RIESZ TRANSFORMS ON GRAPHS FOR $1 \leq p \leq 2$

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Abstract

We prove, for $1 < p \leq 2$, the $L^p$-boundedness of Riesz transforms on graphs satisfying the doubling property and a on-diagonal estimate of the Markov kernel. In [6], Coulhon and Duong proved the analogous result on Riemannian manifolds. We follow closely Coulhon and Duong's work. However, the discrete setting creates difficulties which do not appear in [6].

1. Introduction

This paper deals with Riesz transforms on graphs endowed with suitable Markov kernels. In this setting, one may define a discrete gradient $\nabla$ and a "Laplace operator" $\Delta$. The issue is to know whether $\|\nabla f\|_p$ and $\|(I - P)^{\frac{1}{2}} f\|_p$ are comparable uniformly in $f$. It is clear when $p = 2$. The question arises when $p \neq 2$ and is equivalent to the $L^p$-continuity of the operator $\nabla \Delta^{-\frac{1}{2}}$, which is called the Riesz transform.

Let $\Gamma$ be an infinite graph, endowed with a measure $m$ satisfying

\[ \forall x \in \Gamma, \ m(x) > 0. \]

We assume that $\Gamma$ is connected and locally uniformly finite, which means that

\[ \sup_{x \in \Gamma} N(x) < \infty \]

where, for $x \in \Gamma$, $N(x)$ is the number of neighbours of $x$. We also assume that $\Gamma$ is endowed with its natural distance $d$.

Denote by $B(x, r)$ the closed ball of center $x$ and of radius $r$, and by $V(x, r)$ its volume. We assume that $\Gamma$ has the doubling property, i. e. there exists $C > 0$ such that

\[ V(x, 2r) \leq CV(x, r), \ \forall x \in \Gamma, \ r > 0. \]

That property implies that there exists $D > 0$ such that

\[ V(x, \theta r) \leq C\theta^D V(x, r), \ \forall x \in \Gamma, \ r > 0, \ \theta > 1. \]
Let \( p \) be a Markov kernel on \( \Omega \), i.e. a non-negative map defined on \( \Omega \times \Omega \) such that

\[
\sum_{y \in \Omega} p(x, y) = 1, \quad \forall x \in \Omega.
\]

Assume that \( p \) is reversible with respect to \( m \), which means that

\[
m(x)p(x, y) = m(y)p(y, x) \quad \text{for all} \quad x, y \in \Omega.
\]

We also assume that there exists \( r_0 > 0 \) such that

\[
p(x, y) = 0 \quad \text{whenever} \quad d(x, y) \geq r_0
\]

and that

\[
\inf_{d(x, y) \leq 1} p(x, y) > 0.
\]

The iterated kernel \( p_k \) is defined by

\[
p_k(x, y) = \sum_z p_{k-1}(x, z)p(z, y).
\]

Assume that the following upper estimate holds for \( p_k \): there exists \( C > 0 \) such that

\[
p_k(x, x) \leq \frac{Cm(x)}{V(x, \sqrt{k})}, \quad \forall k \in \mathbb{N}^*, \quad x \in \Omega.
\]

This upper estimate, together with the doubling property, implies a Gaussian upper bound for \( p_k \). Namely, there exists \( \alpha > 0 \) and \( C_\alpha > 0 \) such that, for any \( k \in \mathbb{N}^* \) and any \( x, y \in \Omega \),

\[
p_k(x, y) \leq \frac{C_\alpha m(y)}{V(x, \sqrt{k})} \exp\left[-\alpha \frac{d^2(x, y)}{k}\right].
\]

This is shown by Coulhon and Grigor’yam in [8], Theorem 1.1. More precisely, define, if \( \Omega \subset \Omega \) is finite and non-empty,

\[
\lambda_1(\Omega) = \inf\left\{ \frac{\|\nabla f\|_2^2}{\|f\|_2^2} : f \in c_0(\Omega) \right\},
\]

where \( c_0(\Omega) \) is the set of all real-valued functions defined on \( \Omega \) and supported in \( \Omega \). Then, say that \( \Omega \) satisfies a relative Faber-Krahn inequality if there exists \( a > 0, \nu > 0 \) such that, for any \( x \in \Omega \), any \( r \geq \frac{1}{2} \) and any \( \Omega \subset B(x, r) \), finite and non-empty,
Then, Theorem 1.1 in [8] states the equivalence between:
1) the doubling property together with (7),
2) the doubling property together with (8),
3) the relative Faber-Krahn inequality.

