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SIMULTANEOUS APPROXIMATION IN THE DIRICHLET SPACE

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1. Introduction

Let $\mathscr{D}$ denote all analytic functions f in the unit disc D such that the Dirichlet integral

$$D(f) = \iint |f'|^2 \, dx \, dy < \infty$$

is finite. We characterize the subsets F of D with the following property: Whenever $f \in \mathscr{D}$, there are polynomials p_n, $n = 1, 2, \ldots$ such that

$$D(f - p_n) \to 0$$

and

$$\sup(|p_n(z)|, z \in F) \to \sup(|f(z)|, z \in F)$$

as $n \to \infty$. We also characterize those sets $F \subset D$ with the property

$$\sup(|f(z)|, z \in F) = \sup(|f(z)|, z \in D)$$

for all $f \in \mathscr{D}$. The characterizations involve geometric properties of F as well as certain concepts from potential theory. Let T denote the unit circle. The set $\overline{F} \cap T$, consisting of all limit points of F on T is split into a disjoint union

$$\overline{F} \cap T = F_{\mathrm{t}} \cup F_{\mathrm{nt}}$$

where F_{t} and F_{nt} are called the tangential and non-tangential part of $\overline{F} \cap T$. A point w belongs to F_{nt} if and only if there is $z_n \in F$ converging to w such that $|w - z_n|(1 - |z_n|)^{-1}$ remains bounded as $n \to \infty$. If $z \in D$, $z \neq 0$, let $I_z = \{\, w \in T : |w - z| < 2(1 - |z|) \,\}$. For $m = 1, 2, \ldots,$ let

$$S_m(F) = \cup I_z, \quad z \in F, \quad |z| > 1 - m^{-1}$$

We shall identify T with the interval $[-\pi, \pi)$ on the real line $\mathscr{R}$ when speaking about the capacity $\operatorname{cap}(B)$ or the linear measure $|B|$ of a subset B of T. The

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capacity we use is the so called Bessel capacity $C^{\frac{1}{2},2}(B)$ as defined in [1], page 21. We remark that this capacity is equivalent to logaritmic capacity. We can now formulate our main results:

THEOREM 1.1. *If F is a subset of the unit disc D, the following statements are equivalent*

(a) *If $f \in \mathscr{D}$, there are polynomials p_n, $n = 1, 2, \ldots$, such that $D(f - p_n) \to 0$ and* $\sup(|p_n|, z \in F) \to \sup(|f(z)|, z \in F)$ *as $n \to \infty$.*
(b) *For $m = 1, 2, \ldots$, $S_m(F) \cup F_{\mathrm{nt}}$ is thick at almost all $z \in \overline{F} \cap T$. The exceptional set has zero capacity.*
(c) *For $m = 1, 2, \ldots$, $S_m(F)$ is thick at allmost all $z \in \overline{F} \cap T$. The exceptional set has zero capacity.*

The word "thick" in Theorem 1.1 should be understood in the sense of potential theory in connection with Wiener's criterion. See Section 2 for more details. Theorem 1.1 solves the problem of characterizing the Farrell sets for $\mathscr{D}$. This is part of a general problem in approximation theory raised by L. A. Rubel. See [16] and [18]. Results about Farrell sets for other function spaces can for example be found in [12], [11], [17] and [20].

THEOREM 1.2. *Let F be a subset of D. The following statements are equivalent*

(a) *For all $f \in \mathscr{D}$ the supremum of $|f|$ on F equals the supremum of $|f|$ on D.*
(b) $\mathrm{cap}(T) = \mathrm{cap}(T \setminus F_{\mathrm{t}})$.

Condition (b) in Theorem 1.2 appeared in a theorem by L. V. Ahlfors and A. Beurling characterizing removable singularities for analytic functions with finite Dirichlet integral. See [8], page 82–85, for details. We use their theorem as well as Theorem 1.1 to deduce Theorem 1.2.

If $\mathscr{A}$ is any space of functions in D, a set $F \subset D$ is called a set of determination for $\mathscr{A}$ if

$$\sup\{\,|f(z)|, z \in F\,\} = \sup\{\,|f(z)|, z \in D\,\}$$

whenever $f \in \mathscr{A}$. Sets of determination are known for various function spaces. In particular we mention [5] (bounded harmonic functions), [13] (differences of positive harmonic functions) and [6], (bounded analytic functions). For extensions to higher dimensions see [10]. It is interesting to compare Theorem 1.2 with the result L. Brown, A. L. Shields and K. Zeller found: F in a set of determination for $H^\infty(D)$ if and only if $|T \setminus F_{\mathrm{t}}| = |T|$. ($H^\infty$ denotes the space of all bounded analytic functions in D). The rest of the paper is organized as follows: In Section 2 we give some background about $\mathscr{D}$ and potential theory. In Section 3 we state and prove a theorem relating Farrell sets to other approximation problems. Then Theorem 1.1 is proved in Section 4. We make extensive use of the Bessel potential space $L^{\frac{1}{2},2}(\mathscr{R})$. A convenient reference in this connection is the recent book [1]. In Section 5 we add some final remarks including a proof of Theorem 1.2.

