x_d \in \mathfrak{m}$ are said to be a system of parameters for $Y$ if and only if $\mathfrak{m} \in \text{Anc}_R K(x_1, \ldots, x_d; Y)$.

A sequence $x = x_1, \ldots, x_n$ is said to be a $Y$-parameter sequence if and only if it is part of a system of parameters for $Y$.

Lemma 3.2. Let $Y$ belong to $\mathcal{D}_b(R)$ and set $d = \dim_R Y + \inf Y$. The next two conditions are equivalent for elements $x_1, \ldots, x_d \in \mathfrak{m}$.

(i) $x_1, \ldots, x_d$ is a system of parameters for $Y$.

(ii) For every $j \in \{0, \ldots, d\}$ there is an equality:

$$\dim_R K(x_1, \ldots, x_j; Y) = \dim_R Y - j;$$

and $x_{j+1}, \ldots, x_d$ is a system of parameters for $K(x_1, \ldots, x_j; Y)$.  
Proof. (i) ⇒ (ii): Assume that \( x_1, \ldots, x_d \) is a system of parameters for \( Y \), then
\[
- \inf K(x_1, \ldots, x_d; Y) = \dim_R K(x_1, \ldots, x_d; Y)
\]
\[
= \dim_R K(x_{j+1}, \ldots, x_d; K(x_1, \ldots, x_j; Y))
\]
\[
\geq \dim_R K(x_1, \ldots, x_j; Y) - (d - j) \quad \text{by 2.8 (b)}
\]
\[
\geq \dim_R Y - j - (d - j) \quad \text{by 2.8 (b)}
\]
\[
= \dim_R Y - d - \inf Y.
\]

By 2.7 it now follows that \( - \inf Y = \dim_R K(x_1, \ldots, x_j; Y) - (d - j) \), so
\[
\dim_R K(x_1, \ldots, x_j; Y) = d - j - \inf Y = \dim_R Y - j,
\]
as desired. It also follows that \( d(K(x_1, \ldots, x_j; Y)) = d - j \), and since
\[
\frak{m} \in \Anc_R K(x_1, \ldots, x_d; Y) = \Anc_R K(x_{j+1}, \ldots, x_d; K(x_1, \ldots, x_j; Y)),
\]
we conclude that \( x_{j+1}, \ldots, x_d \) is a system of parameters for \( K(x_1, \ldots, x_j; Y) \).

(ii) ⇒ (i): If \( \dim_R K(x_1, \ldots, x_j; Y) = \dim_R Y - j \) then \( d(K(x_1, \ldots, x_j; Y)) = d - j \) and if \( x_{j+1}, \ldots, x_d \) is a system of parameters for \( K(x_1, \ldots, x_j; Y) \) then \( \frak{m} \) belongs to
\[
\Anc_R K(x_{j+1}, \ldots, x_d; K(x_1, \ldots, x_j; Y)) = \Anc_R K(x_1, \ldots, x_d; Y),
\]
so \( x_1, \ldots, x_d \) must be a system of parameters for \( Y \).

Proposition 3.3. Let \( Y \in \mathcal{D}_b(R) \). The following conditions are equivalent for a sequence \( x = x_1, \ldots, x_n \) in \( \frak{m} \).

(i) \( x \) is a \( Y \)-parameter sequence.

(ii) For each \( j \in \{0, \ldots, n\} \) there is an equality:
\[
\dim_R K(x_1, \ldots, x_j; Y) = \dim_R Y - j;
\]
and \( x_{j+1}, \ldots, x_n \) is a \( K(x_1, \ldots, x_j; Y) \)-parameter sequence.

(iii) There is an equality:
\[
\dim_R K(x_1, \ldots, x_n; Y) = \dim_R Y - n.
\]

Proof. It follows by 3.2 that (i) implies (ii), and (iii) follows from (ii). Now, set \( K = K(x; Y) \) and assume that \( \dim_R K = \dim_R Y - n \). Choose, by 2.9,
$s = s(K) = \dim_R K + \inf K$ elements $w_1, \ldots, w_s$ in $\mathfrak{m}$ such that $\mathfrak{m}$ belongs to $\text{Anc}_R K(w_1, \ldots, w_s; K) = \text{Anc}_R K(x_1, \ldots, x_n, w_1, \ldots, w_s; Y)$. Then, by 2.7, we have

$$n + s = (\dim_Y Y - \dim_R K) + (\dim_R K + \inf K) = \dim_R Y + \inf Y = d,$$

so $x_1, \ldots, x_n, w_1, \ldots, w_s$ is a system of parameters for $Y$, whence $x_1, \ldots, x_n$ is a $Y$-parameter sequence.