The linear operator \( P \) is defined by
\[
Pf(x) = \sum_y p(x,y)f(y)
\]
and we notice that
\[
P^k f(x) = \sum_y p_k(x,y)f(y).
\]

The reversibility assumption means that \( P \) is self-adjoint on \( L^2(\Omega, m(x)) \).

Denote by \( \|f\|_p \) the \( L^p \) norm of a function \( f \), that is to say
\[
\|f\|_p = \left[ \sum_x |f(x)|^p m(x) \right]^{\frac{1}{p}}
\]
and by \( \|f\|_{1,\infty} \) the quantity
\[
\|f\|_{1,\infty} = \sup_{\lambda>0} \lambda m(\{x \in \Gamma | |f(x)| > \lambda\}).
\]

Notice that \( L^p(\Gamma) \subset L^q(\Gamma) \) whenever \( 1 \leq p < q < \infty \).

The gradient of a function \( f \) is defined by
\[
\nabla f(x) = \left[ \frac{1}{2} \sum_y p(x,y) |f(y) - f(x)|^2 \right]^{\frac{1}{2}}
\]
where the sum may be restricted to the ball \( B(x,r_0) \).

The Riesz transform \( T \) is defined as
\[
T = \nabla (I - P)^{-\frac{1}{2}},
\]
where the unbounded linear operator \( (I - P)^{-\frac{1}{2}} \) is defined by means of spectral theory. Notice that \( T \) is a subadditive operator, which means that
\[
|T(f + g)(x)| \leq |Tf(x)| + |Tg(x)|.
\]
Indeed, \( (I - P)^{-\frac{1}{2}} \) is linear and \( \nabla \) is subadditive.

We intend to show the following result:
**Theorem 1.** Let $\Gamma$ be a connected, infinite, locally uniformly finite graph. Assume that $\Gamma$ has the doubling property. Let $p$ be a reversible Markov kernel satisfying (5) and (6). Assume also the on-diagonal upper estimate (7). Then the Riesz transform $T$ is weak $(1, 1)$ and bounded on $L^p$ if $1 < p \leq 2$, which means that for all $p \in [1, 2]$, there exists $C_p > 0$ such that $\forall f$ with finite support,

$$\|\nabla f\|_p \leq C_p \|(I - P)^{\frac{1}{2}}f\|_p, \quad 1 < p \leq 2$$

and, $\forall f$ with finite support,

$$\|\nabla f\|_{1, \infty} \leq C_1 \|(I - P)^{\frac{1}{2}}f\|_1.$$ 

A result about discrete Riesz transforms is obtained by Hebisch and Saloff-Coste in [13]. They prove the $L^p$ boundedness of Riesz transforms on finitely generated groups with polynomial growth, for $1 < p \leq 2$, by using Calderon-Zygmund theory on spaces of homogeneous type, but their approach fails to get such a result when $2 < p < \infty$.

An analogous approach is not suitable for the proof of Theorem 1. Indeed, it would require a pointwise estimate for the gradient of $p_k(x, y)$. Such an estimate is false in general, because, in conjunction with the Gaussian upper estimate, it would imply a Gaussian lower bound for $p_k$ which does not hold under the assumptions of Theorem 1. The strategy of the proof is much inspired by [6], where Coulhon and Duong show that, for $1 < p \leq 2$, the Riesz transforms are $L^p$-bounded on Riemannian manifolds with the doubling property and an on-diagonal upper bound of the heat kernel. First, we notice the $L^2$ continuity. Then, we use the Calderon-Zygmund decomposition in order to get that Riesz transforms are of weak type $(1, 1)$. We conclude by interpolation.

In [6], Coulhon and Duong give a counterexample for the $L^p$ boundedness of the Riesz transform on Riemannian manifolds under the same assumptions when $p > 2$. In section 4, we shall prove that the analogous counterexample (i.e. two copies of $\mathbb{Z}^n$ linked together by an edge) also works in our discrete setting.