2. Potential theory and the space $\mathscr{D}$

If $w \in T$ and $\alpha > 1$, we define

$$\Gamma_\alpha(w) = \{ z \in D : |w - z| \leq \alpha(1 - |z|) \}$$

LEMMA 2.1. *If $f \in \mathscr{D}$, the limit $f(w) = \lim f(z)$, $z \to w$, $z \in \Gamma_\alpha(w)$, exists for almost all $w \in T$. The exceptional set has zero capacity.*

PROOF. See [4] or [21] page 344. More general results can be found in [1] page 161 or [8] page 55.

We shall need that the Dirichlet integral can be expressed as a boundary integral

$$D(f) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|\tilde{f}(s - t) - \tilde{f}(s)|^2}{|e^{it} - 1|^2}\, ds\, dt \tag{1}$$

where $\tilde{f}$ denotes the 2π periodic function corresponding to f by $\tilde{f}(t) = f(e^{it})$, $t \in \mathscr{R}$.

We also consider the Bessel potential space $L^{\frac{1}{2},2}(\mathscr{R})$ and recall that a function g belongs to $L^{\frac{1}{2},2}(\mathscr{R})$ if and only if

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|g(x - y) - g(x)|^2}{y^2}\, dx\, dy \tag{2}$$

is finite.

If $\phi \in C^\infty(\mathscr{R})$ has compact support , it follows directly from (2) that $\phi g \in L^{\frac{1}{2},2}(\mathscr{R})$ whenever $g \in L^{\frac{1}{2},2}(\mathscr{R})$. Our next result is certainly well known and easy to prove.

LEMMA 2.2. *Let $\phi \in C^\infty(\mathscr{R})$ have compact support on $(-\pi, \pi)$*

(a) *If g is defined on $\mathscr{R}$ by $g = \tilde{f}$ for some $f \in \mathscr{D}$, then $\phi g \in L^{\frac{1}{2},2}(\mathscr{R})$.*
(b) *Let $h \in L^{\frac{1}{2},2}(\mathscr{R})$ be real-valued. Define $u = \phi g$ on $(-\pi, \pi)$. Then $u = \Re \tilde{f}$ for some $f \in \mathscr{D}$.*

PROOF. Since $|\frac{e^{it}-1}{t}| \to 1$ as $t \to 0$, the integrals (1) and (2) are simultaneously bounded for ϕg. The same argument applies to $u = \phi h$ as defined in (b) and then Douglas formula (See [3]), gives

$$\iint_D |\nabla u|^2\, dx\, dy < \infty \tag{3}$$

where u is extended to D by Poissons formula. If v is a harmonic conjugate to u in D, the Cauchy-Riemann equations give $u + iv \in \mathscr{D}$.

The reason why we formulated Lemma 2.2 is that we wish to apply various results about $L^{\frac{1}{2},2}(\mathscr{R})$ as expressed in the recent book [1] to the Dirichlet space $\mathscr{D}$. The corresponding results for the Bessel potential space $L^{\frac{1}{2},2}(T)$ do not seem equally accessible through the existing literature.

If $B \subset \mathscr{R}$ and $x_0 \in \mathscr{R}$, we say that B is *thin at* x_0 if

$$\int_0^1 \frac{\operatorname{cap}(B \cap (x_0 - r, x_0 + r))}{r}\, dr < \infty$$

If the above integral diverges, we say that B is *thick at* x_0. See [1] page 166 for more details.

We end this section by a remark about truncation of functions in $L^{\frac{1}{2},2}(\mathscr{R})$ and $\mathscr{D}$. If $u \in L^{\frac{1}{2},2}(\mathscr{R})$ is realvalued and M is a real constant, the function

$$u_M(x) = \min(u(x), M), \qquad x \in \mathscr{R}$$

is called a truncation of u. From (2) it is easy to see that $u_M \in L^{\frac{1}{2},2}(\mathscr{R})$. From (1) it follows that $\Re f$ may be truncated in the same way if $f \in \mathscr{D}$. Even functions in $\mathscr{D}$ may be truncated. For more details see [7].

3. Farrell sets and other approximation problems

We use the notation $\|g\|_B = \sup(|f(z)|, z \in B)$ if g is a function defined on a set B. We also use the norm

$$\|f\| = \left[|f(0)|^2 + \iint_D |f'|^2\, dx\, dy \right]^{\frac{1}{2}}, \qquad f \in \mathscr{D}$$

making $\mathscr{D}$ a Hilbert space.

