MATH. SCAND. 90 (2002), 139–151

ON SOME SEMILINEAR EQUATIONS OF SCHRÖDINGER TYPE

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Abstract

We study the initial value problem for some semilinear pseudo-differential equations of the form $\partial_t u + iH(x, D_x)u = F(u, \nabla u)$. The assumptions we make on H are trivially satisfied by Δ, thus our equations generalize Schrödinger type equations. A local existence theorem is proved in some weighted Sobolev spaces.

0. Introduction

In this paper we consider the initial value problem for some nonlinear evolution equations of the form

$$\partial_t u + iH(x, D_x)u = F(u, \nabla u) \tag{1}$$

where H is a uniformly elliptic pseudo-differential operator of order 2 with real symbol.

We assume that the nonlinear term $F : \mathsf{C} \times \mathsf{C}^n \to \mathsf{C}$ satisfies: $F(u, q) \in \mathscr{C}^\infty(\mathsf{R}^2 \times \mathsf{R}^{2n})$ and $|F(u, q)| \le C(|u|^2 + |q|^2)$ near the origin.

The simplest model we have in mind is the one with $H(x, \xi) = |\xi|^2$, that is (1) generalizes semilinear Schrödinger equations.

Most papers on semilinear Schrödinger equations are concerned with the case $F(u)$ or $F(u, \nabla u)$ but $\operatorname{Im} \frac{\partial F}{\partial q_j} = 0$, $j = 1, \dots, n$. Some troubles arise when one works with classical energy methods in the general case: even in the linear case some difficulties arise owing to the imaginary part of the coefficients of $\partial_{x_j} u$. Correspondingly all the papers about the wellposedness of the Cauchy problem in L^2 or Sobolev spaces for linear Schrödinger equations give necessary or sufficient conditions on the imaginary part of the first order terms of the operator. (See [7], [8], [9], [12]).

In [2] Chihara succeeded in proving local existence in some weighted Sobolev spaces for the semilinear Schrödinger equations in the case $n = 1$. In [3] he generalized the result to higher space dimension. Our paper studies more

Received June 16, 1998; in revised form February 11, 1999.

general operators of Schrödinger type and thus it generalizes [3]. We need the following additional assumption on H :

(2) $\exists\, c > 0$ such that $\{H, p\}(x, \xi) \geq c\langle x\rangle^{-2}|\xi|$ for large $|\xi|$,

where $p(x, \xi) = \langle\xi\rangle^{-1} \sum_{j=1}^{n} \xi_j \operatorname{arctg} x_j$ and $\{.,.\}$ denotes the Poisson's bracket, i.e. $\{H, p\} = \sum_{j=1}^{n}(\partial_{\xi_j} H \partial_{x_j} p - \partial_{\xi_j} p \partial_{x_j} H)$.

A condition similar to (2) can be found in the literature on Schrödinger equations (see (A2) in [5] for example). Such conditions are used to eliminate – in some sense – the bad first order term.

1. Notation

For $x \in \mathbb{R}^n$ let $\langle x\rangle = (1 + |x|^2)^{1/2}$ and $\langle D_x\rangle = (1 - \Delta_x)^{1/2}$.

Let $\|\,\|$ denote the L^2-norm.

For $m, p \in \mathbb{R}$ let $\|f\|_{m,p} = \|\langle x\rangle^p \langle D_x\rangle^m f\|$ and let $H^{m,p} = \{f \in \mathscr{S}'(\mathbb{R}^n);\ \|f\|_{m,p} < \infty\}$.

Note that $H^{m,0}$ is the usual Sobolev space H^m.

In the sequel if ℓ is a sufficiently large integer we shall denote $H^{m+\ell,0} \cap H^{m+1,1} \cap H^{m,2}$ by $\Xi^{m,\ell}$.

We shall use the following notation for pseudo-differential operators. The space of the symbols $\sigma(x, \xi) \in \mathscr{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\sup_{\substack{x,\xi \in \mathbb{R}^n \\ \alpha,\beta \in \mathbb{N}^n}} \left|\partial_\xi^\alpha D_x^\beta \sigma(x, \xi)\right| \langle\xi\rangle^{|\alpha|-m} < \infty$$

will be denoted by S^m. The calculus for the corresponding pseudo-differential operators can be found in Kumano-go's book [11].

2. The main result

Consider the following Cauchy problem for an equation of Schrödinger type:

(3) $\partial_t u + iH(x, D_x)u = F(u, \nabla_x u)$ in $]0, \infty) \times \mathbb{R}^n,\ u(t = 0) = u_\circ$

We make the following assumptions:

(H1) H has a real symbol;

(H2) there exists $c_\circ > 0$ such that $|H(x, \xi)| \geq c_\circ|\xi|^2\ \forall x, \xi \in \mathbb{R}^n$;

(H3) $\exists c > 0$ such that $\{H, p\}(x, \xi) \geq c\langle x\rangle^{-2}|\xi|$ for large $|\xi|$, where $\{.,.\}$ denotes the Poisson's bracket and $p(x, \xi) = \langle\xi\rangle^{-1} \sum_{j=1}^{n} \xi_j \operatorname{arctg} x_j$.