We now recover a classical result (e.g., see [2, Prop. A.4]):

**Corollary 3.4.** Let $M$ be an $R$-module. The following conditions are equivalent for a sequence $x = x_1, \ldots, x_n$ in $\mathfrak{m}$.

(i) $x$ is an $M$-parameter sequence.

(ii) For each $j \in \{0, \ldots, n\}$ there is an equality:

$$\dim_R M/(x_1, \ldots, x_j)M = \dim_R M - j;$$

and $x_{j+1}, \ldots, x_n$ is an $M/(x_1, \ldots, x_j)M$-parameter sequence.

(iii) There is an equality:

$$\dim_R M/(x_1, \ldots, x_n)M = \dim_R M - n.$$

**Proof.** By [6, Prop. 3.12] and [5, (16.22)] we have

$$\dim_R K(x_1, \ldots, x_j; M)
= \sup \{ \dim_R (M \otimes_R H_\ell(K(x_1, \ldots, x_j))) - \ell \mid \ell \in \mathbb{Z} \}
= \sup \{ \dim_R (M \otimes_R H_\ell(K(x_1, \ldots, x_j))) - \ell \mid \ell \in \mathbb{Z} \}
= \dim_R (M \otimes_R R/(x_1, \ldots, x_j)).$$

**Theorem 3.5.** Let $Y \in \mathcal{D}_K^+(R)$. The following hold for a sequence $x = x_1, \ldots, x_n$ in $\mathfrak{m}$.

(a) There is an inequality:

$$\text{amp}_K(x; Y) \geq \text{amp}_Y;$$

and equality holds if and only if $x$ is a $Y$-sequence.

(b) There is an inequality:

$$\text{cmd}_R K(x; Y) \geq \text{cmd}_R Y;$$

and equality holds if and only if $x$ is a $Y$-parameter sequence.
(c) If $x$ is a maximal $Y$-sequence, then

$$\text{amp } Y \leq \text{cmd}_R K(x; Y).$$

(d) If $x$ is a system of parameters for $Y$, then

$$\text{cmd}_R Y \leq \text{amp } K(x; Y).$$

**Proof.** In the following $K$ denotes the Koszul complex $K(x; Y)$.

(a): Immediate by 2.7 and [4, Prop. 5.1].
(b): By [4, Thm. 4.7 (a)] and 2.8 (b) we have

$$\text{cmd}_R K = \dim_R K - \text{depth}_R K = \dim_R K + n - \text{depth}_R Y \geq \text{cmd}_R Y,$$

and by 3.3 equality holds if and only if $x$ is a $Y$-parameter sequence.

(c): Suppose $x$ is a maximal $Y$-sequence, then

$$\text{amp } Y = \sup Y - \inf K \quad \text{by 2.7}$$

$$= -\text{depth}_R K - \inf K \quad \text{by [4, Thm. 5.4]}$$

$$\leq \text{cmd}_R K \quad \text{by (1.2)}.$$

(d): Suppose $x$ is a system of parameters for $Y$, then

$$\text{amp } K = \sup K + \dim_R K$$

$$\geq \dim_R K - \text{depth}_R K \quad \text{by (1.3)}$$

$$= \text{cmd}_R Y \quad \text{by (b)}.$$

**Theorem 3.6.** The following hold for $Y \in \mathcal{D}_R(R)$.

(a) The next four conditions are equivalent.

(i) There is a maximal $Y$-sequence which is also a $Y$-parameter sequence.

(ii) $\text{depth}_R Y + \sup Y \leq \dim_R Y + \inf Y$.

(ii') $\text{amp } Y \leq \text{cmd}_R Y$.

(iii) There is a maximal strong $Y$-sequence which is also a $Y$-parameter sequence.

(b) The next four conditions are equivalent.

(i) There is a system of parameters for $Y$ which is also a $Y$-sequence.

(ii) $\dim_R Y + \inf Y \leq \text{depth}_R Y + \sup Y$.

(ii') $\text{cmd}_R Y \leq \text{amp } Y$. 


(iii) There is a system of parameters for $Y$ which is also a strong $Y$-sequence.

(c) The next four conditions are equivalent.

(i) There is a system of parameters for $Y$ which is also a maximal $Y$-sequence.

(ii) $\dim_R Y + \inf Y = \depth_R Y + \sup Y$.

(ii') $\cmd_R Y = \amp Y$.

(iii) There is a system of parameters for $Y$ which is also a maximal strong $Y$-sequence.

Proof. Let $Y \in D^f(R)$, set $n(Y) = \depth_R Y + \sup Y$ and $d(Y) = \dim_R Y + \inf Y$.