The main result in this section is

THEOREM 3.1. *The following statements are equivalent for a subset F of D:*

(a) *If $f \in \mathscr{D}$ there are polynomials p_n such that $\|p_n - f\| \to 0$ and $\|p_n\|_F \to \|f\|_F$ as $n \to \infty$.*

(b) *If $f \in \mathscr{D}$ and u is a uniformly continuous function on F, there are polynomials p_n such that $\|p_n - f\| \to 0$ and $\|p_n - u\|_F \to \|f - u\|_F$ as $n \to \infty$.*

PROOF. We only need to prove (a) $\Rightarrow$ (b). We follow an idea that previously was used in a similar situation for the Hardy space $H^p(D)$, $0 < p < \infty$. (See [12]). Our next lemma is useful in order to localize the approximation problem in Theorem 3.1.

LEMMA 3.2. *Let F be a Farrell set for $\mathscr{D}$. Suppose $f \in \mathscr{D}$ is bounded on F and that $|f| \leq t$ on $F \cap \Delta$, for some disc Δ and some $t > 0$. Then $|f(e^{i\theta})| \leq t$ for almost all $e^{i\theta} \in \overline{F} \cap \Delta \cap T$. The exceptional set has zero capacity.*

PROOF. Let $z_0 \in \overline{F} \cap \Delta \cap T$ and choose a smooth function ϕ such that $\phi = 0$ near z_0, $\phi \leq 0$ on T, and define $f_\phi = f e^{\phi + i\tilde{\phi}}$ where $\tilde{\phi}$ is a harmonic conjugate to the (harmonic) extension of ϕ to D. We may choose ϕ such that $|f_\phi| \leq t$ on F and $f_\phi \in \mathscr{D}$. If $\{q_\nu\}$ are polynomials such that $\|f_\phi - q_\nu\| \to 0$, it is well known that $\text{cap}\{\theta : |f_\phi(e^{i\theta}) - q_\nu(e^{i\theta})| > \epsilon\} \to 0$ as $\nu \to \infty$ for any $\epsilon > 0$. See for example ([1], Prop 2.3.8). Hence if $\|q_\nu\|_F \to \|f_\phi\|_F \leq t$, it follows that $|f_\phi| \leq t$ on $\overline{F} \cap T$, except for a set of zero capacity. Since $|f_\phi| = |f|$ on T near z_0, Lemma 3.2 follows.

Our next lemma is the key both to prove Theorem 3.1 and to obtain the geometric characterization in Theorem 1.1.

LEMMA 3.3. *Let F be a subset of the unit disc D. The following statements are equivalent:*

(a) *F is a Farrell set for $\mathscr{D}$.*
(b) *If $f \in \mathscr{D}$, then $|f| \leq \|f\|_F$ on $\overline{F} \cap T$ with the exception of a set of zero capacity.*
(c) *If $f \in \mathscr{D}$ is bounded on F and Δ is an open disc, there are polynomials $\{q_\nu\}$ such that $\|f - q_\nu\| \to 0$, $\|q_\nu\|_F \to \|f\|_F$ and $\|q_\nu\|_{F\cap\Delta} \to \|f\|_{F\cap\Delta}$*

PROOF. By Lemma 3.2, we only have to prove (b) $\Rightarrow$ (c). Let $f \in \mathscr{D}$ be bounded on F and assume $\|f\| = 1$. Given $\epsilon > 0$, we define

$$P_\epsilon = \{\, p \in P : \|p\| \leq 1,\ \|p\|_F \leq \|f\|_F + \epsilon,\ \|p\|_{F\cap\Delta} \leq \|f\|_{F\cap\Delta} + \epsilon \,\}.$$

Here P denotes the set of polynomials. If $K \subset D$ is compact, we prove that f is uniformly approximable on K by functions from P_ϵ. Let μ denote a measure on K satisfying

$$|\mu(p)| \leq 1, \qquad p \in P_\epsilon$$

The restrictions of functions in P_ϵ to the space $C(K)$ consisting of all continuous functions on K, is a convex set. By the separation theorem for convex sets and the Riesz representation theorem, it is sufficient to prove that $|\mu(f)| \leq 1$.

To obtain this we consider the Banach space

$$L = \mathscr{D} \times C(\overline{F\backslash\Delta}) \times C(\overline{F \cap \Delta})$$

with the norm

$$N\{a, b, c\} = \max\Big\{\, \|a\|, (t_1 + \epsilon)^{-1}\|b\|_{\overline{F\backslash\Delta}}, (t_2 + \epsilon)^{-1}\|c\|_{\overline{F\cap\Delta}} \,\Big\}$$

where $t_1 = \|f\|_F$ and $t_2 = \|f\|_{F\cap\Delta}$. By the Hahn-Banach theorem the linear functional $p \to \mu(p)$ admits a normpreserving extension from P to L. We represent this extension by a triple (g, μ_1, μ_2) where $g \in \mathscr{D}$ (note that $\mathscr{D}$ is its own dual), and μ_1 and μ_2 are measures on $\overline{F\backslash\Delta}$ and $\overline{F \cap \Delta}$ respectively.

