Abstract

We study the initial value problem for some semilinear pseudo-differential equations of the form

$$\partial_t u + iH(x, D_x) u = F(u, \nabla u).$$

The assumptions we make on $H$ are trivially satisfied by $\gamma E\delta$, thus our equations generalize Schrödinger type equations. A local existence theorem is proved in some weighted Sobolev spaces.

0. Introduction

In this paper we consider the initial value problem for some nonlinear evolution equations of the form

$$\partial_t u + iH(x, D_x) u = F(u, \nabla u)$$

where $H$ is a uniformly elliptic pseudo-differential operator of order 2 with real symbol.

We assume that the nonlinear term $F : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}$ satisfies: $F(u, q) \in \mathcal{C}^\infty(\mathbb{R}^2 \times \mathbb{R}^{2n})$ and $|F(u, q)| \leq C(|u|^2 + |q|^2)$ near the origin.

The simplest model we have in mind is the one with $H(x, \xi) = |\xi|^2$, that is (1) generalizes semilinear Schrödinger equations.

Most papers on semilinear Schrödinger equations are concerned with the case $F(u) \text{ or } F(u, \nabla u)$ but $\text{Im } \frac{\partial F}{\partial q_j} = 0, j = 1, \ldots, n$. Some troubles arise when one works with classical energy methods in the general case: even in the linear case some difficulties arise owing to the imaginary part of the coefficients of $\partial_j u$. Correspondingly all the papers about the wellposedness of the Cauchy problem in $L^2$ or Sobolev spaces for linear Schrödinger equations give necessary or sufficient conditions on the imaginary part of the first order terms of the operator. (See [7], [8], [9], [12]).

In [2] Chihara succeeded in proving local existence in some weighted Sobolev spaces for the semilinear Schrödinger equations in the case $n = 1$. In [3] he generalized the result to higher space dimension. Our paper studies more

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general operators of Schrödinger type and thus it generalizes [3]. We need the following additional assumption on $H$:

(2) \( \exists c > 0 \) such that \( \{H, p\}(x, \xi) \geq c \langle x \rangle^{-2} |\xi| \) for large \( |\xi| \),

where \( p(x, \xi) = \langle \xi \rangle^{-1} \sum_{j=1}^{n} \xi_j \arctg x_j \) and \( \{\ldots\} \) denotes the Poisson’s bracket, i.e. \( \{H, p\} = \sum_{j=1}^{n} (\partial_{\xi_j} H \partial_{x_i} p - \partial_{\xi_j} p \partial_{x_i} H) \).

A condition similar to (2) can be found in the literature on Schrödinger equations (see (A2) in [5] for example). Such conditions are used to eliminate – in some sense – the bad first order term.

1. Notation

For \( x \in \mathbb{R}^n \) let \( \langle x \rangle = (1 + |x|^2)^{1/2} \) and \( \langle D_x \rangle = (1 - \Delta_x)^{1/2} \).

Let \( \| \| \) denote the \( L^2 \)-norm.

For \( m, p \in \mathbb{R} \) let \( \|f\|_{m,p} = \|\langle x \rangle^p \langle D_x \rangle^m f\| \) and let \( H^{m,p} = \{f \in \mathcal{S}'(\mathbb{R}^n); \|f\|_{m,p} < \infty\} \).

Note that \( H^{m,0} \) is the usual Sobolev space \( H^m \).

In the sequel if \( \ell \) is a sufficiently large integer we shall denote \( H^{m+\ell,0} \cap H^{m+1,1} \cap H^{m,2} \) by \( \mathcal{X}^{m,\ell} \).

We shall use the following notation for pseudo-differential operators. The space of the symbols \( \sigma(x, \xi) \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n) \) such that

\[
\sup_{x, \xi \in \mathbb{R}^n} \left| \partial_\xi^\alpha D_x^\beta \sigma(x, \xi) \right| \langle \xi \rangle^{m-|\alpha|} < \infty
\]

will be denoted by \( S^m \). The calculus for the corresponding pseudo-differential operators can be found in Kumano-go’s book [11].

2. The main result

Consider the following Cauchy problem for an equation of Schrödinger type:

(3) \( \partial_t u + iH(x, D_x)u = F(u, \nabla_x u) \) in \( [0, \infty) \times \mathbb{R}^n, u(t = 0) = u_0 \).

We make the following assumptions:

(H1) \( H \) has a real symbol;

(H2) there exists \( c_0 > 0 \) such that \( |H(x, \xi)| \geq c_0 |\xi|^2 \) \( \forall x, \xi \in \mathbb{R}^n \);

(H3) \( \exists c > 0 \) such that \( \{H, p\}(x, \xi) \geq c \langle x \rangle^{-2} |\xi| \) for large \( |\xi| \), where \( \{\ldots\} \) denotes the Poisson’s bracket and \( p(x, \xi) = \langle \xi \rangle^{-1} \sum_{j=1}^{n} \xi_j \arctg x_j \).

