TOEPLITZ OPERATORS ON GENERALIZED BERGMAN-HARDY SPACES

WOLFGANG LUSKY

Abstract
We study the Toeplitz operators $T_f : H_2 \to H_2$ for $f \in L_1$, on a class of spaces $H_2$ which includes, among many other examples, the Hardy and Bergman spaces as well as the Fock space. We investigate the space $X$ of those elements $f \in L_1$ with $\lim_j ||T_j - T_f|| = 0$ where $(f_j)$ is a sequence of vector-valued trigonometric polynomials whose coefficients are radial functions. For these $T_f$ we obtain explicit descriptions of their essential spectra. Moreover, we show that $f \in X$, whenever $T_f$ is compact, and characterize these functions in a simple and straightforward way. Finally, we determine those $f \in L_1$ where $T_f$ is a Hilbert-Schmidt operator.

1. Introduction
Let $T^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_k| = 1, k = 1, \ldots, n\}$ and consider the normalized Haar measure $d\varphi$ on $T^n$. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ we use the following notation. Put $z^m = \prod_{j=1}^n z_j^{m_j}$. We write $r \cdot z = (r z_1, \ldots, r z_n)$ if $r = (r_1, \ldots, r_n)$. Furthermore we put $z = r \cdot \exp(i\varphi)$ if $z_j = r_j e^{i\varphi_j}$ and $\varphi = (\varphi_1, \ldots, \varphi_n)$. Finally, we define $|m| = |m_1| + \ldots + |m_n|.$

Let $\mu$ be a bounded positive measure on $\mathbb{R}^n_+$ with supp $\mu \cap$ interior of $\mathbb{R}_+^n \neq \emptyset$ and consider, for $f, g : \mathbb{C}^n \to \mathbb{C},$

$$\langle f, g \rangle = \int \int f(r \cdot \exp(i\varphi)) \overline{g(r \cdot \exp(i\varphi))} d\varphi d\mu(r), \quad ||f||_2 = \sqrt{\langle f, f \rangle}.$$

We only deal with those $\mu$ which are such that all polynomials on $\mathbb{C}^n$ are elements of $L_2(d\varphi \otimes d\mu)$. (This is always satisfied if $\mu$ has compact support.)

Let $H_2(\mu)$ be the closure of the subspace of all polynomials in $L_2(d\varphi \otimes d\mu)$. $H_2(\mu)$ may be interpreted as a space of holomorphic functions where

$$M_2(f, r) := \left( \int |f(r \cdot \exp(i\varphi))|^2 d\varphi \right)^{1/2}.$$

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is $L_2(\mu)$-bounded with respect to $r$.

**Examples.** Let $\lambda$ be the Lebesgue measure on $\mathbb{R}^n$.

1. $d\mu(r) = (\prod_{j=1}^{n} r_j) e^{-\sum_{j=1}^{n} r_j^2/2} d\lambda(r)$. Here $H_2(\mu)$ is the Fock space ([8]).

2. $d\mu(r) = (\prod_{j=1}^{n} r_j) 1_{[0,1]^n}(r) d\lambda(r)$. Here $H_2(\mu)$ can be identified with the Bergman space on the polydisc $D^n$ ([5,6,11]), i.e.

$$H_2(\mu) \cong \left\{ f : D^n \to \mathbb{C} : f \text{ holomorphic, } \int_{D^n} |f|^2 d\tilde{\lambda} < \infty \right\},$$

where $\tilde{\lambda}$ is the Lebesgue measure on $\mathbb{C}^n$.

3. $\mu = \delta_{(1,\ldots,1)}$ (Dirac measure at $(1, \ldots, 1)$). Here $H_2(\mu)$ yields the classical Hardy space on the polydisc $D^n$ ([6,11]), i.e.

$$H_2(\mu) \cong \left\{ f : D^n \to \mathbb{C} : f \text{ holomorphic, } \sup_{r \in [0,1]} M_2(f, r) < \infty \right\}.$$

4. $\mu = \sum_{j=1}^{\infty} 2^{-k} f_k \nu_k$, where $\nu_k$ is a product of measures of the preceding kind and the $f_k \in L_1(d\nu_k)$ are non-negative.

It is one of our goals to give a unifying approach to these and to similar examples.

1.1. **Definition.** Let $f \in L_\infty := L_\infty(d\varphi \otimes d\mu)$ and consider the orthogonal projection $P : L_2(d\varphi \otimes d\mu) \to H_2(\mu)$. The Toeplitz operator $T_f : H_2(\mu) \to H_2(\mu)$ is defined by $T_f h = P(f \cdot h)$, $h \in H_2(\mu)$.

Clearly, $||T_f|| \leq ||f||_\infty$. However, equality does not hold in general.

A function $f : \mathbb{C}^n \to \mathbb{C}$ is called radial if $f(r \cdot \exp(i\varphi)) = f(r)$ for all $r \cdot \exp(i\varphi) \in \mathbb{C}^n$. $f$ is called angular if $f(r \cdot \exp(i\varphi)) = f(\exp(i\varphi))$ whenever $r \cdot \exp(i\varphi) \in \mathbb{C}^n \setminus \{0\}$. Put, for $k \in \mathbb{Z}^n$,

$$\xi_k(r \cdot \exp(i\varphi)) = \prod_{j=1}^{n} e^{ik_j \varphi_j}.$$ 

So $\xi_k$ is angular.

Note that any $f \in L_\infty(d\varphi \otimes d\mu)$ has a Fourier series expansion

$$\sum_{k \in \mathbb{Z}^n} F_k \cdot \xi_k,$$

where the Fourier coefficients $F_k$ are radial functions. Here

$$F_k(r) = \int f(r \cdot \exp(i\varphi)) \xi_{-k}(r \cdot \exp(i\varphi)) d\varphi.$$ 

This series converges, for fixed $r$, $\mu-a.e.$ in the $L_2(d\varphi)$-sense. Using the dominated convergence theorem we see that the series converges to $f$ in $L_2 := L_2(d\varphi \otimes d\mu)$. We sometimes write $f \overset{(L_2)}{=} \sum_k F_k \xi_k$.