(H4) $\sup_{x,\xi \in \mathbb{R}^n} \left|\partial_\xi^\alpha D_x^\beta H(x, \xi)\right|.\langle\xi\rangle^{|\alpha+\beta|-2} < \infty, \quad \forall \alpha, \beta \in \mathbb{N}^n.$

Moreover we make the following assumptions on the nonlinear term:

(F1) $F : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}$ belongs to $\mathscr{C}^\infty(\mathbb{R}^2 \times \mathbb{R}^{2n})$;

(F2) there exists $C > 0$ such that $|F(u, q)| \leq C(|u|^2 + |q|^2)$ near $(u, q) = (0, 0)$.

In the following section we prove the following

THEOREM 2.1. *For any initial datum $u_\circ \in \Xi^{m,\ell}$ (where m and ℓ are sufficiently large integers) there exists a time $T > 0$ such that the Cauchy problem* (3) *has a solution $u \in \mathscr{C}([0, T]; \Xi^{m,\ell})$.*

To prove this theorem at first we consider a parabolic regularization of our problem which depends on a viscosity parameter $\varepsilon > 0$. The regularized problem is solved by linearization in §4. Finally a solution of (3) is obtained as a zero limit of the solution of the regularized problem.

3. Parabolic regularization

For any $\varepsilon \in]0, 1]$ let us consider

$$\begin{cases} \partial_t u^\varepsilon - \varepsilon \Delta_x u^\varepsilon + i H(x, D_x) u^\varepsilon = F(u^\varepsilon, \nabla_x u^\varepsilon) \\ u^\varepsilon(0, x) = u_o(x) \end{cases} \tag{4}$$

in $]0, +\infty) \times \mathbb{R}^n$, where H, F and $u_\circ$ are as in §2.

Let P_ε denote the linear operator $\partial_t - \varepsilon \Delta_x + i H(x, D_x)$. Let us first construct a fundamental solution $S_\varepsilon(t)$ for P_ε. Consider the following eikonal equation:

$$\begin{cases} \partial_t \phi(t, s; x, \xi) + H(x, \nabla_x \phi(t, s; s, \xi)) \\ \phi(s, s; x, \xi) = x.\xi \end{cases} \tag{5}$$

Then we have the following

LEMMA 3.1. *If H satisfies (H1) and (H4), then there exists $T > 0$ such that for every $t, s \in [-T, T]$ the following estimate is true:*

$$\sup_{x \in \mathbb{R}^n} \left| \partial_\xi^\alpha \partial_x^\beta (\phi(t, s; x, \xi) - x.\xi) \right| \leq C'_{\alpha,\beta} |t - s| \langle \xi \rangle^{2 - |\alpha + \beta|} \tag{6}$$

$\forall \alpha, \beta \in \mathbb{N}^n$, $\forall \xi \in \mathbb{R}^n$ *with large $|\xi|$, and for some $C'_{\alpha,\beta}$.*

PROOF. The proof follows the lines of Theorem 4.1 in [11]. At first we prove inductively that the solutions $q(t, s; y, \xi)$ and $p(t, s; y, \xi)$ of the Hamilton's equations

$$\begin{cases} \dfrac{dq}{dt} = \nabla_\xi H(q, p) \quad \dfrac{dp}{dt} = -\nabla_x H(q, p) \\ (q, p)|_{t=s} = (y, \xi) \end{cases}$$

satisfy the following estimates, for every $\alpha, \beta \in \mathbb{N}^n$:

$$\sup_{y\in\mathbb{R}^n}\left|\partial_\xi^\alpha\partial_y^\beta(q(t,s;y,\xi)-y)\right| \le C''_{\alpha,\beta}|t-s|\langle\xi\rangle^{1-|\alpha+\beta|}$$

$$\sup_{y\in\mathbb{R}^n}\left|\partial_\xi^\alpha\partial_y^\beta(p(t,s;x,\xi)-\xi)\right| \le C''_{\alpha,\beta}|t-s|\langle\xi\rangle^{1-|\alpha+\beta|}$$

Denoting the inverse mapping of $y \to x = q(t,s;y,\xi)$ by $Y(t,s;x,\xi)$, we can prove that, if $T > 0$ is sufficiently small, then for every $\alpha, \beta \in \mathbb{N}^n$, $t, s \in [-T, T]$, $\xi \in \mathbb{R}^n$ with large $|\xi|$, and for some $A_{\alpha,\beta}$, the following inequality holds:

$$\sup_{y\in\mathbb{R}^n}\left|\partial_\xi^\alpha\partial_x^\beta(Y(t,s;x,\xi)-x)\right| \le A_{\alpha,\beta}|t-s|\langle\xi\rangle^{1-|\alpha+\beta|}$$

Finally we construct the solution of (5) setting

$$\phi(t,s;x,\xi) = \psi(t,s;Y(t,s;x,\xi),\xi),$$

where

$$\psi(t,s;y,\xi) = y.\xi + \int_s^t (p.\nabla_\xi H - H)(\tau, q(\tau,s;y,\xi), p(\tau,s;y,\xi))\,d\tau.$$

Consequently, we get (6).