Since we are dealing with a normpreserving extension, we can write (with $\langle h, g\rangle = h(0)\overline{g(0)} + \iint_D h'\overline{g'}\,dx\,dy$ for $f, g \in \mathscr{D}$)

$$\mu(z^n) = \langle z^n, g\rangle + \mu_1(z^n) + \mu_2(z^n), \qquad n \geq 0 \tag{4}$$

and

$$\|g\| + (t_1 + \epsilon)|\mu_1| + (t_2 + \epsilon)|\mu_2| \leq 1 \tag{5}$$

where $|\lambda|$ denotes the total variation of the measure λ.

By (4) and a "restoration theorem" of S. Hruscev ([15], page 440), it follows that $\mu_1(B) = \mu_2(B) = 0$ if $B \subset T$ has zero capacity. Actually we can not apply Hruscev's theorem directly since it deals with measures supported on T. But if we define measures λ_1 and λ_2 on T by

$$\lambda_i(\phi) = \iint_D \phi\, d\mu_i, \qquad i = 1, 2.$$

for all continuous ϕ on $\overline{D}$ being harmonic in D, then λ_1 and λ_2 are uniquely defined measures on T being absolutely continuous with respect to linear measure on T. The desired conclusion about μ_1 and μ_2 follows by applying Hruscev's theorem to λ given by

$$\lambda = \sum_{i=1}^{2} \lambda_i(B) + \mu_i(B)$$

if B is a Borel set contained in T.

To complete the proof, let us first assume f is bounded on D. Let $f_r \in \mathscr{D}$ be defined by $f_r(z) = f(rz)$, $z \in D$, $0 < r < 1$. Since the Taylor series of f_r converge uniformly to f_r near $\overline{D}$, we conclude that

$$\mu(f_r) = \langle f_r, g\rangle + \mu_1(f_r) + \mu_2(f_r)$$

for all r, $0 < r < 1$. Letting $r \to 1$ we get by dominated convergence that

$$\mu(f) = \langle f, g\rangle + \mu_1(f) + \mu_2(f)$$

By combining the hypothesis (b) in Lemma 3.3 and the inequality (5), we conclude that $|\mu(f)| \leq 1$.

Since ϵ and K were arbitrary, we can so far conclude that there are polynomials q_ν converging weakly to f in $\mathscr{D}$ and satisfying the two last requirements in (c) in Lemma 3.3. By taking convex combinations of polynomials from $\{q_\nu\}$ we get (c) in Lemma 3.3 provided f is bounded. By a representation formula for the Dirichlet integral due to L. Carleson [7], any f in $\mathscr{D}$ may be approximated pointwise in D by bounded functions $f_n \in D$ such that $\|f_n\|$ is bounded and $|f_n(z)| \leq |f(z)|$ for $z \in D$. Hence (c) follows and Lemma 3.3 is proved.

We now prove (a) $\Rightarrow$ (b) in Theorem 3.1 by a constructive argument. Let ϕ denote a smooth function with compact support in the complex plane $\mathscr{C}$. If h is locally integrable on $\mathscr{C}$, we recall the definition of the T_ϕ-operator (See [9], page 210)

$$T_\phi h(w) = \phi(w) + \frac{1}{\pi} \iint \frac{h(z)}{z - w} \frac{\partial \phi}{\partial \bar{z}} \, dx \, dy(z)$$

If h belongs to $L^p(dx\,dy)$ locally, $p > 2$, we have

$$\frac{\partial}{\partial \overline{w}} T_\phi h = \phi \frac{\partial h}{\partial \overline{w}} \tag{6}$$

and

$$\frac{\partial}{\partial w} T_\phi h = \frac{\partial}{\partial w}(\phi h) + Th \tag{7}$$

where

$$Th(w) = \frac{-1}{\pi} \iint \frac{h(z)}{(z-w)^2} \, dx \, dy(z)$$

denotes the planar Hilbert transform. We refer to [2], page 85–90 and [9] for more details.

Let us now consider $f \in \mathscr{D}$ being bounded on F and g uniformly continuous on F. For the moment we assume f admits analytic continuation across $T\backslash\overline{F}$. Given $\epsilon > 0$, we cover $\overline{F}$ by a finite collection of open discs $\{\Delta_j\}$ such that $|g(z) - g(w)| < \epsilon$ if $z, w \in \Delta_j \cap F$ for some j. We select constants $g_j = g(z_j)$ where $z_j \in \Delta_j \cap F$.

By Lemma 3.2 and Lemma 3.3, we find for each j polynomials q_ν^j such that

$$\|q_\nu^j - (f - g_j)\| \to 0$$

and

$$\|q_\nu^j\|_{F \cap \Delta_j} \to \|f - g_j\|_{F \cap \Delta_j}$$

and such that $\|q_\nu^j\|_F$ is bounded independent of ν and j.