Define
Then \( \{ e_m : m \in \mathbb{Z}_+^n \} \) is a complete ON-system for \( H_2(\mu) \). For \( h = \sum_{i \in \mathbb{Z}_+^n} \beta_i e_i \in H_2(\mu) \) put \( P_j h = \sum_{|j| < j} \beta_i e_i, \ j \in \mathbb{Z}_+ \), in particular, \( P_0 = 0 \).

1.2. **Proposition.** Let \( f \in L_\infty \) and \( h \in H_2(\mu) \).
If \( f = \sum_{k \in \mathbb{Z}_+^n} F_k \xi_k \), \( F_k \) radial, and \( h = \sum_{i \in \mathbb{Z}_+^n} \beta_i e_i \) then we have

\[
T_f h = \sum_{m \in \mathbb{Z}_+^n} \left( \sum_{i \in \mathbb{Z}_+^n} \frac{\int F_m i r^{m+1} d\mu}{\sqrt{\int r^{2m} d\mu \int r^{2i} d\mu}} \beta_i \right) e_m.
\]

In particular, for radial \( F \),

\[
T_F h = \sum_{m \in \mathbb{Z}_+^n} \left( \frac{\int F_{m} r^{m} d\mu}{\int r^{2m} d\mu} \beta_m e_m. \right.
\]

**Proof.** By definition of \( T_f \) we obtain

\[
T_f h = \sum_{m \in \mathbb{Z}_+^n} \langle f \cdot h, e_m \rangle e_m.
\]

Using the Fourier expansion of \( f \) and the fact that

\[
\langle F_k \xi_k e_l, e_m \rangle = \begin{cases} \frac{\int F_{k} r^{m} d\mu}{\sqrt{\int r^{2m} d\mu \int r^{2i} d\mu}} & \text{if } k + l = m \\ 0 & \text{else} \end{cases}
\]

we derive the first assertion. The second equation follows from the first one by putting \( l = m \).

2. **The spaces \( X \) and \( X_c \)**

Let \( \mathcal{L}(H_2(\mu)) \) be the space of all bounded linear operators on \( H_2(\mu) \) and \( \mathcal{K} \subset \mathcal{L}(H_2(\mu)) \) the space of all compact operators. Moreover let \( q : \mathcal{L}(H_2(\mu)) \to \mathcal{L}(H_2(\mu))/\mathcal{K} \) be the quotient map and define \( \tau : L_\infty \to \mathcal{L}(H_2(\mu)) \) by \( \tau(f) = T_f, \ f \in L_\infty \). \( \tau \) is a linear map.

Recall that \( \mathcal{L}(H_2(\mu)) \) is the dual Banach space for the trace class operators on \( H_2(\mu) \). With respect to this duality, \( \mathcal{L}(H_2(\mu)) \) is the bidual of \( \mathcal{K} \) ([7]).

Functions of the form \( \sum_{|k| \leq j} F_k \xi_k \) for some integer \( j \) and radial \( L_\infty \)-functions \( F_k \) will be called \( L_\infty(d\mu) \)-valued trigonometric polynomials.

Now we introduce our main objects of study.
2.1. **Definition.** Put

\[ X = \{ f \in L_\infty : \text{there is a sequence of } L_\infty(\mu)\text{-valued trigonometric polynomials } f_j \text{ with } \lim_j ||qT_f - qT_j|| = 0 \}, \]

\[ X_c = \{ f \in L_\infty : \text{there is a sequence of } L_\infty(\mu)\text{-valued trigonometric polynomials } f_j \text{ with } \lim_j ||f_j - f||_\infty = 0 \}. \]

We have \( X_c \subset X \). Note, \( X_c \) contains all \( L_\infty(\mu)\)-valued trigonometric polynomials. So there are many discontinuous functions which are elements of \( X_c \) (and hence of \( X \)), for example all radial \( L_\infty \)-functions. The most important property of \( X \) is the following: If \( T_f \) is compact then \( f \) is always an element of \( X \) (by definition of \( X \)).

If \( n = 1 \) we give an explicit description of the maximal ideal space of the \( C^* \)-algebra generated by \( \{ qT_f : f \in X \} \), which turns out to be commutative under some restrictions on \( \mu \) (Theorem 5.3.). In particular we describe \( ||qT_f|| \) and determine the essential spectrum of \( T_f \) for \( f \in X \) (Corollary 5.4.). Finally, for arbitrary \( n \), we characterize those \( f \in X \) where \( T_f \) is compact and those \( f \in L_\infty \) where \( T_f \) is a Hilbert-Schmidt operator (section 6).

2.1. **Lemma.** (a) Let \( f, f_j \in L_\infty \) such that \( \lim_j ||f - f_j||_2 = 0 \) and \( \sup_j ||f_j||_\infty < \infty \). Then, for any \( h \in H_2(\mu) \), we have \( \lim_j T_f h = T_f h \). Furthermore, \( T_f = w^* - \lim_j T_f \) with respect to the \( w^*\)-topology on \( L(H_2(\mu)) \).

(b) Assume that, for \( f, f_j \in L_\infty, \lim_j ||qT_f - qT_j|| = 0 \) and \( \lim_j T_f h = T_f h, \ h \in H_2(\mu) \). Then there is a sequence of convex combinations \( g_k \) of \( f_j \) such that \( \lim_k ||T_f - T_g_k|| = 0 \).