Now we are going to construct a Fourier integral operator whose phase is $\phi(t,s;x,\xi)$ and whose amplitude $\sigma(t,s;x,\xi) \sim \sum_{j=0}^{\infty}\sigma_{2j}(t,s;x,\xi)$ is found by solving the following transport equations:

$$(T_0)\quad \begin{cases} \partial_t\sigma_0(t) + \nabla_\xi H(x,\nabla_x\phi(t,s;x,\xi)).\nabla_x\sigma_0(t) + c_\varepsilon(t,x,\xi)\sigma_0(t) = 0 \\ \sigma_0(s) = 1 \end{cases}$$

where

$$\begin{aligned} &c_\varepsilon(t,x,\xi) \\ &\quad = \frac{1}{2}\sum_{ki}\partial^2_{\xi_k\xi_i}H(x,\nabla_x\phi(t,s;x,\xi))\partial^2_{x_kx_i}\phi(t,s;x,\xi) + \varepsilon|\nabla_x\phi(t,s;x,\xi)|^2, \end{aligned}$$

and for $j \ge 1$

$$(T_{2j})\quad \begin{cases} \partial_t\sigma_{2j}(t) + \nabla_\xi H(x,\nabla_x\phi(t,s;x,\xi)).\nabla_x\sigma_{2j}(t) \\ \qquad\qquad + c_\varepsilon(t,x,\xi)\sigma_{2j}(t) = -ib_j(t,x,\xi) \\ \sigma_{2j}(s) = 0 \end{cases}$$

with

$$
\begin{aligned}
& b_j(t,x,\xi) \\
&= \sum_{k=1}^{j} \sum_{|\gamma|=k+1} \frac{1}{\gamma!} D_z^\gamma \big\{ \partial_\xi^\gamma H(x, \tilde{\nabla}_x \phi(t,s;x,z,\xi)) \sigma_{2j-2k}(t,s;z,\xi) \big\}_{z=x} \\
&\quad - 2\varepsilon \nabla_x \phi(t,s;x,\xi) . \nabla_x \sigma_{2j-2}(t,s;x,\xi) \\
&\quad + i\varepsilon \Delta_x \sigma_{2j-2}(t,s;x,\xi) \\
&\quad - \varepsilon \Delta_x \phi(t,s;x,\xi) \sigma_{2j-2}(t,s;x,\xi)
\end{aligned}
$$

being $\tilde{\nabla}_x \phi(t,s;x,z,\xi) = \int_0^1 \nabla_x \phi(t,s;\theta z + (1-\theta)x, \xi)\, d\theta$.

We can prove inductively that there exists an increasing sequence C_n^* such that:

$$
\begin{aligned}
(7) \quad \big| \partial_\xi^\alpha \partial_x^\beta \sigma_{2j}(t,s;x,\xi) \big| &\le \exp\big(-3\varepsilon |t-s||\xi|^2/4\big) . C_*^{|\alpha+\beta|+6j} \langle \xi \rangle^{-|\alpha+\beta|-2j} \\
&\quad \cdot \sum_{k=0}^{|\alpha+\beta|+2j} \frac{\{2\varepsilon |t-s||\xi|^2\}^k}{k!}
\end{aligned}
$$

for every $\alpha, \beta \in \mathbb{N}^n$ and for every $j \in \mathbb{N}$. We can write:

$$
\sum_{k=0}^{|\alpha+\beta|+2j} \frac{\{2\varepsilon |t-s||\xi|^2\}^k}{k!} \le 8^{|\alpha+\beta|+2j} \exp\big(\varepsilon |t-s||\xi|^2/4\big),
$$

so that (7) becomes:

$$
\big| \partial_\xi^\alpha \partial_x^\beta \sigma_{2j}(t,s;x,\xi) | \le \exp\big(-\varepsilon |t-s||\xi|^2/2) C_{\alpha,\beta,j}^{**} \langle \xi \rangle^{-|\alpha+\beta|-2j}.
$$

Finally, as in Lemma 3.2 in [11], we can construct a symbol which is equivalent to the formal series of the symbols σ_{2j}. Thus we obtain a fundamental solution of P_ε in the form of a Fourier integral operator $S^\varepsilon(t)$ with phase ϕ and amplitude σ^ε such that:

$$
(8) \qquad \big| \partial_\xi^\alpha \partial_x^\beta \sigma^\varepsilon(t,s;x,\xi) \big| \le \exp\big(-\varepsilon |t-s||\xi|^2/2) . C_{\alpha,\beta} \langle \xi \rangle^{-|\alpha+\beta|}.
$$

Now we can prove the following

PROPOSITION 3.2. *If m, ℓ are sufficiently large then for any $u_o \in \Xi^{m,\ell}$ there exists a time $T_\varepsilon = T(\varepsilon, \|u_o\|_{\Xi} m, \ell) > 0$ such that* (4) *has a unique solution $u^\varepsilon \in \mathscr{C}([0, T_\varepsilon]; \Xi^{m,\ell})$.*

PROOF. Let $\varphi(x)$ be 1, x_j $(j=1,\dots,n)$ or $|x|^2$ and let $\alpha \in \mathbb{N}^n$ be such that

$$|\alpha| \le \begin{cases} m+\ell & \text{if } \phi(x)=1 \\ m+1 & \text{if } \phi(x)=x_j \\ m & \text{if } \phi(x)=|x|^2 \end{cases}$$