(H4) \( \sup_{x, \xi \in \mathbb{R}^n} \left| \partial_\xi^\alpha D_x^\beta H(x, \xi) \right| \langle \xi \rangle^{m+|\alpha|} < \infty, \) \( \forall \alpha, \beta \in \mathbb{N}^n \).
Moreover we make the following assumptions on the nonlinear term:

(F1) \( F : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C} \) belongs to \( \mathcal{C}^\infty (\mathbb{R}^2 \times \mathbb{R}^{2n}) \);

(F2) there exists \( C > 0 \) such that \(|F(u, q)| \leq C(|u|^2 + |q|^2)\) near \((u, q) = (0, 0)\).

In the following section we prove the following

**Theorem 2.1.** For any initial datum \( u_o \in \mathbb{Z}^{m,\ell} \) (where \( m \) and \( \ell \) are sufficiently large integers) there exists a time \( T > 0 \) such that the Cauchy problem (3) has a solution \( u \in \mathcal{C}([0, T]; \mathbb{Z}^{m,\ell}) \).

To prove this theorem at first we consider a parabolic regularization of our problem which depends on a viscosity parameter \( \varepsilon > 0 \). The regularized problem is solved by linearization in §4. Finally a solution of (3) is obtained as a zero limit of the solution of the regularized problem.

### 3. Parabolic regularization

For any \( \varepsilon \in [0, 1] \) let us consider

\[
\begin{align*}
\partial_t u^\varepsilon - \varepsilon \Delta_s u^\varepsilon + iH(x, D_x)u^\varepsilon &= F(u^\varepsilon, \nabla_s u^\varepsilon) \\
u^\varepsilon(0, x) &= u_o(x)
\end{align*}
\]

in \( [0, +\infty) \times \mathbb{R}^n \), where \( H, F \) and \( u_o \) are as in §2.

Let \( P_{\varepsilon} \) denote the linear operator \( \partial_t - \varepsilon \Delta_s + iH(x, D_x) \). Let us first construct a fundamental solution \( S_{\varepsilon}(t) \) for \( P_{\varepsilon} \). Consider the following eikonal equation:

\[
\begin{align*}
\partial_t \phi(t, s; x, \xi) + H(x, \nabla_s \phi(t, s; x, \xi)) &= x.\xi \\
\phi(s, s; x, \xi) &= x.\xi
\end{align*}
\]

Then we have the following

**Lemma 3.1.** If \( H \) satisfies (H1) and (H4), then there exists \( T > 0 \) such that for every \( t, s \in [-T, T] \) the following estimate is true:

\[
\sup_{x \in \mathbb{R}^n} \left| \partial^\alpha_\xi \partial^\beta_x (\phi(t, s; x, \xi) - x.\xi) \right| \leq C_{\alpha, \beta}'|t - s|(|\xi|^{2-|\alpha+\beta|})
\]

\( \forall \alpha, \beta \in \mathbb{N}^n, \forall \xi \in \mathbb{R}^n \) with large \( |\xi| \), and for some \( C_{\alpha, \beta}' \).

**Proof.** The proof follows the lines of Theorem 4.1 in [11]. At first we prove inductively that the solutions \( q(t, s; y, \xi) \) and \( p(t, s; y, \xi) \) of the Hamilton’s equations

\[
\begin{align*}
\frac{dq}{dt} &= \nabla_\xi H(q, p) \\
\frac{dp}{dt} &= -\nabla_s H(q, p) \\
(q, p)|_{t=s} &= (y, \xi)
\end{align*}
\]
satisfy the following estimates, for every $\alpha, \beta \in \mathbb{N}^n$:

$$\sup_{y \in \mathbb{R}^n} |\partial_\xi^\alpha \partial_y^\beta (q(t, s; y, \xi) - y)| \leq C_{\alpha, \beta}'' |t - s| \langle \xi \rangle^{1-|\alpha + \beta|}$$

$$\sup_{y \in \mathbb{R}^n} |\partial_\xi^\alpha \partial_y^\beta (p(t, s; x, \xi) - \xi)| \leq C_{\alpha, \beta}'' |t - s| \langle \xi \rangle^{1-|\alpha + \beta|}$$

Denoting the inverse mapping of $y \mapsto x = q(t, s; y, \xi)$ by $Y(t, s; x, \xi)$, we can prove that, if $T > 0$ is sufficiently small, then for every $\alpha, \beta \in \mathbb{N}^n$, $t, s \in [-T, T]$, $\xi \in \mathbb{R}^n$ with large $|\xi|$, and for some $A_{\alpha, \beta}$, the following inequality holds:

$$\sup_{y \in \mathbb{R}^n} |\partial_\xi^\alpha \partial_y^\beta (Y(t, s; x, \xi) - x)| \leq A_{\alpha, \beta} |t - s| \langle \xi \rangle^{1-|\alpha + \beta|}$$

Finally we construct the solution of (5) setting

$$\phi(t, s; x, \xi) = \psi(t, s; Y(t, s; x, \xi), \xi),$$

where

$$\psi(t, s; y, \xi) = y \cdot \xi + \int_s^t (p, \nabla_{t'} H - H)(t', q(t', s; y, \xi), p(t', s; y, \xi)) dt'.$$

Consequently, we get (6).