**Proof.** (a) Fix \( h \in H_2(\mu) \) and take, for \( \epsilon > 0 \), \( \tilde{h} \in L_\infty \) with \( ||h - \tilde{h}||_2 \leq \epsilon \). We have

\[ ||T_f h - T_f \tilde{h}||_2 \leq \epsilon \sup \ ||f - f_j||_\infty + ||\tilde{h}||_\infty ||f_j - f||_2. \]

Hence

\[ \lim_{j \to \infty} \sup \ ||T_f h - T_f j||_2 \leq \epsilon \sup \ ||f - f_j||_\infty. \]

We obtain \( \lim_j ||T_f h - T_f j||_2 = 0 \) since \( \epsilon \) was arbitrary. For the second part of (a) let \( T \) be a trace class operator on \( H_2(\mu) \) with complete ON-systems \( (f_k), (g_l) \) and singular numbers \( \lambda_k \) such that

\[ Th = \sum_k \lambda_k \langle h, f_k \rangle g_k, \ h \in H_2(\mu), \ \text{and} \ \sum_k |\lambda_k| < \infty. \]

Then according to the duality on \( L(H_2(\mu)) \) ([7]),
\[ \langle T, T_f \rangle := \text{trace}(TT_f) = \sum_m \langle TT_f; g_m, g_m \rangle = \sum_k \lambda_k \langle T_f; g_k, f_k \rangle. \]

Since \( \lim_j \langle T_f; g_k, f_k \rangle = \langle T_f; g_k, f_k \rangle \) for all \( k \) we see that \( \lim_j \langle T, T_f \rangle = \langle T, T_f \rangle. \)

(b) We find \( K_j \in \mathcal{K} \) with \( \lim_j \| (T_f - T_j) P_k \| = 0 \) for all \( k \). Since \( T_f \to T_f \) in the strong operator topology, applying the basis projections \( P_k \), we obtain \( \lim_j \| (T_f - T_f) P_k \| = 0 \) for all \( k \). Moreover \( \lim_j \| (T_f - T_j) P_k + K_j P_k \| = 0 \), so \( \lim_j \| K_j h \|_2 = 0 \) for all \( h \in H_2(\mu) \). We infer, as in (a), that \( K_j \to 0 \) weakly since \( \mathcal{K}^* \) is the space of all trace class operators. By Mazur’s theorem ([2]) there is a suitable sequence \( H_k = \sum_{j=a_k}^{b_k} \lambda_{j,k} K_j \) of convex combinations of \( K_j \) with \( \lim_k \| H_k \| = 0 \) and \( a_k \to \infty \). Denote the corresponding convex combinations of the \( f_j \) by \( g_k \). We conclude \( \lim_k \| T_f - T_g_k \| = 0 \).

For \( f = \sum_{k \in \mathbb{Z}^n} F_k \xi_k \), \( F_k \) radial, define the Cesaro means \( \sigma_j f \) by

\[ \sigma_j f = \sum_{|k| \leq j} \frac{j - |k|}{j} F_k \xi_k. \]

We always have \( \| \sigma_j f \|_p \leq \| f \|_p \), if \( p = 2 \) or \( p = \infty \) and \( \lim_j \| f - \sigma_j f \|_2 = 0 \) ([3], apply \( \sigma_j \) to the function \( f_z(w) = f(wz) \) for fixed \( z \in \mathbb{C}^n \) and \( w \in \mathbb{C} \)).

Put, for \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{T}^n \) and \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \),

\[ f_{\lambda}(z) = f(\lambda_1 z_1, \ldots, \lambda_n z_n). \]

Then we obtain \( \| f \|_p = \| f_{\lambda} \|_p \) if \( p = 2 \) or \( p = \infty \).

Let \( T \in \mathcal{Z}_r \). Frequently, we make use of the fact that

\[ \| q T \| = \inf_j \| T(id - P_j) \| = \inf_j \| (id - P_k) T \|. \]

2.2. Lemma. We have

(a) \( T_{f_{\lambda}} h = (T_{f_{\lambda}})_{h} \) if \( \lambda \in \mathbb{T}^n \) and \( h \in H_2(\mu) \),

(b) \( \| T_{\sigma f} \| \leq \| T \| \) and \( \| q T_{\sigma f} \| \leq \| q T \| \) for every \( j \in \mathbb{Z}_+ \).

Proof. (a) Here \( f_{\lambda} \equiv \sum_k F_k \lambda^k \xi_k \) if \( f \equiv \sum_k F_k \xi_k \). Hence, (a) follows from Proposition 1.2.

(b) Let \( \Gamma_j(w) \) be the Fejer kernel with

\[ \Gamma_j(w) = \sum_{|k| \leq j} \frac{\lambda - |k|}{j} w^k, \quad w \in \mathbb{T}. \]

Extending the preceding notation we define, for \( h \in H_2(\mu) \) and \( w \in \mathbb{T} \),

\[ h_w(z) = h(w \cdot z), \quad \text{if } z \in \mathbb{C}^n, \]
(i.e. $h_w = h_{(w,...,w)}$ in the former notation). Then, using Fubini’s theorem and Cauchy-Schwarz inequality, we have, for $h \in H_2(\mu)$,

$$\|T_{\sigma,f}h\|_2^2 = \iint \left| \iint (T_j h e^{-i\psi})_{\mu} \Gamma_j(e^{-i\psi}) d\psi \right|^2 d\varphi d\mu \leq \sup \|T_j h e^{-i\psi}\|_2^2.$$ 

This implies $\|T_{\sigma,f}\| \leq \|T_f\|$. Moreover, if $h \in (\text{id} - P_j) H_2(\mu)$ then $h_\lambda \in (\text{id} - P_j) H_2(\mu)$ for any $\lambda \in \mathbb{T}^n$. Hence the preceding yields $\|T_{\sigma,f}(\text{id} - P_j)\| \leq \|T_j(\text{id} - P_j)\|$ for any $j$ from which we infer $\|qT_{\sigma,f}\| \leq \|qT_f\|$. 