We fix u in a class that will be defined in the continuation of this proof and consider

$$\begin{cases} \partial_t v - \varepsilon\Delta_x v + iH(x,D_x)v = F(u,\nabla_x u) \\ v(0,x)=u_o(x) \end{cases} \tag{9}$$

Applying $\varphi(x)\partial_x^\alpha$ to (9) we get:

$$\begin{aligned} \partial_t(\varphi(x)\partial_x^\alpha v) &- \varepsilon\Delta_x(\varphi(x)\partial_x^\alpha v) + iH(x,D_x)(\varphi(x)\partial_x^\alpha v) \\ &= -\varepsilon(\Delta_x\varphi(x)\partial_x^\alpha v + 2\nabla_x\varphi(x).\nabla_x\partial_x^\alpha v) - i\big[\varphi(x)\partial_x^\alpha, H(x,D_x)\big]v \\ &\qquad + \varphi(x)\partial_x^\alpha F(u,\nabla_x u) \end{aligned} \tag{10}$$

and

$$\varphi(x)\partial_x^\alpha v(0,x) = \varphi(x)\partial_x^\alpha u_o(x), \tag{11}$$

where [., .] denotes the usual commutator.

Let us consider the fundamental solution $S^\varepsilon(t)$ of P_ε that we constructed above. Then going back to (10) we can write:

$$\begin{aligned} \varphi\partial_x^\alpha v(t) = S^\varepsilon(t)(\varphi\partial_x^\alpha u_o) &+ \varepsilon\int_0^t S^\varepsilon(t-\tau)\big(\Delta_x\varphi\partial_x^\alpha v + 2\nabla_x\varphi.\nabla_x\partial_x^\alpha v\big)(\tau)\,d\tau \\ &- i\int_0^t S^\varepsilon(t-\tau)\big[\varphi\partial_x^\alpha, H(x,D_x)\big]v(\tau)\,d\tau \\ &+ \int_0^t S^\varepsilon(t-\tau)\big(\varphi\partial_x^\alpha F(u,\nabla_x u)\big)(\tau)\,d\tau. \end{aligned}$$

Let Φ^ε be a solution operator of (9) defined by $\Phi^\varepsilon(u)=v$; then

$$\begin{aligned} \varphi\partial_x^\alpha\Phi^\varepsilon(u)(t) &= S^\varepsilon(t)(\varphi\partial_x^\alpha u_o) \\ &\quad + \varepsilon\int_0^t S^\varepsilon(t-\tau)\big(\Delta_x\varphi\partial_x^\alpha\Phi^\varepsilon(u) + 2\nabla_x\varphi.\nabla_x\partial_x^\alpha\Phi^\varepsilon(u)\big)(\tau)\,d\tau \\ &\quad - i\int_0^t S^\varepsilon(t-\tau)\big[\varphi\partial_x^\alpha, H(x,D_x)\big]\Phi^\varepsilon(u)(\tau)\,d\tau \\ &\quad + \int_0^t S^\varepsilon(t-\tau)\big(\varphi\partial_x^\alpha F(u,\nabla_x u)\big)(\tau)\,d\tau. \end{aligned}$$

Taking (8) into account and adapting Th. 2.3 in Ch. 10 of [11] we obtain, for some constant $c_\sigma > 0$, the following estimate:

$$\left\|\varphi\partial_x^\alpha \Phi^\varepsilon(u)(t)\right\| \leq c_\sigma\left(\left\|\varphi\partial_x^\alpha u_o\right\|+\varepsilon\int_0^t I_1(\tau)\,d\tau+\int_0^t I_H(\tau)\,d\tau\right)+\int_0^t I_F(\tau)\,d\tau,$$

where

$$\begin{aligned}
I_1(\tau) &= \left\|\Delta_x\varphi\partial_x^\alpha\Phi^\varepsilon(u)(\tau)\right\| + 2\left\|\nabla_x\varphi.\nabla_x\partial_x^\alpha\Phi^\varepsilon(u)(\tau)\right\| \\
I_H(\tau) &= \left\|\left[\varphi\ \partial_x^\alpha,\ H(x,D_x)\right]\Phi^\varepsilon(u)(\tau)\right\| \\
I_F(\tau) &= \left\|S^\varepsilon(t-\tau)\big(\varphi(x)\partial_x^\alpha F(u,\nabla_x u)\big)(\tau)\right\|.
\end{aligned}$$

Let $B_r(T) = \{u \in L^\infty([0,T]),\ \Xi^{m,\ell});\ \|u\|_{m,\ell,T} = \sup_{t\in[0,T]}\|u(t)\|_\Xi m,\ell \leq r\}$ where $r > 0$ is such that $\|u_o\|_\Xi m,\ell < r/(2c_\sigma)$, and assume $u \in B_r(T)$. It follows immediately that

$$I_1(\tau) \leq c'\left\|\Phi^\varepsilon(u)\right\|_{m,\ell,T}$$

and since, in view of (H4), we can write

$$\left[\varphi(x)\partial_x^\alpha,\ H(x,D_x)\right] = \varphi(x)R_{|\alpha|}(x,D_x)+\nabla\varphi(x).R'_{|\alpha|+1}(x,D_x)+R''_{\alpha|}(x,D_x),$$

where the subscripts denote the order of the operators, then we have

$$I_H(\tau) \leq C''\left\|\Phi^\varepsilon(u)\right\|_{m,\ell,T}.$$

If we choose $\varphi(x) = 1, x_j, |x|^2$ and $|\alpha| < m+\ell, m+1, m$ respectively, then we have:

$$I_F(\tau) \leq C'_r\left\|\varphi\langle D_x\rangle^{|\alpha|+1}u(\tau)\right\| \leq C''_r\|u(\tau)\|_\Xi m,\ell.$$