(a): A maximal $Y$-sequence is of length $n(Y)$, cf. [4, Cor. 5.5], and the length of a $Y$-parameter sequence is at most $d(Y)$. Thus, (i) implies (ii) which in turn is equivalent to (ii'). Furthermore, a maximal strong $Y$-sequence is, in particular, a maximal $Y$-sequence, cf. [4, Cor. 5.7], so (iii) is stronger than (i). It is now sufficient to prove the implication (ii) $\Rightarrow$ (iii): We proceed by induction. If $n(Y) = 0$ then the empty sequence is a maximal strong $Y$-sequence and a $Y$-parameter sequence. Let $n(Y) > 0$; the two sets $\Ass_R Y$ and $W_0(Y)$ are both finite, and since $0 < n(Y) \leq d(Y)$ none of them contain $m$. We can, therefore, choose an element $x \in m - \cup_{\Ass_R Y \cup W_0(Y)} p$, and $x$ is then a strong $Y$-sequence, cf. [4, Def. 3.3], and a $Y$-parameter sequence, cf. 3.3 and 2.8. Set $K = K(x; Y)$, by [4, Thm. 4.7 and Prop. 5.1], respectively, 2.8 and 2.7 we have

$$\depth_R K + \sup K = n(Y) - 1 \leq d(Y) - 1 = \dim_R K + \inf K.$$  

By the induction hypothesis there exists a maximal strong $K$-sequence $w_1, \ldots, w_{n-1}$ which is also a $K$-parameter sequence, and it follows by [4, 3.5] and 3.3 that $x, w_1, \ldots, w_{n-1}$ is a strong $Y$-sequence and a $Y$-parameter sequence, as wanted.

The proof of (b) is similar to the proof of (a), and (c) follows immediately by (a) and (b).

Theorem 3.7. The following hold for $Y \in D^f(R)$:

(a) If $\amp Y = 0$, then any $Y$-sequence is a $Y$-parameter sequence.

(b) If $\cmd_R Y = 0$, then any $Y$-parameter sequence is a strong $Y$-sequence.

Proof. The empty sequence is a $Y$-parameter sequence as well as a strong $Y$-sequence, this founds the base for a proof by induction on the length $n$ of
the sequence \( x = x_1, \ldots, x_n \). Let \( n > 0 \) and set \( K = K(x_1, \ldots, x_{n-1}; Y) \); by 2.8 (a) we have
\[
\dim R K(x_1, \ldots, x_n; Y) = \dim R K(x_n; K) = \sup \{ \dim R/v - \inf K_p \mid v \in \text{Supp}_K K \cap V(x_n) \}.
\]

Assume that \( \text{amp} Y = 0 \). If \( x \) is a \( Y \)-sequence, then \( \text{amp} K = 0 \) by 3.5 (a) and \( x_n \not\in Z R K \), cf. [4, Def. 3.3]. As \( Z R K = \bigcup_{v \in \text{Ass}_R K} v \), cf. [4, 2.5], it follows by (b) and (c) in 2.3 that \( x_n \) is not contained in any prime ideal \( v \in \mathcal{W}_0(K) \); so from (*) we conclude that \( \dim R K(x_n; K) < \dim R K \), and it follows by 2.8 (b) that \( \dim R K(x_n, K) = \dim R K - 1 \). By the induction hypothesis \( \dim R K = \dim R Y - (n - 1) \), so \( \dim R K(x_1, \ldots, x_n; Y) = \dim R Y - n \) and it follows by 3.3 that \( x \) is a \( Y \)-parameter sequence. This proves (a).

We now assume that \( \text{cmd} R Y = 0 \). If \( x \) is a \( Y \)-parameter sequence then, by the induction hypothesis, \( x_1, \ldots, x_{n-1} \) is a strong \( Y \)-sequence, so it is sufficient to prove that \( x_n \not\in Z R K \), cf. [4, 3.5]. By 3.3 it follows that \( x_n \) is a \( K \)-parameter sequence, so \( \dim R K(x_n; K) = \dim R K - 1 \) and we conclude from (*) that \( x_n \not\in \bigcup_{v \in \mathcal{W}_0(K)} v \). Now, by 3.5 (b) we have \( \text{cmd} R K = 0 \), so it follows from 2.3 (d) that \( x_n \not\in \bigcup_{v \in \text{Ass}_R K} v = Z R K \). This proves (b).