Now we are going to construct a Fourier integral operator whose phase is $\phi(t, s; x, \xi)$ and whose amplitude $\sigma(t, s; x, \xi) \sim \sum_{j=0}^\infty \sigma_{2j}(t, s; x, \xi)$ is found by solving the following transport equations:

$$(T_0) \begin{cases} \partial_t \sigma_0(t) + \nabla_{\xi} H(x, \nabla_x \phi(t, s; x, \xi)).\nabla_{\xi} \sigma_0(t) + c_s(t, x, \xi) \sigma_0(t) = 0 \\ \sigma_0(s) = 1 \end{cases}$$

where

$$c_s(t, x, \xi) = \frac{1}{2} \sum_{ki} \partial_{\xi_k \xi_i}^2 H(x, \nabla_x \phi(t, s; x, \xi)) \partial_{x_k x_i}^2 \phi(t, s; x, \xi) + \varepsilon |\nabla_x \phi(t, s; x, \xi)|^2,$$

and for $j \geq 1$

$$(T_{2j}) \begin{cases} \partial_t \sigma_{2j}(t) + \nabla_{\xi} H(x, \nabla_x \phi(t, s; x, \xi)).\nabla_{\xi} \sigma_{2j}(t) + c_s(t, x, \xi) \sigma_{2j}(t) = -ib_j(t, x, \xi) \\ \sigma_{2j}(s) = 0 \end{cases}$$
\[ b_j(t, x, \xi) = \sum_{k=1}^{j} \sum_{|\gamma| = k+1} \frac{1}{\gamma!} D_{\gamma} \{ a_{\gamma}^{\xi} H(x, \hat{\nabla}_x \phi(t, s; x, z, \xi)) \sigma_{2j-2k}(t, s; z, \xi) \}_{z=x} \]

\[ - 2e \nabla_x \phi(t, s; x, \xi) \cdot \nabla_x \sigma_{2j-2}(t, s; x, \xi) \]

\[ + i e \Delta_x \sigma_{2j-2}(t, s; x, \xi) \]

\[ - e \Delta_x \phi(t, s; x, \xi) \sigma_{2j-2}(t, s; x, \xi) \]

being \[ \hat{\nabla}_x \phi(t, s; x, z, \xi) = \int_0^1 \nabla_x \phi(t, s; \theta x + (1 - \theta) x, \xi) \ d\theta. \]

We can prove inductively that there exists an increasing sequence \( C^*_n \) such that:

\[ |\partial^\alpha \xi \partial^\beta_x \sigma^2_j(t, s; x, \xi)| \leq \exp\left(-3e |t - s| ||\xi||^2/4\right) C_n^{(\alpha + \beta) + 6} \cdot \sum_{k=0}^{\lfloor (\alpha + \beta) + 2 \rfloor} \frac{(2e |t - s| ||\xi||^2)^k}{k!} \]

for every \( \alpha, \beta \in \mathbb{N}^n \) and for every \( j \in \mathbb{N} \). We can write:

\[ \sum_{k=0}^{\lfloor (\alpha + \beta) + 2 \rfloor} \frac{(2e |t - s| ||\xi||^2)^k}{k!} \leq 8^{(\alpha + \beta) + 2} \exp(e |t - s| ||\xi||^2/4), \]

so that (7) becomes:

\[ |\partial^\alpha \xi \partial^\beta_x \sigma^2_j(t, s; x, \xi)| \leq \exp(-e |t - s| ||\xi||^2/2) C_n^{(\alpha + \beta) + 2} \cdot \sum_{k=0}^{\lfloor (\alpha + \beta) + 2 \rfloor} \frac{(2e |t - s| ||\xi||^2)^k}{k!} C^{\alpha + \beta}_n \]

Finally, as in Lemma 3.2 in [11], we can construct a symbol which is equivalent to the formal series of the symbols \( \sigma_j \). Thus we obtain a fundamental solution of \( P_\epsilon \) in the form of a Fourier integral operator \( S_\epsilon(t) \) with phase \( \phi \) and amplitude \( \sigma_\epsilon \) such that:

\[ |\partial^\alpha \xi \partial^\beta_x \sigma_\epsilon(t, s; x, \xi)| \leq \exp(-e |t - s| ||\xi||^2/2) C_\alpha,\beta(\xi)^{-|\alpha + \beta|}. \]