In the cases $|\alpha| = m+\ell, m+1, m$ respectively, we can obtain the following estimates. Let $\hat\alpha = (\alpha_1,\ldots,\alpha_k-1,\ldots,\alpha_n)$ for some $k \in \{1,\ldots,n\}$. Then

$$\begin{aligned}
I_F(\tau) &\leq \left\|S^\varepsilon(t-\tau)\big(\partial_{x_k}(\varphi\partial_x^{\hat\alpha}F(u,\nabla_x u))(\tau)\big)\right\| \\
&\qquad + \left\|S^\varepsilon(t-\tau)\big(\partial_{x_k}\varphi\partial_x^{\hat\alpha}F(u,\nabla_x u)(\tau)\big)\right\| \\
&\leq \sup_{\xi\in\mathrm{R}^n}\big(|\xi|e^{-\varepsilon|\xi|^2(t-\tau)/2}\big)\hat C\left\|\varphi\partial_x^{\hat\alpha}F(u,\nabla_x u)(\tau)\right\| \\
&\qquad + c_\sigma\left\|\partial_{x_k}\varphi\partial_x^{\hat\alpha}F(u,\nabla_x u)(\tau)\right\| \\
&\leq \hat C/\big(\sqrt{\varepsilon(t-\tau)}\big)\left\|\varphi\partial_x^{\hat\alpha}F(u,\nabla_x u)(\tau)\right\| + c_\sigma\left\|\partial_{x_k}\varphi\partial_x^{\hat\alpha}F(u,\nabla_x u)(\tau)\right\| \\
&\leq \tilde C_r\big(1+1/\sqrt{\varepsilon(t-\tau)}\big)\|u(\tau)\|_\Xi m,\ell.
\end{aligned}$$

Summing up we get the following estimate:

$$\left\|\Phi^{\varepsilon}(u)\right\|_{m,\ell,T} \leq c_{\sigma}\|u_o\|_{\Xi}m,\ell + C^{*}T\|\Phi^{\varepsilon}(u)\|_{m,\ell,T} + C_r\big(T + 2\sqrt{T/\varepsilon}\big)r.$$

Hence, if we choose a sufficiently small T_{ε}, we get

$$\|\Phi^{\varepsilon}(u)\|_{m,\ell,T} \leq r \quad \forall T \leq T_{\varepsilon}.$$

If $u, u' \in B_r(T)$ a similar computation gives:

$$\left\|\Phi^{\varepsilon}(u) - \Phi^{\varepsilon}(u')\right\|_{m,\ell,T} \leq (C_r/(1 - C^{*}T))\big(T + \sqrt{T/\varepsilon}\big)\|u - u'\|_{m,\ell,T}.$$

Then Φ^{ε} is a contraction mapping on $B_r(T)$, $\forall T \leq T_{\varepsilon}$.

4. Linearization and uniform energy estimates

In this section we write (4) in the form of a system. Then we diagonalize the system. Finally we are able to obtain energy estimates by applying a method which is now almost classic in the theory of linear equations of Schrödinger type.

Let $w = {}^{t}\big(\varphi\partial_x^{\alpha}u, \varphi\partial_x^{\alpha}\bar{u}\big)$. Then (4) can be written in the following form:

$$(\partial_t - \varepsilon\Delta + i\mathscr{H} - i\mathscr{B})w = G(u) \tag{12}$$

where

$$\mathscr{H}(x, D_x) = \begin{pmatrix} H(x, D_x) & 0 \\ 0 & -H(x, D_x) \end{pmatrix}$$

$$\mathscr{B}(x, D_x) = \begin{pmatrix} \displaystyle\sum_{j=1}^{n} \frac{\partial F}{\partial q_j}(u, \nabla u) D_{x_j} & \displaystyle\sum_{j=1}^{n} \frac{\partial F}{\partial \bar{q}_j}(u, \nabla u) D_{x_j} \\ \displaystyle\sum_{j=1}^{n} \overline{\frac{\partial F}{\partial q_j}(u, \nabla u)} D_{x_j} & \displaystyle\sum_{j=1}^{n} \overline{\frac{\partial F}{\partial \bar{q}_j}(u, \nabla u)} D_{x_j} \end{pmatrix}$$

and $G(u) = {}^t(g(u), \overline{g(u)})$ with

(13)