2.3. Proposition. We obtain

$$X = \{ f \in L_\infty : \lim_j \|T_f - T_{\sigma,f}\| = 0 \text{ for some } L_\infty(d\mu)-\text{valued trigonometric polynomials } f_j \} = \{ f \in L_\infty : \lim_j \|T_f - T_{\sigma,f}\| = 0 \}.$$ 

Proof. Put

$$Y = \{ f \in L_\infty : \lim_j \|T_f - T_{\sigma,f}\| = 0 \text{ for some } L_\infty(d\mu)-\text{valued trigonometric polynomials } f_j \}.$$ 

Then clearly, $Y \subset X$. Conversely, let $f \in X$ and let $f_j$ be $L_\infty(d\mu)$-valued trigonometric polynomials with $\lim_j \|qT_f - qT_{f,j}\| = 0$. We obtain easily $\lim_k \|f_j - \sigma_k f_j\|_\infty = 0$ for each $j$. Fix $\epsilon > 0$ and $j$ with $\|qT_{f-j}\| \leq \epsilon/3$ and find $k_j$ with $\|f_j - \sigma_k f_j\|_\infty \leq \epsilon/3$ for all $k \geq k_j$. We conclude, using Lemma 2.2(b),

$$\|qT_f - qT_{\sigma,f}\| \leq \|qT_{f-j}\| + \|qT_{j-\sigma,f}\| + \|qT_{\sigma(f-j)}\| \leq \epsilon.$$ 

Thus $\lim_k \|qT_f - qT_{\sigma,f}\| = 0$. In view of Lemma 2.1. we find suitable convex combinations $g_j$ of the $\sigma_k f$ such that $\lim_k \|T_f - T_{g_k}\| = 0$. This yields the first part of the proposition. Finally, a 3e--proof as before now shows that even $\lim_k \|T_f - T_{\sigma,f}\| = 0$.

3. Conditions on the measure $\mu$

Before we come to the main results in sections 4 and 5 we discuss moment conditions on $\mu$ which are needed in the proofs lateron. Here we restrict ourselves to the case of $n = 1$. So let $\mu$ be a measure on $\mathbb{R}_+$. 

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3.1. Definition. Consider

\begin{align*}
(\text{I}) & \quad \lim_{m \to \infty} \int \left| \frac{s_{m-k}}{r^{m-k}d\mu} - \frac{s_m}{r^m d\mu} \right| d\mu(s) = 0 \text{ for all } k \in \mathbb{Z}_+ \\
(\text{II}) & \quad \lim_{m \to \infty} \int \frac{r^m d\mu}{r^{m-k} d\mu} \int \frac{r^{m-l-k} d\mu}{r^m d\mu} = 1 \text{ for all } k, l \in \mathbb{Z}_+
\end{align*}

Examples. If \( \mu \) is a Dirac measure then (I) and (II) are satisfied. An elementary calculation shows that \( \mu \) of the Fock space (section 1) satisfies (I) and (II), too. Similarly \( d\mu(r) = e^{-r} dr \) fulfils the conditions of Definition 3.1. The next Proposition implies that the measure of the Bergman space is also included. Indeed, we have

3.2. Proposition. Let \( \mu \) have bounded support and assume that \( a = \sup(\text{supp } \mu) \). Then \( \mu \) satisfies (I) and (II).

Proof. We show

\[(\ast) \quad \lim_{m \to \infty} \int \frac{r^{m-k} d\mu}{r^m d\mu} a^{-k} = \text{ for all } k \in \mathbb{Z}_+.
\]

(II) is a direct consequence of (\( \ast \)). By assumption, for \( 0 < \delta < 1 \), we have \( 0 < \int_{(1-\delta)a}^{a} r d\mu \). Moreover,

\[\mu([0,a]) \leq \mu([0,(1-\delta)a]) + \mu([(1-\delta)a,a]).\]

Hence

\[a^{-k} \leq \frac{\int_{0}^{a} r^{m-k} d\mu}{\int_{0}^{a} r^{m} d\mu} \leq \frac{(1-\delta)^m a^m}{(1-\delta/2)^m a^m} (1-\delta)^{-k} a^{-k} \int_{0}^{(1-\delta)a} r d\mu + (1-\delta)^{-k} a^{-k} \int_{(1-\delta)a}^{a} r^m d\mu.
\]

The right-hand side converges to \( (1-\delta)^{-k} a^{-k} \) as \( m \to \infty \). Since \( \delta \) was arbitrary we obtain (\( \ast \)) and hence (II). To prove (I) observe that

\[\int \left| \frac{s_{m-k}}{r^{m-k} d\mu} - \frac{s_m}{r^m d\mu} \right| d\mu = \frac{\int s_{m-k} \left| 1 - s^k \frac{r^{m-k} d\mu}{r^m d\mu} \right| d\mu}{\int r^{m-k} d\mu}.
\]

With \( C = \sup_{m}(\int_{0}^{a} r^{m-k} d\mu / \int_{0}^{a} r^m d\mu) \) and \( 0 < \delta < 1 \) as above we obtain
\[0 \leq \frac{\int_0^a s^{m-k} \left| 1 - s^k \frac{\int_0^a r^{m-k} d\mu}{\int_0^a r^m d\mu} \right| d\mu}{\int_0^a r^{m-k} d\mu} \leq \frac{(1 - \delta)^{m-k} d^{m-k}}{(1 - \delta/2)^{m-k} d^{m-k}} (1 + a^k C) \frac{\int_0^a (1-\delta)^{m-k} d\mu}{\int_0^a (1-\delta/2)^{m-k} d\mu} + \max \left( \frac{a^k \int_0^a r^{m-k} d\mu}{\int_0^a r^m d\mu} - 1, \left| 1 - a^k (1 - \delta)^k \frac{\int_0^a r^{m-k} d\mu}{\int_0^a r^m d\mu} \right| \right) \frac{\int_0^a r^{m-k} d\mu}{\int_0^a (1-\delta)^{m-k} d\mu}.\]

With \((*)\) the right-hand side tends to \(1 - (1 - \delta)^k\) as \(m \to \infty\). Since \(\delta\) was arbitrary we obtain (I).