Now we can prove the following

**Proposition 3.2.** If \( m, \ell \) are sufficiently large then for any \( u_0 \in \mathbb{E}^m, \ell \) there exists a time \( T_\epsilon = T(\epsilon, \|u_0\|_{\mathbb{E}^m, \ell}) > 0 \) such that (4) has a unique solution \( u^\epsilon \in \mathbb{C}([0, T_\epsilon]; \mathbb{E}^m, \ell) \).
Proof. Let $\varphi(x)$ be $1$, $x_j$ ($j = 1, \ldots, n$) or $|x|^2$ and let $\alpha \in \mathbb{N}^n$ be such that
\[
|\alpha| \leq \begin{cases} 
m + \ell & \text{if } \varphi(x) = 1 \\
m + 1 & \text{if } \varphi(x) = x_j \\
m & \text{if } \varphi(x) = |x|^2
\end{cases}
\]
We fix $u$ in a class that will be defined in the continuation of this proof and consider
\[
\begin{aligned}
\partial_t v - \varepsilon \Delta_x v + iH(x, D_x)v &= F(u, \nabla_x u) \\
v(0, x) &= u_o(x)
\end{aligned}
\]
Applying $\varphi(x)\partial^\alpha_x$ to (9) we get:
\[
\begin{aligned}
\partial_t (\varphi(x)\partial^\alpha_x v) - \varepsilon \Delta_x (\varphi(x)\partial^\alpha_x v) + iH(x, D_x)(\varphi(x)\partial^\alpha_x v) \\
&= -\varepsilon(\Delta_x \varphi(x)\partial^\alpha_x v + 2\nabla_x \varphi(x).\nabla_x \partial^\alpha_x v) - i[\varphi(x)\partial^\alpha_x, H(x, D_x)]v \\
&\quad + \varphi(x)\partial^\alpha_x F(u, \nabla_x u)
\end{aligned}
\]
and
\[
\begin{aligned}
\varphi(x)\partial^\alpha_x v(0, x) &= \varphi(x)\partial^\alpha_x u_o(x),
\end{aligned}
\]
where $[\ldots]$ denotes the usual commutator.
Let us consider the fundamental solution $S^\varepsilon(t)$ of $P_\varepsilon$ that we constructed above. Then going back to (10) we can write:
\[
\begin{aligned}
\varphi \partial^\alpha_x v(t) &= S^\varepsilon(t)(\varphi \partial^\alpha_x u_o) + \varepsilon \int_0^t S^\varepsilon(t - \tau)(\Delta_x \varphi \partial^\alpha_x v + 2\nabla_x \varphi.\nabla_x \partial^\alpha_x v)(\tau) d\tau \\
&\quad - i \int_0^t S^\varepsilon(t - \tau)[\varphi \partial^\alpha_x, H(x, D_x)]v(\tau) d\tau \\
&\quad + \int_0^t S^\varepsilon(t - \tau)(\varphi \partial^\alpha_x F(u, \nabla_x u))(\tau) d\tau.
\end{aligned}
\]
Let $\Phi^\varepsilon$ be a solution operator of (9) defined by $\Phi^\varepsilon(u) = v$; then
\[
\begin{aligned}
\varphi \partial^\alpha_x \Phi^\varepsilon(u)(t) &= S^\varepsilon(t)(\varphi \partial^\alpha_x u_o) \\
&\quad + \varepsilon \int_0^t S^\varepsilon(t - \tau)(\Delta_x \varphi \partial^\alpha_x \Phi^\varepsilon(u) + 2\nabla_x \varphi.\nabla_x \partial^\alpha_x \Phi^\varepsilon(u))(\tau) d\tau \\
&\quad - i \int_0^t S^\varepsilon(t - \tau)[\varphi \partial^\alpha_x, H(x, D_x)]\Phi^\varepsilon(u)(\tau) d\tau \\
&\quad + \int_0^t S^\varepsilon(t - \tau)(\varphi \partial^\alpha_x F(u, \nabla_x u))(\tau) d\tau.
\end{aligned}
\]
Taking (8) into account and adapting Th. 2.3 in Ch. 10 of [11] we obtain, for some constant $c_\sigma > 0$, the following estimate:

$$
\| \partial_x^n \Phi'(u)(t) \| \leq c_\sigma \left( \| \partial_x^n u_0 \| + \epsilon \int_0^t I_1(\tau) \, d\tau + \int_0^t I_H(\tau) \, d\tau \right) + \int_0^t I_F(\tau) \, d\tau,
$$

where

$$
I_1(\tau) = \| \Delta_x \partial_x^n \Phi'(u)(\tau) \| + 2 \| \nabla_x \varphi \cdot \nabla_x \partial_x^n \Phi'(u)(\tau) \|
$$

$$
I_H(\tau) = \left\| \left[ \varphi \partial_x^n, H(x, D_x) \right] \Phi'(u)(\tau) \right\|
$$

$$
I_F(\tau) = \left\| S^\epsilon(t - \tau) \left( \varphi(x) \partial_x^n F(u, \nabla_x u) \right)(\tau) \right\|.
$$

Let $B_r(T) = \{ u \in L^\infty([0, T]), \mathbb{Z}^{m, \ell}; \| u \|_{m, \ell, T} = \sup_{t \in [0, T]} \| u(t) \|_{\mathbb{Z}^m, \ell} \leq r \}$ where $r > 0$ is such that $\| u_0 \|_{\mathbb{Z}^m, \ell} < r / (2c_\sigma)$, and assume $u \in B_r(T)$. It follows immediately that

$$
I_1(\tau) \leq c' \left\| \Phi'(u) \right\|_{m, \ell, T}
$$

and since, in view of (H4), we can write

$$
\left[ \varphi(x) \partial_x^n, H(x, D_x) \right] = \varphi(x) R_{|\alpha|} (x, D_x) + \nabla \varphi(x). R'_{|\alpha|+1} (x, D_x) + R''_{|\alpha|} (x, D_x),
$$

where the subscripts denote the order of the operators, then we have

$$
I_H(\tau) \leq C'' \left\| \Phi'(u) \right\|_{m, \ell, T}.
$$

If we choose $\varphi(x) = 1, x_j, |x|^2$ and $|\alpha| < m + \ell, m + 1, m$ respectively, then we have:

$$
I_F(\tau) \leq C' \left\| \varphi(D_x)^{|\alpha|+1} u(\tau) \right\| \leq C'' \| u(\tau) \|_{\mathbb{Z}^m, \ell}.
$$