$$
\begin{aligned}
g(u) &= -\varepsilon\big(\Delta_x\varphi(x)\partial_x^\alpha u + 2\nabla_x\varphi(x).\nabla_x\partial_x^\alpha u\big) - i\big[\varphi(x)\partial_x^\alpha, H(x, D_x)\big]u \\
&\quad + \varphi(x)\sum_{\gamma\le\hat\alpha}\binom{\hat\alpha}{\gamma}\left(\partial_x^\gamma\left(\frac{\partial F}{\partial u}(u, \nabla_x u)\right)\partial_x^{\alpha-\gamma} + \partial_x^\gamma\left(\frac{\partial F}{\partial \bar u}(u, \nabla_x u)\right)\partial_x^{\alpha-\gamma}\bar u\right) \\
&\quad + \varphi(x)\sum_{j=1}^n\sum_{0<\gamma\le\hat\alpha}\binom{\hat\alpha}{\gamma}\left(\partial_x^\gamma\left(\frac{\partial F}{\partial q_j}(u, \nabla_x u)\right)\partial_{x_j}\partial^{\alpha-\gamma}u\right. \\
&\qquad \left. + \partial_x^\gamma\left(\frac{\partial F}{\partial \bar q_j}(u, \nabla_x u)\right)\partial_{x_j}\partial^{\alpha-\gamma}\bar u\right) \\
&\quad - \sum_{j=1}^n\partial_{x_j}\varphi(x)\left(\frac{\partial F}{\partial q_j}(u, \nabla_x u)\partial_x^\alpha u + \frac{\partial F}{\partial \bar q_j}(u, \nabla_x u)\partial_x^\alpha\bar u\right)
\end{aligned}
$$

if $|\alpha| > 0$ and $\hat\alpha = (\alpha_1, \dots, \alpha_k - 1, \dots)$ for some $k \in \{1, \dots, n\}$.

Let $u(t) \in \Xi^{m,\ell}$ be such that $\sup_{t\in[0,T]}\|u(t)\|_{\Xi^{m-1,\ell}} \le r$. Since F is quadratic, there exists a constant c_r such that

$$
\begin{aligned}
\left|\frac{\partial F}{\partial q_j}(u, \nabla u)(t, x)\right| &\le c_r(|u(t, x)| + |\nabla_x u(t, x)|) \\
&\le Cc_r\langle x\rangle^{-2}\|\langle x\rangle^2 u(t, x)\|_{H^{[n/2]+2}} \\
&\le Cc_r\langle x\rangle^{-2}\|u(t)\|_{\Xi^{m-1,\ell}}
\end{aligned}
\tag{14}
$$

if $m \ge [n/2] + 3$ and analogously

$$
\left|\frac{\partial F}{\partial \bar q_j}(u, \nabla u)(t, x)\right| \le Cc_r\langle x\rangle^{-2}\|u(t)\|_{\Xi^{m-1,\ell}}.
$$

Moreover taking (14) into account we can prove

$$
\|G(u(t))\| \le C'_r\|u(t)\|_{\Xi^{m,\ell}}. \tag{15}
$$

Now define the operator $L(t) = L(t, x, D_x)$ whose symbol is

$$
\ell(t, x, \xi) = \begin{pmatrix} H(x, \xi) - b_{11}(t, x, \xi) & -b_{12}(t, x, \xi) \\ -b_{21}(t, x, \xi) & -H(x, \xi) - b_{22}(t, x, \xi) \end{pmatrix}
$$

where $(b_{ik})_{i,k=1,2}$ are the entries of $\mathscr{B}$. Note that $b_{ik}(t,x,\xi)=\sum_{j=1}^{n} b_{ikj}(t,x)\xi_j$ with $|b_{ikj}(t,x)|\le rc_r\langle x\rangle^{-2}$ $\forall t\in[0,T]$ in view of (14). Let

$$\tilde{\lambda}(t,x,\xi)=\begin{pmatrix} 0 & \frac{1}{2}b_{12}(t,x,\xi)/H(x,\xi) \\ -\frac{1}{2}b_{21}(t,x,\xi)/H(x,\xi) & 0 \end{pmatrix}$$

In view of (H2) $\tilde{\lambda}(t)\in\left(S^{-1}\right)^{2\times 2}$ $\forall t\in[0,T]$. Let $\lambda(t,x,\xi)=I+\tilde{\lambda}(t,x,\xi)$ and $\lambda'(t,x,\xi)=I-\tilde{\lambda}(t,x,\xi)$ where I is the identity, and let $\tilde{\Lambda}(t)=\tilde{\lambda}(t,x,D_x)$, $\Lambda(t)=\lambda(t,x,D_x)$, $\Lambda'(t)=\lambda'(t,x,D_x)$ denote the corresponding pseudo-differential operators. Then we have the following

LEMMA 4.1. *Under the assumptions above there exists* $c_o(t)\in\left(S^0\right)^{2\times 2}$ $\forall t\in[0,T]$ *such that*

$$\Lambda(t)(L(t)v)=L^d(t)\Lambda(t)v+c_o(t)v$$

where $L^d(t)=\ell^d(t,x,D_x)$ *and*

$$\ell^d(t,x,\xi)=\begin{pmatrix} h(x,\xi)-b_{11}(t,x,\xi) & 0 \\ 0 & -h(x,\xi)-b_{22}(t,x,\xi) \end{pmatrix}.$$