**Semi-dualizing Complexes** 3.8. We recall two basic definitions from [3]:

A complex \( C \in \mathcal{D}_b(R) \) is said to be **semi-dualizing** for \( R \) if and only if the homothety morphism \( \chi^R_C : R \to \mathsf{RHom}_R(C, C) \) is an isomorphism [3, (2.1)].

Let \( C \) be a semi-dualizing complex for \( R \). A complex \( Y \in \mathcal{D}_b(R) \) is said to be **\( C \)-reflexive** if and only if the **dagger dual** \( Y^C = \mathsf{RHom}_R(Y, C) \) belongs to \( \mathcal{D}_b(R) \) and the **biduality morphism** \( \delta^C_Y : Y \to \mathsf{RHom}_R(\mathsf{RHom}_R(Y, C), C) \) is invertible in \( \mathcal{D}(R) \) [3, (2.7)].

Relations between dimension and depth for \( C \)-reflexive complexes are studied in [3, sec. 3], and the next result is an immediate consequence of [3, (3.1) and (2.10)].

Let \( C \) be a semi-dualizing complex for \( R \) and let \( Z \) be a \( C \)-reflexive complex. The following holds for \( v \in \text{Spec } R \): If \( v \in \text{Anc}_R Z \) then \( v \in \text{Ass}_R Z^C \), and the converse holds in \( C \) is Cohen-Macaulay.

A **dualizing complex**, cf. [7], is a semi-dualizing complex of finite injective dimension, in particular, it is Cohen-Macaulay, cf. [3, (3.5)]. If \( D \) is a dualizing complex for \( R \), then, by [7, Prop. V.2.1], all complexes \( Y \in \mathcal{D}_b(R) \) are \( D \)-reflexive; in particular, all finite \( R \)-modules are \( D \)-reflexive and, therefore, [4, 5.10] is a special case of the following:
Theorem 3.9. Let $C$ be a Cohen-Macaulay semi-dualizing complex for $R$, and let $x = x_1, \ldots, x_n$ be a sequence in $m$. If $Y$ is $C$-reflexive, then $x$ is a $Y$-parameter sequence if and only if it is a $R\text{Hom}_R(Y, C)$-sequence; that is

$$x \text{ is a } Y\text{-parameter sequence } \iff x \text{ is a } R\text{Hom}_R(Y, C)\text{-sequence.}$$

Proof. We assume that $C$ is a Cohen-Macaulay semi-dualizing complex for $R$ and that $Y$ is $C$-reflexive, cf. 3.8. The desired biconditional follows by the next chain, and each step is explained below (we use the notation $-^c$ introduced in 3.8).

$$x \text{ is a } Y\text{-parameter sequence } \iff \text{cmd}_R K(x; Y) = \text{cmd}_R Y$$
$$\iff \text{amp} K(x; Y)^c = \text{amp} Y^c$$
$$\iff \text{amp} K(x; Y^c) = \text{amp} Y^c$$
$$\iff x \text{ is a } Y^c\text{-sequence.}$$

The first biconditional follows by 3.5 (b) and the last by 3.5 (a). Since $K(x)$ is a bounded complex of free modules (hence of finite projective dimension), it follows from [3, Thm. (3.17)] that also $K(x; Y)$ is $C$-reflexive, and the second biconditional is then immediate by the CMD-formula [3, Cor. (3.8)]. The third one is established as follows:

$$K(x; Y)^c \simeq R\text{Hom}_R(K(x) \otimes_R^L Y, C)$$
$$\simeq R\text{Hom}_R(K(x), Y^c)$$
$$\simeq R\text{Hom}_R(K(x), R \otimes_R^L Y^c)$$
$$\simeq R\text{Hom}_R(K(x), R) \otimes_R^L Y^c$$
$$\sim K(x) \otimes_R^L Y^c$$
$$\simeq K(x; Y^c),$$

where the second isomorphism is by adjointness and the fourth by, so-called, tensor-evaluation, cf. [1, (1.4.2)]. It is straightforward to check that $\text{Hom}_R(K(x), R)$ is isomorphic to the Koszul complex $K(x)$ shifted $n$ degrees to the right, and the symbol $\sim$ denotes isomorphism up to shift.

If $C$ is a semi-dualizing complex for $R$, then both $C$ and $R$ are $C$-reflexive complexes, cf. [3, (2.8)], so we have an immediate corollary to the theorem:

Corollary 3.10. If $C$ is a Cohen-Macaulay semi-dualizing complex for $R$, then the following hold for a sequence $x = x_1, \ldots, x_n$ in $m$.

(a) $x$ is a $C$-parameter sequence if and only if it is an $R$-sequence.
(b) *x* is an *R*-parameter sequence if and only if it is a *C*-sequence.

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**REFERENCES**