When (9) holds, one may naturally wonder if $\|(I - P)^{\frac{1}{2}}f\|_q$ can be controlled by $\|\nabla f\|_q$ where $q$ is the conjugate exponent of $p$. In the continuous setting, it is well-known that, for every $p \in ]1, +\infty[$, the inequality

$$\|\nabla f\|_p \leq C_p \|\Delta^\frac{1}{2} f\|_p, \quad \forall f \in C_0^\infty(M)$$

implies by duality that

$$\|\Delta^\frac{1}{2} f\|_q \leq C'_q \|\nabla f\|_q, \quad \forall f \in C_0^\infty(M)$$
where \( \frac{1}{p} + \frac{1}{q} = 1 \). For a proof of this fact, see, for instance, [1] or [2]. In our discrete setting, we have a corresponding result. For every \( p \in [1, +\infty[ \), the inequality

\[
\|\nabla f\|_p \leq C_p \|(I - P)^{\frac{1}{2}} f\|_p, \quad \forall f \in L^p(\Gamma) \cap L^2(\Gamma)
\]

implies that

\[
\|(I - P)^{\frac{1}{2}} f\|_q \leq C_q \|\nabla f\|_q, \quad \forall f \in L^2(\Gamma) \cap L^2(\Gamma)
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

In order to prove that (10) implies (11), we will follow Bakry’s ideas for the continuous setting. Let us start with the following lemma:

**Lemma 1.** Let \((T_t)_{t \geq 0}\) be a continuous semigroup of self-adjoint contracting operators on \(L^2(X)\), where \((X, \mu)\) is a \(\sigma\)-finite measured space. Assume also that \((T_t)_{t \geq 0}\) is a continuous semi-group on \(L^p(X)\) for every \( p \in \mathbb{N}, +\infty \], and that \(T_t\) contracts \(L^1(X)\) and \(L^\infty(X)\). Denote by \(A\) the infinitesimal generator of \(T_t\), and define \(L^p_0 \) as \(L^p \cap (\ker A)^\perp \) (the orthogonal space of \( \ker(A) \) in \( L^2(X) \)), for \( 1 \leq p \leq \infty \). Then, for \( 1 < p < \infty \), \( \{(Af) f \in L^p(X) \cap D_p(A)\} \) is dense in \( L^1_0(X) \cap L^\infty(X) \) for the \(L^p\) norm (\(D_p(A)\) is the domain of \(A\) in \(L^p(X)\)).

The proof of this lemma will be given in an appendix.

Assume now that (10) holds and take \( f \in L^2(\Gamma) \). We may write that

\[
\|(I - P)^{\frac{1}{2}} f\|_q = \sup_{g \in L^q, \|g\| \leq 1} |\langle (I - P)^{\frac{1}{2}} f, g \rangle|.
\]

Define now

\[
A = -(I - P)^{\frac{1}{2}}
\]

\[
= - \sum_{k=0}^{\infty} a_k P^k
\]

where the \( a_k \)'s are given by

\[
(1 - x)^{\frac{1}{2}} = \sum_{k=0}^{\infty} a_k x^k.
\]

The operator \(A\) generates a semigroup which satisfies the requirements of Lemma 1. Indeed, if \( S = I - (I - P)^{\frac{1}{2}} \), one has \( S = - \sum_{k=1}^{\infty} a_k P^k \), so that
\[ \|S\|_{p\to p} \leq \sum_{k=1}^{\infty} (-a_k) \|P^k\|_{p\to p} \]

\[ \leq 1. \]

It follows that

\[ \|e^{-t(I-P)^{1/2}}\|_{p\to p} \leq e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \|S^k\|_{p\to p} \]

\[ \leq 1. \]