In the cases $|\alpha| = m + \ell, m + 1, m$ respectively, we can obtain the following estimates. Let $\alpha = (\alpha_1, \ldots, \alpha_k - 1, \ldots, \alpha_n)$ for some $k \in \{1, \ldots, n\}$. Then

$$
I_F(\tau) \leq \left\| S^\epsilon(t - \tau) \left( \partial_x^n \varphi \partial_x^n \hat{\varphi}_x^\epsilon F(u, \nabla_x u) \right)(\tau) \right\|
$$

$$
+ \left\| S^\epsilon(t - \tau) \left( \partial_x^n \varphi \partial_x^n \hat{\varphi}_x^\epsilon F(u, \nabla_x u) \right)(\tau) \right\|
$$

$$
\leq \sup_{\xi \in \mathbb{R}^n} \left( \| e^{-|\xi|^2(t - \tau)} \right) \hat{C} \left\| \varphi \partial_x^n \hat{\varphi}_x^\epsilon F(u, \nabla_x u) \right\|
$$

$$
+ c_\sigma \left\| \partial_x^n \varphi \partial_x^n \hat{\varphi}_x^\epsilon F(u, \nabla_x u) \right\|
$$

$$
\leq \hat{C} \left( \sqrt{\epsilon(t - \tau)} \right) \| u(\tau) \|_{\mathbb{Z}^m, \ell} + c_\sigma \left\| \partial_x^n \varphi \partial_x^n F(u, \nabla_x u) \right\|
$$

$$
\leq \hat{C}_r \left( 1 + 1 / \sqrt{\epsilon(t - \tau)} \right) \| u(\tau) \|_{\mathbb{Z}^m, \ell}.
$$
Summing up we get the following estimate:

\[ \| \Phi^\epsilon(u) \|_{m, \ell, T} \leq c_\sigma \| u_0 \|_{\Xi m, \ell} + C^* T \| \Phi^\epsilon(u) \|_{m, \ell, T} + C_r (T + 2\sqrt{T/\epsilon}) r. \]

Hence, if we choose a sufficiently small \( T_\varepsilon \), we get

\[ \| \Phi^\epsilon(u) \|_{m, \ell, T} \leq r \quad \forall T \leq T_\varepsilon. \]

If \( u, u' \in B_r(T) \) a similar computation gives:

\[ \| \Phi^\epsilon(u) - \Phi^\epsilon(u') \|_{m, \ell, T} \leq (C_r / (1 - C^* T))(T + \sqrt{T/\epsilon}) \| u - u' \|_{m, \ell, T}. \]

Then \( \Phi^\epsilon \) is a contraction mapping on \( B_r(T) \), \( \forall T \leq T_\varepsilon. \)

4. Linearization and uniform energy estimates

In this section we write (4) in the form of a system. Then we diagonalize the system. Finally we are able to obtain energy estimates by applying a method which is now almost classic in the theory of linear equations of Schrödinger type.

Let \( w = \imath (\psi \partial_x^u u, \psi \partial_{\bar{u}} u) \). Then (4) can be written in the following form:

\[
(\partial_t - \epsilon \Delta + i \mathcal{H} - i \mathcal{B})w = G(u)
\]

where

\[
\mathcal{H}(x, D_x) = \begin{pmatrix} H(x, D_x) & 0 \\ 0 & -H(x, D_x) \end{pmatrix}
\]

\[
\mathcal{B}(x, D_x) = \begin{pmatrix}
\sum_{j=1}^n \frac{\partial F}{\partial q_j}(u, \nabla u)D_{x_j} & \sum_{j=1}^n \frac{\partial F}{\partial \bar{q}_j}(u, \nabla u)D_{x_j} \\
\sum_{j=1}^n \frac{\partial F}{\partial q_j}(u, \nabla u)D_{x_j} & \sum_{j=1}^n \frac{\partial F}{\partial \bar{q}_j}(u, \nabla u)D_{x_j}
\end{pmatrix}
\]
and $G(u) = \ell'(g(u), \overline{g(u)})$ with

\begin{equation}
g(u)

= -\varepsilon \left( \Delta_x \varphi(x) \partial_x^\alpha u + 2\nabla_x \varphi(x) \cdot \nabla_x \partial_x^\alpha u \right) - i \left[ \varphi(x) \partial_x^\alpha, H(x, D_x) \right] u