4. The algebra generated by \(q^r(X)\)

Here we study \(q^r(X) \subset \mathcal{L}(H_2(\mu))/\mathcal{R}\). Again, let \(n = 1\). At first we show

4.1. Proposition. Let \(\mu\) satisfy (I) and (II). Then for any radial \(F\) and \(k, l \in \mathbb{Z}\) we have

(a) \[q(T_{F^k}) = q(T_F) \cdot q(T_{e_l}) = q(T_{e_l}) \cdot q(T_F) \quad \text{and} \]

(b) \[q(T_{e_l}^k) = q(T_{e_l}) \cdot q(T_{e_l}).\]

**Proof.** Let \(h = \sum_{i \in \mathbb{Z}_+} \beta_i e_i \in H_2(\mu)\). Then, in view of Proposition 1.2.,

\[T_{F^k \xi} h = \sum_{m \geq \max(k,0)} \frac{\int F_r^{2m-k} d\mu}{\sqrt{\int r^{2m} d\mu} \int r^{2m-2k} d\mu} \beta_{m-k} e_m \quad \text{and} \]

\[T_{\xi} h = \sum_{m \geq \max(k,0)} \frac{\int r^{2m-k} d\mu}{\sqrt{\int r^{2m} d\mu} \int r^{2m-2k} d\mu} \beta_{m-k} e_m.\]

Hence

\[T_F T_{\xi} h = \sum_{m \geq \max(k,0)} \left( \frac{\int F_r^{2m} d\mu}{\sqrt{\int r^{2m} d\mu} \int r^{2m-2k} d\mu} \right) \left( \frac{\int r^{2m-k} d\mu}{\sqrt{\int r^{2m} d\mu} \int r^{2m-2k} d\mu} \right) \beta_{m-k} e_m.\]

We obtain

\[(T_{F^k} - T_F T_{\xi}) h = \sum_{m \geq \max(k,0)} \frac{\int F(s)^{2m-k} \left( 1 - s^k \frac{\int r^{2m-k} d\mu}{\sqrt{\int r^{2m} d\mu} \int r^{2m-2k} d\mu} \right) d\mu(s)}{\sqrt{\int r^{2m} d\mu} \int r^{2m-2k} d\mu} \beta_{m-k} e_m.\]

So, for \(j \in \mathbb{Z}_+\) and the basis projections \(P_j\) (section 1),
\[
\| (\text{id} - P_j)(T_{F_{\xi_k}} - T_{F_k}) \| \leq \sup_{m \geq j} \frac{\int F_s^{2m-k} \left( 1 - s^k \frac{\int r^{2m-k} d\mu}{\int r^{2m-k} d\mu} \right) d\mu}{\sqrt{\int r^{2m-k} d\mu}} \leq \| F \|_\infty \sup_{m \geq j} \frac{\int s^{2m-k} \left| 1 - s^k \frac{\int r^{2m-k} d\mu}{\int r^{2m-k} d\mu} \right| d\mu}{\int r^{2m-k} d\mu}.
\]

(Here we used the Cauchy-Schwarz inequality.) In view of condition (I) the right-hand side tends to 0 as \( j \to \infty \). This implies \( T_{F_{\xi_k}} - T_F T_{\xi_k} \in \mathcal{H} \). Similarly we obtain

\[
(T_{F_{\xi_k}} - T_{\xi_k} T_F) h = \sum_{m \geq \max(k,0)} \int F_s^{2m-k} \left( s^k - \frac{\int r^{2m-2k} d\mu}{\int r^{2m-2k} d\mu} \right) d\mu \beta_{m-k} e_m
\]

and

\[
\| (\text{id} - P_j)(T_{F_{\xi_k}} - T_{\xi_k} T_F) \| \leq \| F \|_\infty \sup_{m \geq j} \frac{\int s^{2m-2k} \left| s^k - \frac{\int r^{2m-2k} d\mu}{\int r^{2m-2k} d\mu} \right| d\mu}{\int r^{2m-k} d\mu}.
\]

Again by (I), \( T_{F_{\xi_k}} - T_{\xi_k} T_F \in \mathcal{H} \). We conclude (a). To prove (b) we derive from Proposition 1.2.

\[
T_{\xi_k} T_{\xi_k} h = \sum_{m \geq \max(k+l,l,0)} \left( \int \frac{r^{2m-l} d\mu}{\sqrt{\int r^{2m-l} d\mu} \int r^{2m-2l} d\mu} \right) \left( \int \frac{r^{2m-2l-k} d\mu}{\sqrt{\int r^{2m-2l-k} d\mu} \int r^{2m-2l-2k} d\mu} \right) \beta_{m-k-l} e_m
\]

and hence, for \( j \in \mathbb{Z}_+ \) with \( j > \max(k+l,l,0) \),

\[
(\text{id} - P_j)(T_{\xi_{j+k}} - T_{\xi_j} T_{\xi_k}) h = \sum_{m \geq j} \left( \int \frac{r^{2m-l-k} d\mu}{\sqrt{\int r^{2m-l-k} d\mu} \int r^{2m-2l-2k} d\mu} \right) \left( 1 - \int \frac{r^{2m-l-k} d\mu}{\sqrt{\int r^{2m-l-k} d\mu} \int r^{2m-2l-2k} d\mu} \right) \beta_{m-k-l} e_m.
\]

This implies
\[ \|(id - P_j)(T_{\xi+k} - T_{\xi}T_{\xi+k})\| \]
\[ \leq \sup_{m \geq j} \left( \frac{\int r^{2m-1-k}d\mu}{\sqrt{\int r^{2m}d\mu \int r^{2m-2l-k}d\mu}} \right) \left( 1 - \frac{\int r^{2m-2l-k}d\mu}{\int r^{2m-1-k}d\mu \int r^{2m-2l}d\mu} \right) \]
\[ \leq \sup_{m \geq j} \left| 1 - \frac{\int r^{2m-1-k}d\mu}{\int r^{2m-2l-k}d\mu \int r^{2m-2l}d\mu} \right| \]

(For the latter estimate we used the Cauchy-Schwarz inequality.) The right-hand side tends to 0 as \( j \to \infty \) according to condition (II). We obtain \( T_{\xi+k} - T_{\xi}T_{\xi+k} \in \mathcal{K} \) which yields (b).

**Remark.** Proposition 4.1.(a) remains valid for arbitrary \( n \) with an analogous proof. However 4.1.(b) is no longer true for \( n > 1 \). Here \( T_{\xi}T_{\xi+k} - id \) is not compact in general.

4.2. **Corollary.** If \( \mu \) satisfies (I) and (II) then \( q\tau(X) \) generates a commutative \( C^* \)-algebra, hence a \( C(K) \)-space.

**Proof.** This is an easy consequence of Proposition 4.1. and the fact that \( \{qT_f : f \in L_\infty(d\mu)\text{-valued trigonometric polynomial}\} \) is dense in \( q\tau(X) \).