Moreover, \( A \) is injective on \( L^2(X) \), so that \( \ker(A) = 0 \), and \( D_p(A) = L^p(A) \). Therefore, thanks to Lemma 1, \( \{(I-P)^{1/2}f\mid f \in L^p(\Gamma)\} \) is dense in \( L^1(\Gamma) \cap L^\infty(\Gamma) \), which is dense in \( L^p(\Gamma) \). Hence, the supremum in (12) may be taken over \( \{(I-P)^{1/2}h\mid h \in L^p(\Gamma)\} \). If \( h \in L^p(\Gamma) \) is such that \( \|(I-P)^{1/2}h\|_p \leq 1 \), one has

\[ |\langle (I-P)^{1/2}f, (I-P)^{1/2}h \rangle| = |\langle (I-P)f, h \rangle| \]

\[ = \left| \sum_{x,y} m(x)p(x,y)[f(x) - f(y)]h(x) \right| \]

\[ = \frac{1}{2} \left| \sum_{x,y} m(x)p(x,y)[f(x) - f(y)][h(x) - h(y)] \right| \]

\[ \leq \frac{1}{2} \sum_x m(x) \left| \sum_y p(x,y)[f(y) - f(x)]|h(y) - h(x)| \right| \]

\[ \leq \frac{1}{2} \sum_x m(x) \left[ \sum_y p(x,y)[f(y) - f(x)]^2 \right]^{1/2} \]

\[ \times \left[ \sum_y p(x,y)|h(y) - h(x)|^2 \right]^{1/2}. \]

Since \( h \in L^p(\Gamma) \), we find that

\[ |\langle (I-P)^{1/2}f, (I-P)^{1/2}h \rangle| \leq C \sum_x m(x) \nabla f(x) \nabla h(x) \]

\[ \leq C \|\nabla f\|_q \|\nabla h\|_p. \]

It follows that
\[ |(I - P)^{1/2}f, (I - P)^{1/2}h| = C\|\nabla f\|_q \|\nabla h\|_p \]
\[ \leq KC_p\|\nabla f\|_q \| (I - P)^{1/2}h\|_p \]
\[ \leq C_p\|\nabla f\|_q. \]

The second line follows from our assumption (10), and the third is true because \( \| (I - P)^{1/2}h \|_p \leq 1 \). Finally, (11) holds.

We now turn to the proof of Theorem 1.

2. Two results in view of Theorem 1

2.1. The Calderon-Zygmund decomposition

We will need the following result, called the Calderon-Zygmund decomposition:

**Theorem 2.** There exists \( C > 0 \) such that, for any \( f \in L^1(\Gamma) \cap L^2(\Gamma) \) and \( \lambda > 0 \), one may write \( f = g + b \) with \( b = \sum b_i \), so that

- a) \( \forall x \in \Gamma, |g(x)| \leq C\lambda \).
- b) \( \forall i \), \( \exists B_i = B(x_i, r_i) \) so that the support of \( b_i \) is contained in \( B_i \), \( \sum x |b_i(x)| \leq C\lambda |B_i| \) and \( \sum x b_i(x) = 0 \).
- c) \( \sum |B_i| \leq \frac{C}{\lambda} \| f \|_1 \).
- d) \( \exists k \in \mathbb{N}^* \) such that every \( x \in \Gamma \) belongs at most to \( k \) balls \( B_i \).

For a proof in the general setting of homogeneous spaces, see [10]. Thanks to conditions b) and c), we see that

\[ \|b\|_1 \leq \sum_i \|b_i\|_1 \leq C\lambda \sum_i |B_i| \leq C\|f\|_1 \]

so that

\[ \|g\|_1 = \|f - b\|_1 \leq (1 + C)\|f\|_1. \]

2.2. Estimates for the kernels

We will also need the following lemma:

**Lemma 2.** There exists \( \beta > 0 \) such that, for any \( x \in \Gamma, l \in \mathbb{N}, k \in \mathbb{N}^* \),

\[ \sum_{y \notin B(x, \sqrt{l})} |\nabla_j p_k(y, x)m(y)| \leq Cm(x)e^{-\frac{\beta}{2k}l}. \]

Lemma 2 will follow from a few technical results.