+ \varphi(x) \sum_{\gamma \leq \hat{\alpha}} \left( \frac{\hat{\alpha}}{\gamma} \right) \left( \partial_x^\gamma \left( \frac{\partial F}{\partial u}(u, \nabla_x u) \right) \partial_x^{\alpha-\gamma} u + \partial_x^\gamma \left( \frac{\partial F}{\partial \bar{u}}(u, \nabla_x u) \right) \partial_x^{\alpha-\gamma} \bar{u} \right)

+ \varphi(x) \sum_{j=1}^n \sum_{0 < \gamma \leq \hat{\alpha}} \left( \frac{\hat{\alpha}}{\gamma} \right) \left( \partial_x^\gamma \left( \frac{\partial F}{\partial q_j}(u, \nabla_x u) \right) \partial_j \partial_x^{\alpha-\gamma} u \right)

+ \left( \partial_x^\gamma \left( \frac{\partial F}{\partial \bar{q}_j}(u, \nabla_x u) \right) \partial_j \partial_x^{\alpha-\gamma} \bar{u} \right)

- \sum_{j=1}^n \partial_j \varphi(x) \left( \frac{\partial F}{\partial q_j}(u, \nabla_x u) \partial_x^\alpha u + \frac{\partial F}{\partial \bar{q}_j}(u, \nabla_x u) \partial_x^\alpha \bar{u} \right)

\end{equation}

if $|\alpha| > 0$ and $\hat{\alpha} = (\alpha_1, \ldots, \alpha_k, -1, \ldots)$ for some $k \in \{1, \ldots, n\}$.

Let $u(t) \in \mathcal{X}_{m,\ell}$ be such that $\sup_{t \in [0, T]} \|u(t)\|_{\mathcal{X}_{m,-1,\ell}} \leq r$. Since $F$ is quadratic, there exists a constant $c_r$ such that

\begin{equation}
\left| \frac{\partial F}{\partial q_j}(u, \nabla u)(t, x) \right| \leq c_r (|u(t, x)| + |\nabla_x u(t, x)|)

\leq C c_r \langle x \rangle^{-2} \|u(t, x)\|_{H^{n/2+1}}

\leq C c_r \langle x \rangle^{-2} \|u(t)\|_{\mathcal{X}_{m,-1,\ell}}
\end{equation}

if $m \geq [n/2] + 3$ and analogously

\begin{equation}
\left| \frac{\partial F}{\partial \bar{q}_j}(u, \nabla u)(t, x) \right| \leq C c_r \langle x \rangle^{-2} \|u(t)\|_{\mathcal{X}_{m,-1,\ell}}.
\end{equation}

Moreover taking (14) into account we can prove

\begin{equation}
\|G(u(t))\| \leq C'_r \|u(t)\|_{\mathcal{X}_{m,\ell}}.
\end{equation}

Now define the operator $L(t) = L(t, x, D_x)$ whose symbol is

$$
\ell(t, x, \xi) = \begin{pmatrix}
H(x, \xi) & -b_{11}(t, x, \xi) & -b_{12}(t, x, \xi) \\
-b_{21}(t, x, \xi) & -H(x, \xi) & -b_{22}(t, x, \xi)
\end{pmatrix}
$$
where \((b_{ik})_{i,k=1,2}\) are the entries of \(B\). Note that \(b_{ik}(t, x, \xi) = \sum_{j=1}^{n} b_{ikj}(t, x)\xi_j\) with \(|b_{ikj}(t, x)| \leq r c_f \langle x \rangle^{-2} \forall t \in [0, T]\) in view of (14). Let

\[
\tilde{\lambda}(t, x, \xi) = \begin{pmatrix}
0 & \frac{1}{2} b_{12}(t, x, \xi)/H(x, \xi) \\
-\frac{1}{2} b_{21}(t, x, \xi)/H(x, \xi) & 0
\end{pmatrix}
\]

In view of (H2) \(\tilde{\lambda}(t) \in (S^{-1})^{2 \times 2} \forall t \in [0, T]\). Let \(\lambda(t, x, \xi) = I + \tilde{\lambda}(t, x, \xi)\) and \(\lambda'(t, x, \xi) = I - \tilde{\lambda}(t, x, \xi)\) where \(I\) is the identity, and let \(\tilde{\Lambda}(t) = \tilde{\lambda}(t, x, D_x)\), \(\Lambda(t) = \lambda(t, x, D_x)\), \(\Lambda'(t) = \lambda'(t, x, D_x)\) denote the corresponding pseudo-differential operators. Then we have the following

**Lemma 4.1.** Under the assumptions above there exists \(c_\alpha(t) \in (S^0)^{2 \times 2}\) \(\forall t \in [0, T]\) such that

\[
\Lambda(t)(L(t)v) = L^d(t)\Lambda(t)v + c_\alpha(t)v
\]

where \(L^d(t) = \ell^d(t, x, D_x)\) and

\[
\ell^d(t, x, \xi) = \begin{pmatrix}
h(x, \xi) - b_{11}(t, x, \xi) & 0 \\
0 & -h(x, \xi) - b_{22}(t, x, \xi)
\end{pmatrix}
\]