Since f is analytic near $\overline{D}\setminus \cup \Delta_j$, there are polynomials p_ν approximating f uniformly near $\overline{D}\setminus \cup \Delta_j$. Let q_ν be given by $q_\nu = p_\nu$ on $\overline{D}\setminus \cup \Delta_j$, and

$$q_\nu(z) = q_\nu^j(z) + g_j, \qquad z \in \Delta_\nu \setminus \cup_1^{\nu-1} \Delta_k$$

for $\nu = 1, 2, \ldots, j$. These "P-splines" q_ν converge to f in the sense that

$$\iint_D |f' - q_\nu'|^2 \, dx\, dy \to 0 \tag{8}$$

as $\nu \to \infty$. Moreover, it is easy to verify that

$$\|q_\nu - g\|_F \le \|f - g\|_F + 2\epsilon \tag{9}$$

for ν sufficiently large.

The only problem with $\{q_\nu\}$ is the discontinuity near $E = \cup \partial \Delta_j \cap \{ z : |z| \le 1 \}$. We modify q_ν near E using the T_ϕ-operator introduced above.

Let $\{D_k\}$ denote a finite covering of E by open discs D_k and let $\{\phi_k\}$ denote smooth functions supported in D_k such that

$$\sum \phi_k = 1$$

near E. We assume $D_k \cap T = \emptyset$ if $1 \le k \le k_0$ and if $k > k_0$ we assume D_k is centered at $\zeta_k \in T$ and that the nontangential limit

$$\lambda_k = \lim f(z), \quad z \to \zeta_k, \quad z \in \Gamma_\alpha(\zeta_k)$$

exists for some large α to be specified. This is possible to obtain by Lemma 2.1. We define

$$\tilde{q}_\nu = q_\nu - \sum_{k=1}^{k_0} T_{\phi_k}(q_\nu - f) - \sum_{k>k_0} T_{\phi_k}\big((q_\nu - \lambda_k)\chi_\delta\big)$$

where χ_δ is the characteristic function of $\{ z : |z| < 1 + \delta \}$ and $\delta = \delta(\nu)$ is a small number to be specified.

CLAIM 1. *$\tilde{q}_\nu$ is analytic near $\overline{D}$.*

PROOF. By (6) we have

$$\frac{\partial}{\partial \overline{w}} \tilde{q}_\nu = \frac{\partial}{\partial \overline{w}} q_\nu \Big(1 - \sum_k \phi_k\Big) = 0$$

near $\overline{D}$.

CLAIM 2. $\|\tilde{q}_\nu - g\|_F \leq \|f - g\|_F + 3\epsilon$.

PROOF. Recall that D_k is centered at $\zeta_k \in T$ if $k > k_0$. We assume that either $\zeta_k \in T \backslash \overline{F}$ or that ζ_k is an interior point relative to $\overline{F} \cap T$. When estimating $\|\tilde{q}_\nu - g\|_F$, the essential contribution comes from

$$\|q_\nu - \sum_{k>k_0} \phi_k(q_\nu - \lambda_k) - g\|_F$$

The terms we have neglected in doing this approximation are

$$\sum_{1\leq k\leq k_0} T_\phi(q_\nu - f)$$

and

$$\sum_{k>k_0} \iint \frac{(q_\nu - \lambda_k)}{(z-w)} \frac{\partial \phi}{\partial \overline{z}} \chi_\delta \, dx\, dy(z)$$

The first of these sums is small when ν is large since $q_\nu \to f$ uniformly on compact subsets of D. The second sum is uniformly small in w if α is large and δ is small, and the discs D_k have small radii for $k > k_0$. We proceed to estimate $\|q_\nu - \sum_{k>k_0} \phi_k(q_\nu - \lambda_k) - g\|_F$:

We may assume that the discs D_k are pairwise disjoint and that $D_k \cap \overline{F} = \emptyset$ if the center $\zeta_k \notin \overline{F}$. By (9) we only have to estimate $\|q_\nu - \phi_k(q_\nu - \lambda_k) - g\|_{F\cap D_k}$. But

$$\begin{aligned}\|q_\nu - \phi_k(q_\nu - \lambda_k) - g\|_{F\cap D_k} &\leq \|(q_\nu - g)(1 - \phi_k) + \phi_k(\lambda_k - g)\|_{F\cap D_k} \\ &\leq \max\{\, \|q_\nu - g\|_{F\cap D_k}, \|\lambda_k - g\|_{F\cap D_k}\| \,\} \leq \|f - g\|_F + 2\epsilon\end{aligned}$$

provided the radius d_k of D_k is sufficiently small. (To get the last inequality, it is essential that ζ_k is interior relative to $\overline{F} \cap T$).

CLAIM 3. $\|\tilde{q}_\nu - f\| \leq \epsilon$ *provided* $\{D_k\}$ *are chosen properly and* ν *is sufficiently large.*

PROOF. We first remark that the planar Hilbert transform is an isometry on $L^2(\mathscr{C})$ ([2], page 89). From (6) and (7) it follows that the T_ϕ-operator maps $\mathscr{D}$ into itself. By (8) we need only estimate

$$\iint_{D\backslash E} |\tilde{q}_\nu{}' - q_\nu'|^2 \, dx\, dy$$

which equals

$$\iint_{D\backslash E} \left| \sum_{k=1}^{k_0} \left\{ T_{\phi_k}(q_\nu - f) - \sum_{k>k_0} T_{\phi_k}((q_\nu - \lambda_k)\chi_\delta) \right\}' \right|^2 dx\, dy \tag{10}$$

The first sum in (10) is easily estimated since $q_\nu \to f$ and $q_\nu^{'} \to f^{'}$ uniformly on the support of ϕ_k for $1 \le k \le k_0$.