PROOF. In what follows we shall denote the symbol of a pseudo-differential operator, say Q, by $\sigma(Q)$. Since $\Lambda'\Lambda=I-\tilde{\Lambda}^2$ we have

$$\Lambda L=\Lambda L(\Lambda'\Lambda+\tilde{\Lambda}^2)=\Lambda L\Lambda'\Lambda+\Lambda L\tilde{\Lambda}^2. \tag{16}$$

where $\sigma(\Lambda L\tilde{\Lambda}^2)(t)\in(S^0)^{2\times 2}$ $\forall t\in[0,T]$. Moreover

$$\begin{aligned}\sigma(\Lambda L\Lambda')(t)&=\sigma(L-L\tilde{\Lambda}+\tilde{\Lambda}L-\tilde{\Lambda}L\tilde{\Lambda})(t)\\ &=\ell(t,.,.)+\sigma(\tilde{\Lambda}L-L\tilde{\Lambda})(t)-\sigma(\tilde{\Lambda}L\tilde{\Lambda})(t)\end{aligned}\tag{17}$$

where $\sigma(\tilde{\Lambda}L\tilde{\Lambda})(t)\in(S^0)^{2\times 2}$ $\forall t\in[0,T]$. Then we have:

$$\sigma(\tilde{\Lambda}L-L\tilde{\Lambda})(t)=\sigma(\tilde{\Lambda}\mathscr{H}-\mathscr{H}\tilde{\Lambda})(t)+\sigma(\tilde{\Lambda}\mathscr{B}-\mathscr{B}\tilde{\Lambda})(t)$$

where $\sigma(\tilde{\Lambda}\mathscr{B}-\mathscr{B}\tilde{\Lambda})(t)\in(S^0)^{2\times 2}$ $\forall t\in[0,T]$. Moreover, if b denotes the symbol of $\mathscr{B}$ and b^d its diagonal, we have:

$$\sigma(\tilde{\Lambda}\mathscr{H}-\mathscr{H}\tilde{\Lambda})(t)=b(t)-b^d(t)+r_0(t),$$

with $r_0(t)\in(S^0)^{2\times 2}$. Denoting $r_0-\sigma(\tilde{\Lambda}L\tilde{\Lambda})+\sigma(\tilde{\Lambda}\mathscr{B}-\mathscr{B}\tilde{\Lambda})$ by z, we obtain

$$\sigma(\Lambda L\Lambda')(t)=\ell(t)+b(t)-b^d(t)+z(t)=\ell^d(t)+z(t).$$

Denoting $Z(t)\Lambda(t) + \Lambda(t)L(t)\tilde{\Lambda}^2(t)$ by $C_o(t)$ and its symbol by $c_0(t)$, we prove our claim in view of (16), (17).

Now we derive energy estimates for the diagonalized system. Define

$$k(x,\xi) = \begin{pmatrix} e^{-Mp(x,\xi)} & 0 \\ 0 & e^{Mp(x,\xi)} \end{pmatrix} \tag{18}$$

where $p(x,\xi) = \langle\xi\rangle^{-1}\sum_{j=1}^{n}\xi_j \operatorname{arctg} x_j$ and $M \geq rc_r/c$, with c_r as in (14), and c as in (H3). Denote the corresponding operator by $K(x, D_x)$. Applying $K\Lambda(t)$ to (12) we get

$$\begin{aligned}\frac{d}{dt}\|K\Lambda(t)w(t)\|^2 &= 2\operatorname{Re}\langle K\partial_t(\Lambda(t)w(t)), K\Lambda(t)w(t)\rangle \\ &= 2\operatorname{Re}\langle K(\varepsilon\Delta\Lambda(t) - i\Lambda(t)L(t) + \varepsilon[\Lambda(t),\Delta] \\ &\qquad + [\partial_t,\Lambda(t)])w(t) + K\Lambda(t)G(u(t)), K\Lambda(t)w(t)\rangle\end{aligned}$$

which, in view of Lemma 4.1, is equal to

$$2\operatorname{Re}\langle K((\varepsilon\Delta - iL^d(t))\Lambda(t) + r_\varepsilon(t))w(t) + K\Lambda(t)G(u(t)), K\Lambda(t)w(t)\rangle$$

where $r_\varepsilon(t) = \varepsilon[\Lambda(t),\Delta] + [\partial_t,\Lambda(t)] - ic_o(t)$ with $c_o(t) \in (S^0)^{2\times 2}$ $\forall t \in [0,T]$. Since the first term in the asymptotic expansion of $\sigma([\Lambda(t),\Delta])(x,\xi)$ is

$$\begin{pmatrix} 0 & -\displaystyle\sum_{j=1}^{0}\xi_j D_{x_j}(b_{12}(t,x,\xi)/H(x,\xi)) \\ \displaystyle\sum_{j=1}^{0}\xi_j D_{x_j}(b_{21}(t,x,\xi)/H(x,\xi)) & 0 \end{pmatrix}$$

which belongs to $(S^0)^{2\times 2}$, then $c_o(t) \in (S^0)^{2\times 2}$ $\forall t \in [0,T]$.