**Lemma 3.** For all \( \gamma > 0, x \in \Gamma, l \in \mathbb{N} \) and \( k \in \mathbb{N}^* \),

\[ \sum_{y \notin B(x, \sqrt{l})} e^{-\gamma \frac{x^2}{x^2}m(y)} \leq C_\gamma V(x, \sqrt{k})e^{-\frac{\gamma}{2}}. \]
Indeed,

\[
\sum_{y \notin B(x,\sqrt{l})} e^{-2\gamma^2(x,y)} m(y) \leq e^{-\gamma^2} \sum_{y} e^{-\gamma^2(x,y)} m(y)
\]

\[
\leq e^{-\gamma^2} \sum_{i=0}^{\infty} \sum_{i^{\sqrt{k}} \leq d(y,x) < (i+1) \sqrt{k}} e^{-\gamma^2(x,y)} m(y)
\]

\[
\leq e^{-\gamma^2} \sum_{i=0}^{\infty} e^{-\gamma^2(x,y)} V(x, (i+1) \sqrt{k})
\]

\[
\leq C e^{-\gamma^2} \sum_{i=0}^{\infty} e^{-\gamma^2(x,y)} (i+1)^D V(x, \sqrt{k})
\]

\[
= CV(x, \sqrt{k}) e^{-\gamma^2}.
\]

The last but one line follows from (3).

As a consequence of Lemma 3 and (8) with \(k = 0\), we get

**Lemma 4.**

\[
\sum_{y} [p_k(y,x)]^2 e^{-\gamma^2(x,y)} m(y) \leq \frac{C m^2(x)}{V(x,\sqrt{k})} \forall \gamma \in ]0, 2\alpha[, k \in \mathbb{N}^+, x \in \Gamma.
\]

We now seek for a result analogous to Lemma 4, replacing \(p_k\) by its gradient. In order to get it, we will use the following result, which depends on (7) and (6):

**Lemma 5.**

\[
|p_{k+1}(x,y) - p_k(x,y)| \leq \frac{c m(y)}{k V(x,\sqrt{k}) V^2(y,\sqrt{k})}.
\]

The main tool used to get this estimate is the following statement:

**Lemma 6.**

\[
\left\| (I - P) \left[ \frac{p_k(x,\cdot)}{m} \right] \right\|_2 \leq \frac{C}{k V^2(x,\sqrt{k})}.
\]

We first consider the case where \(k = 2l\). We write that

\[
(I - P) \left[ \frac{p_k(x,\cdot)}{m} \right] = (I - P) p_l \left[ \frac{p_l(x,\cdot)}{m} \right].
\]

Indeed,
It follows from (14) that
\[ \left\| (I - P) \left[ \frac{p_k(x, \cdot)}{m} \right] \right\|_2 \leq \left\| (I - P) P' \right\|_{2 \to 2} \left[ \frac{p_k(x, \cdot)}{m} \right]_2. \]

On the one hand,
\[
\left\| \frac{p_k(x, \cdot)}{m} \right\|_2^2 = \sum_{y} \frac{p_k(x, y)p_k(x, y)}{m(y)}
\]
\[
= \frac{p_k(x, x)}{m(x)}
\]
\[
\leq \frac{C}{V(x, \sqrt{k})}.
\]

On the other hand, if
\[ P = \int_a^b \lambda dE_\lambda \]
where \( a > -1 \), we get that
\[
(15) \quad \left\| (I - P) P' \right\|_{2 \to 2} \leq \sup_{\lambda \in [a, 1]} (1 - \lambda) |\lambda|^l
\]
\[
\leq \frac{C}{I} = \frac{2C}{k}.
\]

We do the same when \( k = l + (l + 1) \). Therefore, Lemma 6 is proved.

Remark. In this proof, we used the fact that \(-1 \notin \text{Sp}(P)\), which is implied by the on-diagonal lower bound of \( p \), (6). Indeed, for any function \( f \in L^2 \), an elementary computation shows that
\[
\langle (I + P)f, f \rangle = \frac{1}{2} \sum_{x,y} \left| f(x) + f(y) \right|^2 p(x,y)m(x)
\]
\[
\geq 2 \sum_x |f(x)|^2 p(x,x)m(x)
\]
\[
\geq c \|f\|_2^2
\]
where \(c > 0\). The role of \(P\)'s spectrum with respect to the analyticity of \(P\) on \(L^2\) (see (15)) is pointed out in [9], Proposition 3, which claims that either \(-1 \notin \text{Sp}(P)\) and one has \(\|P^k - P^{k+1}\|_{2 \to 2} = 2\) for all \(k \in \mathbb{N}\), or \(-1 \notin \text{Sp}(P)\) and \(P\) is analytic on \(L^2\), which exactly means (15).