The second sum in (10) is easily estimated using (6), (7) and the isometric property of the planar Hilbert transform. By construction we may choose D_k, $k > k_0$, so small that $\|q_\nu\|_{D\cap D_k}$ is bounded independently of k and ν. Using this bound and (8) and that $\delta = \delta(\nu)$ may be chosen as small as we wish, the second sum in (10) gives a small contribution provided the area of $\cup_{k>k_0} D_k$ is sufficiently small.

In the proof we assumed f to admit analytic continuation across $T\backslash\overline{F}$. If $f_1 \in \mathscr{D}$ is arbitrary it is sufficient to find f as above such that

$$\|f - f_1\| + \|f - f_1\|_F$$

is as small as we please. This is easy to obtain since the T_ϕ-operator maps $\mathscr{D}$ into itself. Indeed put $f_1 = 0$ outside D and define

$$f = f_1 - \sum_j T_{\psi_j}(f_1) - R_j$$

Here ψ_j ars smooth functions with compact support disjoint from $\overline{F}$ such that $\sum_j \psi_j \equiv$ near $T\backslash\overline{F}$. We may assume ψ_j is supported on a disc K_j and that no more than M of these discs overlap, where M is a numerical constant. Moreover, if h_j=$T_{\psi_j} f_1$ then R_j denotes the map in $\mathscr{D}$ given by

$$R_j(z) = h_j(r_j z)$$

for some $r_j < 1$. We may choose $\{r_j\}$ such that

$$\|h_j - R_j\| + \|h_j - R_j\|_F \le \epsilon 2^{-j}$$

for $j = 1, 2, \ldots$. The basic property (6) now gives that our function f admits analytic continuation across $T\backslash\overline{F}$ and satisfies

$$\|f_1 - f\| + \|f_1 - f\|_F \le \epsilon$$

For an alternative method to find $f \in \mathscr{D}$ approximating f_1 as above we refer to the proof of Lemma 3.2 in [19].

4. Proof of Theorem 1.1

In the proof of Theorem 1.1 we find it convenient to introduce another concept from potential theory: If B and S are Borel sets in $\mathscr{R}$ and $B \subset S$, we say that B is *representative* for S if $\operatorname{cap}(B \cap \Delta) = \operatorname{cap}(S \cap \Delta)$ for any disc Δ. We refer to Theorem 11.4.2 and Remark 11.4 on page 327 in [1] for more about these sets. By Theorem 11.4.2 in [1], condition (c) in Theorem 1.1 can be restated as: $S_m(F)$ is representative for $S_m(F) \cup \overline{F} \cap T$ for any $m \geq 1$.

To prove (a) $\Rightarrow$ (c), we assume (c) fails while (a) holds. Then $L = S_m(F)$ fails to be representative for $L_1 = S_m(F) \cup \overline{F} \cap T$ for some m. By Theorem 11.4.2 in [1] there is a disc Δ such that $\operatorname{cap} \Delta \cap L < \operatorname{cap} \Delta \cap L_1$. By the definition of Bessel capacity there is $u \in L^{\frac{1}{2},2}(\mathscr{R})$ such that $u \geq 1$ on $L \cap \Delta$ and $u \leq t < 1$ on a set $K \subset (L_1 \backslash L) \cap \Delta$ having positive capacity. We choose $\phi \in C_0^\infty(\Delta)$ such that $0 \leq \phi \leq 1$ and define v by $v = \phi(1-u)$. Then $v \geq 1-t$ on a set $K_1 \subset K$ with $\operatorname{cap} K_1 > 0$. Replacing v by $\min\{v, 1-t\}$ if necessary, we may assume $v \leq (1-t)$ everywhere and $v \leq 0$ on $S_m(F)$. By Lemma 2.2, w defined by

$$w(e^{i\theta}) = v(\theta), \qquad -\pi \leq \theta < \pi$$

extends by Poisson's integral formula to a harmonic function in D having finite Dirichlet integral. It is evident that $\|w\|_F < 1-t$ since if $z \in F$, $|z| > 1-\frac{1}{m}$, the contribution to $w(z)$ coming from integrating the Poisson kernel over $S_m(F)$ is significant.

If $\tilde{w}$ is a harmonic conjugate to w in D and $f = e^{w+i\tilde{w}}$, then $f \in \mathscr{D}$, $\|f\|_F < e^{1-t}$, while $|f| \geq e^{1-t}$ on a set $K_1 \subset \overline{F} \cap T$ having positive capacity. By Lemma 3.3 this contradicts F being a Farrell set and (a) $\Rightarrow$ (c) is proved. That (c) $\Rightarrow$ (b) is trivial.