Let us now examine the symbol of the diagonal matrix $K(\varepsilon\Delta - iL^d(t)) - (\varepsilon\Delta - iL^d(t))K$. A simple calculation shows that it is of the form

$$M\begin{pmatrix} \begin{array}{r}\{p,H\}(x,\xi) \\ +\,2\varepsilon i\xi.\nabla_x p(x,\xi) \\ +\,s_o(t,x,\xi)\end{array} & 0 \\ 0 & \begin{array}{r}\{p,H\}(x,\xi) \\ -\,2\varepsilon i\xi.\nabla_x p(x,\xi) \\ +\,\tilde{s}_o(t,x,\xi)\end{array} \end{pmatrix} k(x,\xi)$$

with $s_o(t), \tilde{s}_o(t) \in S^0$. Thus

$$\begin{aligned}\frac{d}{dt}\|K\Lambda(t)w(t)\|^2 &\\ &\leq -2\operatorname{Re}\langle (iL^d(t) - \varepsilon\Delta + M\{H, p\})K\Lambda(t)w(t), K\Lambda(t)w(t)\rangle \\ &\quad + (C'_\varepsilon\|w(t)\| + \|K\Lambda(t)G(u)\|)\|K\Lambda(t)w(t)\|\end{aligned}$$

In view of the assumption (H3) and of (14), we have

$$\operatorname{Im} b_{kk}(t, x, \xi) + M\{H, p\}(x, \xi) \geq (-c_r r + Mc)\langle x\rangle^{-2}|\xi| \geq 0,$$

for $k = 1, 2$. Then by applying the sharp Gårding inequality we obtain

$$\operatorname{Re}\langle (iL^d(t) + M\{H, p\})K\Lambda(t)w(t), K\Lambda(t)w(t)\rangle \geq -\tilde{C}_r\|K\Lambda(t)w(t)\|^2,$$

for some $\tilde{C}_r > 0$. Hence

$$\begin{aligned}&-2\operatorname{Re}\langle (iL^d(t) - \varepsilon\Delta + M\{H, p\})K\Lambda(t)w(t), K\Lambda(t)w(t)\rangle \\ &\qquad \leq 2\tilde{C}_r\|K\Lambda(t)w(t)\|^2 - 2\varepsilon\|\nabla K\Lambda(t)w(t)\|^2 \leq 2\tilde{C}_r\|K\Lambda(t)w(t)\|^2.\end{aligned}$$

Then we get

$$\begin{aligned}\frac{d}{dt}\|K\Lambda(t)w(t)\|^2 &\leq 2\tilde{C}_r\|K\Lambda(t)w(t)\|^2 \\ &\quad + (C'_\varepsilon\|w(t)\| + \|K\Lambda(t)G(u)\|)\|K\Lambda(t)w(t)\|\end{aligned} \tag{19}$$

5. End of the proof of the theorem

Let

$$\begin{aligned}\tilde{E}(u(t)) = \sum_{|\alpha|=m+\ell}\left\|K\Lambda(t)\partial_x^\alpha u(t)\right\| + \sum_{j=1}^{n}\sum_{|\alpha|=m+1}\left\|K\Lambda(t)(x_j\partial_x^\alpha u(t))\right\| \\ + \sum_{|\alpha|=m}\left\|K\Lambda(t)(|x|^2\partial_x^\alpha u(t))\right\|\end{aligned}$$

Let $\varepsilon \in]0, 1]$ and let $u_\varepsilon \in \mathscr{C}([0, T]; \Xi^{m,\ell})$ be a solution of (4) such that $\sup_{t\in[0,T]}\|u_\varepsilon(t)\|_{\Xi^{m-1,\ell}} \leq r$. Let

$$E(u_\varepsilon(t)) = \tilde{E}(u_\varepsilon(t)) + \|u_\varepsilon(t)\|_{\Xi^{m-1,\ell}}.$$

As in the proof of (4.3) in [3], one can see that $E(u_\varepsilon(t))$ is equivalent to $\|u_\varepsilon(t)\|_{\Xi^{m,\ell}}$; specifically, if $\|u_\varepsilon(t)\|_{\Xi^{m-1,\ell}} \leq r$, then there exists $M_r > 1$ such that

$$M_r^{-1}\|u_\varepsilon(t)\|_{\Xi^{m,\ell}} \leq E(u_\varepsilon(t)) \leq M_r\|u_\varepsilon(t)\|_{\Xi^{m,\ell}}.$$

Now from (19) and (15) we have

$$\frac{d}{dt}\|K\Lambda(t)w(t)\|^2 \leq C_r^{**}E(u_\varepsilon(t))\|K\Lambda(t)w(t)\|,$$

and summing up on $\varphi(x)$ and α we obtain

$$\frac{d}{dt}\tilde{E}(u_\varepsilon(t)) \leq C_r^* E(u_\varepsilon(t)).$$

Thus we finally obtain

$$E(u_\varepsilon(t)) \leq E(u_o)e^{C_r^* t}$$

with C_r^* which is independent of $\varepsilon \in]0, 1]$. Then there exists a time $T > 0$ such that $\{u_\varepsilon\}_{\varepsilon\in]0,1]}$ is bounded in $\mathscr{C}([0, T]; \Xi^{m,\ell})$, and thus by a standard argument we get a solution $u(t) \in \Xi^{m,\ell}$ $\forall t \in [0, T]$ of (3).

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