**Proof.** In what follows we shall denote the symbol of a pseudo-differential operator, say \(Q\), by \(\sigma(Q)\). Since \(\Lambda'\Lambda = I - \tilde{\Lambda}\) we have

\[
(16) \quad \Lambda L = \Lambda (\Lambda' + \tilde{\Lambda}^2) = \Lambda L \Lambda' + \Lambda L \tilde{\Lambda}^2.
\]

where \(\sigma(\Lambda L \tilde{\Lambda}^2)(t) \in (S_0^0)^{2 \times 2} \forall t \in [0, T]\). Moreover

\[
\sigma(\Lambda L \Lambda')(t) = \sigma(L - L \tilde{\Lambda} + \tilde{\Lambda} L - \tilde{\Lambda} L \tilde{\Lambda})(t)
\]

\[
= \ell(t, ..) + \sigma(\tilde{\Lambda} L - L \tilde{\Lambda})(t) - \sigma(\tilde{\Lambda} L \tilde{\Lambda})(t)
\]

where \(\sigma(\tilde{\Lambda} L \tilde{\Lambda})(t) \in (S_0^0)^{2 \times 2} \forall t \in [0, T]\). Then we have:

\[
\sigma(\tilde{\Lambda} L - L \tilde{\Lambda})(t) = \sigma(\tilde{\Lambda} \mathcal{H} - \mathcal{H} \tilde{\Lambda})(t) + \sigma(\tilde{\Lambda} \mathcal{B} - \mathcal{B} \tilde{\Lambda})(t)
\]

where \(\sigma(\tilde{\Lambda} \mathcal{B} - \mathcal{B} \tilde{\Lambda})(t) \in (S_0^0)^{2 \times 2} \forall t \in [0, T]\). Moreover, if \(b\) denotes the symbol of \(\mathcal{B}\) and \(b^d\) its diagonal, we have:

\[
\sigma(\tilde{\Lambda} \mathcal{H} - \mathcal{H} \tilde{\Lambda})(t) = b(t) - b^d(t) + r(t),
\]

with \(r(t) \in (S_0^0)^{2 \times 2}\). Denoting \(r - \sigma(\tilde{\Lambda} L \tilde{\Lambda}) + \sigma(\tilde{\Lambda} \mathcal{B} - \mathcal{B} \tilde{\Lambda})\) by \(z\), we obtain

\[
\sigma(\Lambda L \Lambda')(t) = \ell(t) + b(t) - b^d(t) + z(t) = \ell^d(t) + z(t).
\]
Denoting \( Z(t)\Lambda(t) + \Lambda(t)L(t)\tilde{\Lambda}(t) \) by \( C_0(t) \) and its symbol by \( c_0(t) \), we prove our claim in view of (16), (17).

Now we derive energy estimates for the diagonalized system. Define

\[
(18) 
\begin{pmatrix}
    e^{-Mp(x,\xi)} & 0 \\
    0 & e^{Mp(x,\xi)}
\end{pmatrix}
\]

where \( p(x,\xi) = (\xi)^{-1}\sum_{j=1}^n \xi_j \arctg x_j \) and \( M \geq rc_r/c \), with \( c_r \) as in (14), and \( c \) as in (H3). Denote the corresponding operator by \( K(x, \partial_x) \). Applying \( K\Lambda(t) \) to (12) we get

\[
\frac{d}{dt} \| K\Lambda(t)w(t) \|^2 = 2 \text{Re}(K\partial_t(\Lambda(t)w(t)), K\Lambda(t)w(t))
\]

\[
= 2 \text{Re}(K(\varepsilon\Delta\Lambda(t) - i\Lambda(t)L(t) + \varepsilon[\Lambda(t),\Delta])
\]

\[
+ [\partial_t, \Lambda(t)]w(t) + K\Lambda(t)G(u(t)), K\Lambda(t)w(t))
\]

which, in view of Lemma 4.1, is equal to

\[
2 \text{Re}(K(\varepsilon\Delta - iL^d(t))\Lambda(t) + r_o(t))w(t) + K\Lambda(t)G(u(t)), K\Lambda(t)w(t))
\]

where \( r_o(t) = \varepsilon[\Lambda(t),\Delta] + [\partial_t, \Lambda(t)] - ic_o(t) \) with \( c_o(t) \in (S^0)^{2\times 2} \forall t \in [0, T]. \) Since the first term in the asymptotic expansion of \( \sigma(\Lambda(t),\Delta)(x,\xi) \) is

\[
\begin{pmatrix}
    0 & -\sum_{j=1}^0 \xi_j D_{\xi_j}(b_{12}(t, x, \xi)/H(x, \xi)) \\
    \sum_{j=1}^0 \xi_j D_{\xi_j}(b_{21}(t, x, \xi)/H(x, \xi)) & 0
\end{pmatrix}
\]

which belongs to \( (S^0)^{2\times 2} \), then \( c_o(t) \in (S^0)^{2\times 2} \forall t \in [0, T] \).