To prove (b) $\Rightarrow$ (a), we assume (b) holds while (a) fails. Again by Lemma 3.3 there is $f \in \mathscr{D}$ with $\|f\|_F < 1$ and $|f| \geq 1$ on a set $B \subset \overline{F} \cap T$ with $\operatorname{cap}(B) > 0$. By truncating f if necessary (See [7]), we may assume $\|f\|_D = 1$. Given $\epsilon > 0$, we may even assume $\|f\|_F < \epsilon$ by replacing f by f^N for some integer N.

If $u = \Re(\frac{1}{2}(1+e^{i\alpha} f))$, we can choose α real such that $u \geq 1-\frac{\epsilon}{2}$ on a subset B' of B with $\operatorname{cap}(B') > 0$. We also have $\|u\|_F < \frac{1}{2}(1+\epsilon)$ and $0 \leq u \leq 1$ on $[-\pi, \pi)$.

Recall that by assumption (b) the set $S_m(F) \cup F_{\text{nt}}$ is representative for $S_m(F) \cup \overline{F} \cap T$ for any $m \geq 1$. For each m there is a subset $D_m \subset F_{\text{t}}$ with $\operatorname{cap} D_m = 0$, such that $S_m(F) \cup F_{\text{nt}}$ is thick at z_0 for any $z_0 \in F_{\text{t}} \backslash D_m$. Let

$$D_0 = \cup_1^\infty D_m$$

Then $cap(D_0) = 0$ and any set $S_m(F) \cup F_{\text{nt}}$ is thick at any $z_0 \in F_{\text{t}} \backslash D_0$.

Given such z_0, we may choose m so large that if $I_z \cap B_r(z_0) \neq \emptyset$, then $I_z \subset B_{2r}(z_0)$ whenever $z \in F$ and $|z| > 1-\frac{1}{m}$. (Here $B_t(z) = \{w : |w-z| < t\}$)

We fix such an integer m and shall use the function u constructed above to get a contradiction at our point $z_0 \in F_{\text{t}} \backslash D_0$. Since $u(z) < \frac{1}{2}(1+\epsilon)$ if $I_z \in S_m(F)$, it follows by elementary estimates for the Poisson integral that

$u \leq (1-\epsilon)$ on a set $E_z \subset I_z$ with $|E_z| \geq d|I_z|$ and d is independent of z. Since u is finely continuous outside a subset of T having zero capacity ([1], page 177), we assume in the following that u is finely continuous at z_0.

Let

$$E = \bigcup_{z\in F, |z|>1-\frac{1}{m}} E_z$$

CLAIM 4. *There is a subset E' of E such that $E' \cup F_{\text{nt}}$ is thick at z_0.*

Taking the claim for granted and using that $u \leq (1-\epsilon)$ on $E' \cup F_{\text{nt}}$, it follows by fine continuity that $u(z_0) \leq (1-\epsilon)$. But this contradicts that $u \geq (1-\frac{\epsilon}{2})$ on a set $B' \subset F_{\text{t}}$ of positive capacity.

It remains to prove the claim. With m fixed as above, let $\{J_\nu\}$ denote a countable subcollection of $\{ I_z : z \in F, |z| > 1 - \frac{1}{m} \}$ such that

$$\bigcup_\nu J_\nu = \bigcup_{z\in F,\ |z|>1-\frac{1}{m}} I_z$$

Also let $E_\nu \subset J_\nu$ be the corresponding subset with $|E_\nu| \geq d|J_\nu|$ and define

$$E' = \bigcup_\nu E_\nu$$

Since $S_m(F) \cup F_{\text{nt}}$ is thick at z_0, the above claim follows from Wieners criterion (See [1], page 166) if we can show that

$$\text{cap}((E' \cup F_{\text{nt}}) \cap B_{2r}(z_0)) \geq C_0\, \text{cap}((S_m(F) \cup F_{\text{nt}}) \cap B_r(z_0)) \tag{11}$$

for all sufficiently small r and with $C_0 > 0$ independent of r. By regularity properties of capacity ([1], page 28), it is sufficient to prove (11) if $S_m(F)$ is replaced by an arbitrary finite union $J = \cup_1^N J_\nu$ and E' is replaced by $\cup_1^N E_\nu$. By subadditivity of capacity we may even replace $\{J_\nu\}_{\nu=1}^N$ by a subcollection of pairwise disjoint intervals.