5. **The functions** \( \Phi_\mathcal{U}(f) \)

Here we want to characterize the maximal ideal space of the algebra generated by \( q\tau X \). **Throughout this section let** \( n = 1 \) **and let** \( \mu \) **satisfy (I) and (II).**

Let \( f \in L_\infty = L_\infty(d\varphi \otimes \mu) \). Recall, \( \int f(r \cdot \exp(i\varphi))r^{2m}d\mu(r)/\int r^{2m}d\mu(r) \) is an element of \( L_\infty(d\varphi) = L_\infty^1(d\varphi) \). Let \( \mathcal{U} \) be a free ultrafilter on \( \mathbb{Z}_+ \). The limit along \( \mathcal{U} \) will be denoted by \( \lim_{m,\mathcal{U}} \). Put, for \( z = \exp(i\varphi) \in T \),

\[ \Phi_\mathcal{U}(f)(z) = w^* - \lim_{m,\mathcal{U}} \left( \frac{\int f(r \cdot \exp(i\varphi))r^{2m}d\mu}{\int r^{2m}d\mu} \right) . \]

Then \( \Phi_\mathcal{U} \) is linear in \( f \). Moreover, \( \Phi_\mathcal{U}(f) \in L_\infty(d\varphi) \) and \( ||\Phi_\mathcal{U}(f)||_{\infty} \leq ||f||_{\infty} \).

5.1. **Lemma.** (a) If \( f \overset{(L_2)}{=} \sum_{k \in \mathbb{Z}} F_k \xi_k \) for radial \( F_k \) then we have

\[ \Phi_\mathcal{U}(f) \overset{(L_2)}{=} \sum_{k \in \mathbb{Z}} \left( \lim_{m,\mathcal{U}} \frac{\int F_k(r)r^{2m}d\mu}{\int r^{2m}d\mu} \right) \xi_k . \]

(b) For any \( \mathcal{U} \) there is a suitable sequence \( N \subset \mathbb{Z}_+ \) with \( \Phi_\mathcal{U}(f) = w^* - \lim_{m \in N} \left( \int f(r^{2m}d\mu)/\int r^{2m}d\mu \right) . \)

(c) \( \Phi_\mathcal{U}(f) = f \) if \( f \) is angular.

(d) \( \Phi_\mathcal{U}(F) = \lim_{m,\mathcal{U}} \left( \int F(r)r^{2m}d\mu/\int r^{2m}d\mu \right) \) if \( F \) is radial. Hence \( \Phi_\mathcal{U}(F) \) is a constant function.
(e) Let \( a = \sup \text{supp} \mu \) (a can be \( \infty \)). Assume that \( \lim_{r \to a} f(r \cdot \exp(i\varphi)) \) exists a.e. on \( T \). Then
\[
\Phi_\#(f)(\exp(i\varphi)) = \lim_{r \to a} f(r \cdot \exp(i\varphi)).
\]

**Proof.** Put \( \Phi_m(f) = \int \frac{f r^{2m} dm}{r^{2m} dm} \). Then \( \Phi_m(f) \) is uniformly bounded in \( L_\infty(d\varphi) \) and
\[
\Phi_m(f) = \sum_k \left( \frac{\int F_k r^{2m} d\mu}{\int r^{2m} d\mu} \right) \xi_k.
\]

Since the unit ball of \( L_\infty(d\varphi) \) is \( w^* \)-sequentially compact we find a sequence \( N \in \mathcal{U} \) such that \( \Phi_\#(f) = w^* - \lim_{m \in N} \Phi_m(f) \). The Fourier coefficients of \( \Phi_\#(f) \) are \( \lim_{m \in N} \int F_k r^{2m} dm \). This proves (a) and (b). The remaining assertions are straightforward.

5.2. **Lemma.** For any \( f \in L_\infty \) with \( f \equiv \sum_k F_k \xi_k \), \( F_k \) radial, we have
\[
\lim_{m \in \mathcal{U}} \frac{\int F_k r^{2m} dm}{r^{2m} dm} = \lim_{m \in \mathcal{U}} \langle T_f e_{m-k}, e_m \rangle, \quad k \in \mathbb{Z}.
\]

In particular \( \lim_{m \in \mathcal{U}} \frac{\int F_k r^{2m} dm}{r^{2m} dm} \leq ||q T_f|| \) for all \( k \).

**Proof.** We have, with Proposition 1.2.,
\[
\langle T_f e_{m-k}, e_m \rangle = \frac{\int F_k r^{2m-k} dm}{\sqrt{\int r^{2m} dm} \int r^{2m-2k} dm} =
\]
\[
\left( \frac{\int r^{2m-k} dm \int r^{2m-k} dm}{\int r^{2m} dm \int r^{2m-2k} dm} \right)^{1/2} \left( \frac{\int F_k r^{2m} dm}{\int r^{2m} dm} + \int F_k \frac{r^{2m-k}}{s^{2m-k} \int s^{2m} dm} \right) d\mu.
\]

The first result follows by applying (I) and (II) (with \( l = k \)). Finally, we obtain for any \( j \in \mathbb{Z}_+ \),
\[
|\lim_{m \in \mathcal{U}} \langle T_f e_{m-k}, e_m \rangle| = |\lim_{m \in \mathcal{U}} \langle T_f (\text{id} - P_j) e_{m-k}, e_m \rangle| \leq ||T_f (\text{id} - P_j)||.
\]

Since \( ||q T_f|| = \inf_j ||T_f (\text{id} - P_j)|| \) we infer the second result.

If \( \Phi_\#(f) \in L_\infty(d\varphi) \) can be represented by a continuous function, we shall always identify \( \Phi_\#(f) \) with its continuous representative. For a commutative Banach algebra \( A \) let \( \text{Spec}(A) \) be the maximal ideal space. Finally, let \( \mathcal{A} \) be the closed subalgebra of \( \mathcal{L}(H_2(\mu)) / \mathcal{K} \) generated by \( q \tau X \).
5.3. Theorem. For any \( f \in X \) the function \( \Phi_{\mathcal{A}}(f) \) is continuous. Moreover,

\[
\text{Spec}(\mathcal{A}) \circ q \circ \tau|_X = \{ \Phi_{\mathcal{A}}(\cdot)(z)|_X : z \in T, \ \mathcal{U} \text{ a free ultrafilter on } \mathbb{Z}_+ \}.
\]

Proof. (a): At first, a few introductory remarks.