Let us now examine the symbol of the diagonal matrix \( K(\varepsilon\Delta - iL^d(t)) - (\varepsilon\Delta - iL^d(t))K \). A simple calculation shows that it is of the form

\[
M = \begin{pmatrix}
    \{p, H\}(x, \xi) + 2\varepsilon i\xi. \nabla_x p(x, \xi) + \hat{s}_o(t, x, \xi) & 0 \\
    0 & \{p, H\}(x, \xi) - 2\varepsilon i\xi. \nabla_x p(x, \xi) + \hat{s}_o(t, x, \xi)
\end{pmatrix}
\]

\[
k(x, \xi)
\]
with $s_\alpha(t), \tilde{s}_\alpha(t) \in S^0$. Thus
\[
\frac{d}{dt} \| K \Lambda(t) w(t) \|_2^2 \\
\leq -2 \Re \langle (i L^d(t) - \varepsilon \Delta + M[H, p]) K \Lambda(t) w(t), K \Lambda(t) w(t) \rangle \\
+ (C'_\varepsilon \| w(t) \| + \| K \Lambda(t) G(u) \|) \| K \Lambda(t) w(t) \| 
\]
In view of the assumption (H3) and of (14), we have
\[
\text{Im } b_{kk}(t, x, \xi) + M[H, p](x, \xi) \geq (-c_r + Mc)(x)^{-2} |\xi| \geq 0,
\]
for $k = 1, 2$. Then by applying the sharp Gårding inequality we obtain
\[
\Re \langle (i L^d(t) + M[H, p]) K \Lambda(t) w(t), K \Lambda(t) w(t) \rangle \geq -\tilde{C}_r \| K \Lambda(t) w(t) \|_2^2,
\]
for some $\tilde{C}_r > 0$. Hence
\[
-2 \Re \langle (i L^d(t) - \varepsilon \Delta + M[H, p]) K \Lambda(t) w(t), K \Lambda(t) w(t) \rangle \\
\leq 2\tilde{C}_r \| K \Lambda(t) w(t) \|_2^2 - 2\varepsilon \| \nabla K \Lambda(t) w(t) \|_2^2 \leq 2\tilde{C}_r \| K \Lambda(t) w(t) \|_2^2.
\]
Then we get
\[
\frac{d}{dt} \| K \Lambda(t) w(t) \|_2^2 \leq 2\tilde{C}_r \| K \Lambda(t) w(t) \|_2^2 \\
+ (C'_\varepsilon \| w(t) \| + \| K \Lambda(t) G(u) \|) \| K \Lambda(t) w(t) \|
\]
5. End of the proof of the theorem

Let
\[
\tilde{E}(u(t)) = \sum_{|\alpha| = m+\ell} \| K \Lambda(t) \partial_\alpha^u u(t) \| + \sum_{j=1}^n \sum_{|\alpha| = m+1} \| K \Lambda(t)(x_j \partial_\alpha^u u(t)) \| \\
+ \sum_{|\alpha| = m} \| K \Lambda(t)(|x|^2 \partial_\alpha^u u(t)) \|
\]
Let $\varepsilon \in ]0, 1]$ and let $u_\varepsilon \in C([0, T]; \Xi^{m,\ell})$ be a solution of (4) such that $\sup_{t \in [0, T]} \| u_\varepsilon(t) \|_{\Xi^{m-1,\ell}} \leq r$. Let
\[
E(u_\varepsilon(t)) = \tilde{E}(u_\varepsilon(t)) + \| u_\varepsilon(t) \|_{\Xi^{m-1,\ell}}.
\]
As in the proof of (4.3) in [3], one can see that $E(u_\varepsilon(t))$ is equivalent to $\| u_\varepsilon(t) \|_{\Xi^{m,\ell}}$; specifically, if $\| u_\varepsilon(t) \|_{\Xi^{m-1,\ell}} \leq r$, then there exists $M_r > 1$ such that
\[
M_r^{-1} \| u_\varepsilon(t) \|_{\Xi^{m,\ell}} \leq E(u_\varepsilon(t)) \leq M_r \| u_\varepsilon(t) \|_{\Xi^{m,\ell}}.
\]
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Now from (19) and (15) we have

\[ \frac{d}{dt} \| K \Lambda(t) w(t) \|^2 \leq C_{r}^{\ast} E(u_\varepsilon(t)) \| K \Lambda(t) w(t) \|, \]

and summing up on \( \varphi(x) \) and \( \alpha \) we obtain

\[ \frac{d}{dt} \bar{E}(u_\varepsilon(t)) \leq C_{r}^{\ast} E(u_\varepsilon(t)). \]

Thus we finally obtain

\[ E(u_\varepsilon(t)) \leq E(u_\varepsilon) e^{C_{r}^{\ast} t} \]

with \( C_{r}^{\ast} \) which is independent of \( \varepsilon \in [0, 1] \). Then there exists a time \( T > 0 \) such that \( \{ u_\varepsilon \}_{\varepsilon \in [0, 1]} \) is bounded in \( C([0, T]; \mathbb{R}^{m,\varepsilon}) \), and thus by a standard argument we get a solution \( u(t) \in \mathbb{R}^{m,\varepsilon} \forall t \in [0, T] \) of (3).

REFERENCES