The following lemma immediately gives (11) and completes the proof of Theorem 1.1. (Recall that m was large enough to guarantee $I_\nu \subset B_{2r}(z_0)$ if $I_\nu \cap B_r(z_0) \neq \emptyset$)

LEMMA 4.1. *Let $\{J_\nu\}$ be a finite collection of disjoint intervals and let for each ν, K_ν be a subset of J_ν such that $|K_\nu| \geq d|J_\nu|$ with $d > 0$. Then* $\text{cap}(\cup_\nu K_\nu) \geq C(d)\,\text{cap}(\cup_\nu J_\nu)$ *where $C(d)$ depends only on d.*

The proof of Lemma 4.1 will not be given in detail since it is an immediate consequence of the definition of $C_{\frac{1}{2},2}$-capacity and the lower bounds for the capacitary potential $V^\nu_{\alpha,p}$ given in Lemma 9.8.3 in [1]. The set F in Lemma 9.8.3

corresponds to our set $\cup_\nu K_\nu$ and the measure "ν" in Lemma 9.8.3 is the capacitary measure for F.(see [1], page 21 for more about these concepts). That Lemma 9.8.3 applies in our situation follows since

$$\operatorname{cap}(K_\nu) \geq c\left(\log \frac{2}{|K_\nu|}\right)$$

for some constant c and any $K_\nu \subset (-\pi, \pi]$.

5. Concluding remarks

We indicate the proof of Theorem 1.2: If $\overline{F} \supset T$, we have $S_m(F) \cup F_{\mathrm{nt}} = T \backslash F_m$ where $F_m = F_{\mathrm{t}} \backslash S_m(F)$. By Theorem 1.1 and Theorem 13 in [14], we only have to prove

$$\{\operatorname{cap} T = \operatorname{cap}(T \backslash F_m), m = 1, 2, \ldots\} \Rightarrow \operatorname{cap} T = \operatorname{cap}(T \backslash F_{\mathrm{t}})$$

So we assume $\operatorname{cap} T = \operatorname{cap}(T \backslash F_m)$, $m = 1, 2, \ldots$ and that

$$\operatorname{cap} T > \operatorname{cap}(T \backslash K) \tag{12}$$

for some compact subset K of F_{t}. Then there is an analytic function h_K in $\mathscr{C} \backslash K$ having finite Dirichlet integral in $\mathscr{C} \backslash K$ and being non constant. In fact, a theorem by L. V. Ahlfors and A. Beurling tells that (12) is necessary and sufficient for the existence of such a function h_K.(See [8], page 82–84).

In the following we assume K is minimal in the sense that h_K doesn't extend to be analytic near any $z \in K$. This minimal set is still denoted by K and (12) will hold because of the Ahlfors-Beurling theorem. Let $K_m = K \cap F_m$. Then K_m is compact and $\operatorname{cap} T = \operatorname{cap}(T \backslash K_m)$ for all m. Let $z \in K_m$ and assume Δ is a disc centered at z such that $\Delta \cap K \subset K_m$. If ϕ is a smooth function with compact support in Δ and $\phi \equiv 1$ near z, we define

$$h_\Delta = T_\phi(h_K)$$

By the Ahlfors-Beurling theorem and property (6) of the T_ϕ-operator, $h_\Delta = 0$, but on the other hand $h_K - h_\Delta$ is analytic near z and this contradicts the assumed minimality of K.

We may therefore conclude that $K \backslash K_m$ is dense in K for all m. By the Baire category theorem, the intersection of all $K \backslash K_m$ is dense in K, but on the other hand

$$\bigcap_{m=1}^{\infty} K \backslash K_m = \emptyset$$

We conclude that our initial assumption $\operatorname{cap} T > \operatorname{cap}(T\backslash K)$ was false and this proves Theorem 1.2.

We finally show that in general the m-condition in Theorem 1.1 involving $S_m(F)$ cannot be removed. At the same time we get an example of a Farrell set for $\mathscr{D}$ where $|F_{\mathrm{t}}| > 0$. We start with a compact totally disconnected subset K of T such that $|K| > 0$ and $\operatorname{cap} T = \operatorname{cap}(T\backslash K)$. Such sets exists (see [1], Remark 2, on page 314). Let

$$T\backslash K = \bigcup I_\nu$$

where the open arcs I_ν are disjoint. We push each I_ν slightly into D and obtain an arc Λ_ν, $\nu = 1, 2, \ldots$. Now let

$$F = \bigcup_\nu \Lambda_\nu$$

We can choose Λ_ν such that $F_{\mathrm{t}} = K$ while $F_{\mathrm{nt}} = \emptyset$. Since $\operatorname{cap} T = \operatorname{cap}(T\backslash K)$ it follows that $T\backslash K$ is thick at almost all $z \in K$ with respect to capacity (see [1], Theorem 11.4.2 combined with [14], Theorem 13). But we can choose the sets Λ_ν so close to T that $S_m(F) = (\bigcup_\nu \Lambda_\nu)\backslash L_m, m = 1, 2, \ldots$ for some compact set L_m of $\overline{\bigcup_\nu \Lambda_\nu})$. This means that $S_m(F)$ is thick at almost all $z \in \overline{F}$ and F is a Farrell set for $\mathscr{D}$ by Theorem 1.1.

Let us finally remark that $S_m(f)$ could be replaced by

$$S_m^k(F) = \{\, w : |w - z| < k(1 - |z|) \ \text{ for some } \ z \in F, |z| \geq 1 - m^{-1} \,\}$$

in Theorem 1.1 for any $k > 1$, and the theorem would still remain true. This follows easily from Lemma 4.1.

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