Let \( Y = \text{closed span of } \{ \xi_k : k \in \mathbb{Z} \} \subset L_\infty \). Then, in view of the Weierstrass theorem, \( Y \) can be identified with \( C(T) \), the continuous functions on \( T \). By Proposition 4.1., \( \overline{\tau Y} \) is a commutative \( C^* \)-algebra.

Put \( \mathcal{B} = \text{closed subalgebra of } \mathcal{L}(H_2(\mu)) \) generated by \( \{ T_F : F \text{ radial} \} \). According to Proposition 1.2., \( \mathcal{B} \) is a commutative \( C^* \)-algebra which consists of multipliers, i.e. if \( T \in \mathcal{B} \) then there is a bounded sequence \( (a_k) \) with \( T[\sum_k \beta_k e_k] = \sum_k a_k \beta_k e_k \). Put \( \Phi_k(T) = a_k \). Then \( \Phi_k \in \text{Spec}(\mathcal{B}) \). Moreover \( ||T|| = \sup_k |\Phi_k(T)| \). Hence \( \text{Spec}(\mathcal{B}) = w^*-\text{closure of } \{ \Phi_k : k \in \mathbb{Z}_+ \} \).

The definition of \( \mathcal{A} \) and Proposition 4.1. imply \( \mathcal{A} = q\mathcal{B} \otimes_q \tau Y \). We have \( \text{Spec}(\mathcal{A})|_{q\mathcal{B}} = \text{Spec}(q\mathcal{B}) \) and \( \text{Spec}(\mathcal{A})|_{q\tau Y} = \text{Spec}(q\tau Y) \). Put

\[
\Omega = \{ \Phi_{\mathcal{A}}(\cdot)(z)|_X : z \in T, \ \mathcal{U} \text{ a free ultrafilter on } \mathbb{Z}_+ \}.
\]

(b): Now let \( \Psi \in \text{Spec}(\mathcal{A}) \). For radial \( F \in L_\infty \) and \( k \in \mathbb{Z} \) we obtain, by Proposition 4.1.,

\[
\Psi(qT_{F_\lambda}) = \Psi(qT_F) \cdot \Psi(qT_{\xi_k})
\]

and \( \Psi \circ q \circ \tau|_Y \in \text{Spec}(Y) \). Hence there is \( z \in T \) such that \( \Psi \circ q \circ \tau|_Y \) is the Dirac functional \( \delta_z \). Moreover \( \Psi \circ q|_\mathcal{B} \in \text{Spec}(\mathcal{B}) \) and \( \Psi \circ q|_X = 0 \). This implies, for any \( T \in \mathcal{B} \) and \( j \in \mathbb{Z}_+ \), \( \Psi(qP_j T) = 0 \). Hence there is a free ultrafilter \( \mathcal{U} \) on \( \mathbb{Z}_+ \) with \( \Psi \circ q|_\mathcal{B} = w^* - \text{lim}_{k, \mathcal{U}} \Phi_k \) and therefore \( (\Psi(q\tau)|F) = \Phi_{\mathcal{A}}(F) \) if \( F \) is radial (in view of Proposition 1.2.). Thus, if \( f = \sum |k| F_k \xi_k \) is a \( L_\infty(d\mu) \)-valued trigonometric polynomial we have \( \Psi(qT_f) = \Phi_{\mathcal{A}}(f)(z) \).

(c): Conversely, let \( \mathcal{U} \) be a free ultrafilter on \( \mathbb{Z}_+ \). Then there is \( \Psi \in \text{Spec}(\mathcal{A}) \) with \( \Psi \circ q|_\mathcal{B} = w^* \text{lim}_{m, \mathcal{U}} \Phi_m \). Hence, for radial \( F \), \( \Psi(qT_F) = \Phi_{\mathcal{A}}(F) \). Since \( \Psi \in \text{Spec}(\mathcal{A}) \) there exists some \( z \in T \) with \( \Psi(qT_f) = f(z) = \Phi_{\mathcal{A}}(f)(z) \) if \( f \in Y \). We have

\[
\Phi_{\mathcal{A}}(f)(z) = \Phi_{\mathcal{A}}(f)(\lambda z)
\]

if \( z \in T \) and \( \lambda \in T \). So, using Lemma 2.2.(a) and Proposition 4.1., we obtain, for any \( w \in T \), an element \( \Psi \in \text{Spec}(\mathcal{A}) \) with \( \Psi(qT_f) = \Phi_{\mathcal{A}}(f)(w) \) if \( f \) is a \( L_\infty(d\mu) \)-valued trigonometric polynomial.

(d): (b) and (c) imply that \( \text{Spec}(\mathcal{A}) \circ q \circ \tau \) and \( \Omega \) coincide on the \( L_\infty(d\mu) \)-valued trigonometric polynomials. Now let \( f \in X \) and let \( (f_j) \) be a sequence of \( L_\infty(d\mu) \)-valued trigonometric polynomials with \( \lim_j ||qT_f - qT_{f_j}|| = 0 \). Since \( \mathcal{A} \) is a commutative \( C^* \)-algebra we conclude
\[
\lim_{j} \sup_{\varphi \in \text{Spec}(\varphi)} |\Psi(qT_j) - \Psi(qT_{j_i})| = 0.
\]

(b) and (c) yield \( ||qT_{j_i} - qT_j|| = \sup_{\varphi} ||\Phi_{\varphi}(f_j) - \Phi_{\varphi}(f_k)||_\infty \). This implies that, for any \( \mathcal{U}, (\Phi_{\varphi}(f_j)) \) is a \( || \cdot ||_\infty \)-Cauchy sequence of trigonometric polynomials on \( T \). Let \( \hat{\Phi} = \lim_{j} \Phi_{\varphi}(f_j) \). According to the second assertion of Lemma 5.2., the Fourier coefficients of \( \Phi \) coincide with those of \( \Phi_{\varphi}(f) \). Hence \( \Phi_{\varphi}(f) = \Phi \). In particular, \( \Phi_{\varphi}(f) \) is continuous. Finally, with (b) and (c), \( \text{Spec}(\varphi) \circ q \circ \tau|_X \) and \( \Omega \) coincide.

For \( T \in \mathcal{L}(H_2(\mu)) \) let \( \sigma_{\text{ess}}(T) \) be the spectrum of \( q(T) \) in \( \mathcal{L}(H_2(\mu)) / \mathcal{K} \).

5.4. COROLLARY. Let \( f \in X \). Then

\[
\sigma_{\text{ess}}(T_f) = \{ \Phi_{\varphi}(f)(z) : z \in T, \mathcal{U} \text{ a free ultrafilter on } Z_+ \}.
\]

Moreover, \( ||qT_f|| = \sup_{\varphi} ||\Phi_{\varphi}(f)||_\infty \).

In particular, \( T_f \) is a Fredholm operator if and only if \( \Phi_{\varphi}(f)(z) \neq 0 \) for all \( z \in T \) and all free ultrafilters \( \mathcal{U} \).

5.5. COROLLARY. Let \( f \in L_\infty \) be an angular function. Then

\[
||f||_\infty = ||qT_f|| = ||T_f||.
\]

Moreover, if \( f \) is continuous on \( T \) and angular then \( \sigma_{\text{ess}}(T_f) = f(T) \).

PROOF. If \( f \) is angular and continuous on \( T \) then \( f \in X \) and \( \Phi_{\varphi}(f) = f \). Hence \( \sigma_{\text{ess}}(T_f) = f(T) \) and \( ||f||_\infty = ||qT_f|| = ||T_f|| \). Now let \( f \in L_\infty \) be arbitrarily angular. Then \( \sigma_f \rightarrow f \) a.e. on \( T \) (\cite{10}). Moreover, all \( \sigma_f \) are angular and continuous on \( T \). We obtain, in view of Lemma 2.2.,

\[
||f||_\infty \leq \limsup_j ||\sigma_f||_\infty = \limsup_j ||qT_{\sigma_f}|| \leq ||qT_f|| \leq ||T_f|| \leq ||f||_\infty,
\]

hence equality.

6. Compact Toeplitz operators

Now, again, let \( n \) be an arbitrary positive integer. Throughout this section let \( f \in L_\infty \) and \( f \in L_2(\sum_{k \in \mathbb{Z}} f_k \xi_k) \).

At first we characterize those Toeplitz operators which are Hilbert-Schmidt operators.

6.1. PROPOSITION. \( T_f \) is a Hilbert-Schmidt operator if and only if

\[
\sum_{l \in \mathbb{Z}_+} \sum_{m \in \mathbb{Z}_+} \frac{\int F_{m-l} r^{m+l-1} d\mu^2}{\int r^{2m} d\mu \int r^{2l} d\mu} < \infty.
\]
Proof. $T_f$ is a Hilbert-Schmidt operator if and only if $\sum_{l \in \mathbb{Z}^n} ||T_f e_l||^2 < \infty$. Proposition 1.2. yields

$$||T_f e_l||^2 = \sum_{m \in \mathbb{Z}^n} \frac{\int F_{m-l} r^{2m} d\mu}{\int r^{2m} d\mu} < 1$$

which proves Proposition 6.1.

Now we determine those $f$ among the elements of $X$ where $T_f$ is compact. Recall that $f \in X$ whenever $f \in L^1$ and $T_f$ is compact.

6.2. Proposition. (a) $T_f$ is compact if and only if $f \in X$ and

$$\lim_{m \to \infty} \frac{\int F_k r^{2m-k} d\mu}{\int r^{2m} d\mu} = 0 \text{ for all } k \in \mathbb{Z}^n.$$

(b) Let $n = 1$ and let $\mu$ satisfy (I) and (II). Then $T_f$ is compact if and only if $f \in X$ and

$$\lim_{m \to \infty} \frac{\int F_k r^{2m} d\mu}{\int r^{2m} d\mu} = 0 \text{ for all } k \in \mathbb{Z}.$$

Proof. (a) If $T_f$ is compact then $f \in X$. Proposition 1.2. yields

$$\frac{\int F_k r^{2m-k} d\mu}{\int r^{2m} d\mu} = \langle T_f e_{m-k}, e_m \rangle.$$

Since $(\langle e_{m-k} \rangle$ converges weakly to 0 as $m \to \infty$ and $T_f$ is compact we see that $\lim_{m} \langle T_f e_{m-k}, e_m \rangle = 0$.

Conversely, if $\lim_{m} \langle T_f e_{m-k}, e_m \rangle = 0$ then Proposition 1.2. shows that $T_{F_k \xi_k}$ is compact for all $k$. Hence, by definition, $T_{\alpha f}$ is compact for all $j$. Since $f \in X$ Proposition 2.3. shows that $T_f$ is compact.

(b) follows from Theorem 5.3. and Lemma 5.1.(a). Here $T_f$ is compact if and only if $f \in X$ and $\Phi_{\psi}(f) = 0$ for all $\psi$.

For other conditions which characterize compact Toeplitz operators on the Bergman and on the Fock space see [8,9].

Example. Let $\mu_1 = 1_{[0,1]} d\lambda + \delta_1$ and $\mu_2 = \delta_1$ ($\lambda$ the Lebesgue measure on $\mathbb{R}_+$). It follows from the maximum principle that $H_2(\mu_1)$ and $H_2(\mu_2)$ are isomorphic and can be identified as sets of holomorphic functions. There are many non-trivial compact Toeplitz operators on $H_2(\mu_1)$, for example $T_F$ with $F(r) = 1_{[0,1/2]}(r)$. On the other hand, in view of Corollary 5.5., the only compact Toeplitz operator on $H_2(\mu_2)$ is the zero operator.
6.3. Corollary. Let n = 1 and let μ satisfy (I) and (II). If Tf is compact then all T_F_k are compact.

REFERENCES