Let us deduce Lemma 5 from Lemma 6. Choose an integer \(l \sim \frac{k}{2}\). One has
\[
|p_{k+1}(x,y) - p_k(x,y)| \leq \sum_z \frac{|p_{l+1}(x,z) - p_l(x,z)|}{m(z)} p_l(z,y)m(z)
\]
\[
\leq \left\| \frac{p_{l+1}(x,.) - p_l(x,.)}{m} \right\|_2 \|p_l(.,y)\|_2.
\]
On the one hand, according to Lemma 6,
\[
\left\| \frac{p_{l+1}(x,.) - p_l(x,.)}{m} \right\|_2 = \left\| (I - P) \left[ \frac{p_l(x,.)}{m} \right] \right\|_2
\]
\[
\leq \frac{C}{lV^2(x, \sqrt{l})}
\]
\[
\leq \frac{C}{kV^2(x, \sqrt{k})},
\]
the last line being valid thanks to the doubling property. On the other hand,
\[
\|p_l(.,y)\|_2^2 = p_{2l}(y,y)m(y)
\]
\[
\leq \frac{Cm^2(y)}{V(y, \sqrt{2l})}
\]
\[
\leq \frac{Cm^2(y)}{V(y, \sqrt{k})}.
\]
Thus, Lemma 5 is proved.

**Remark.** The question naturally arises to know if this estimate about the
temporal difference may be improved, and, namely, if one can get a Gaussian estimate:

\[
|p_{k+1}(x, y) - p_k(x, y)| \leq \frac{Cm(y)}{kV(x, \sqrt{k})} \exp \left[ -c \frac{d^2(x, y)}{k} \right].
\]

In the continuous setting, the analogous estimate, i.e., an estimate on time derivatives of the heat kernel, follows rather easily from the on-diagonal estimate. In a discrete setting, the proof of (16) was given by Christ in [5] in the case of a polynomial volume growth. Christ’s proof, which is difficult, may be adapted to the case of the doubling property, so that (16) does hold. But it is unnecessary for our purpose.

**Lemma 7.** \[ \sum_y |\nabla_y p_k(y, x)|^2 e^{-\varphi(x,y)} m(y) \leq \frac{C_\gamma m^2(x)}{kV(x, \sqrt{k})} \forall \gamma \in \mathbb{R}, \alpha, k \in \mathbb{N}. \]

Define

\[
I(k, x) = \sum_y |\nabla_y p_k(y, x)|^2 e^{-\varphi(x,y)} m(y).
\]

The very definition of \( \nabla \) shows that

\[
I(k, x) = \frac{1}{2} \sum_{y,z} |p_k(y, x) - p_k(z, x)|^2 e^{-\varphi(x,y)} m(y).
\]

Remember that this sum may be restricted to the \((y, z)\) such that \(d(y, z) \leq r_0\), since \(p(y, z) = 0\) if it is not the case. To estimate the analogous quantity in a continuous setting, Grigor’yan, in [12], Theorem 1.1, makes several integrations by parts. Such computations do not work very well in a discrete setting. We replace them by computations about \(I(k, x)\) inspired by the proof of the estimate of \(\nabla p^{(n)}\) in Theorem 5.1 of [13], and using the temporal estimate given by Lemma 5:

\[
I(k, x) = \sum_{d(y, z) \leq r_0} p_k(y, x)[p_k(y, x) - p_k(z, x)]m(y)p(y, z)e^{-\varphi(x,y)}
\]

\[- \sum_{d(y, z) \leq r_0} p_k(z, x)[p_k(y, x) - p_k(z, x)]m(y)p(y, z)e^{-\varphi(x,y)}
\]

\[= \sum_{d(y, z) \leq r_0} p_k(y, x)[p_k(y, x) - p_k(z, x)]m(y)p(y, z)e^{-\varphi(x,y)}
\]

\[+ \sum_{d(y, z) \leq r_0} p_k(y, x)[p_k(y, x) - p_k(z, x)]m(y)p(y, z)e^{-\varphi(x,y)}.
\]
In the last line, we inverted $y$ and $z$ and we used the reversibility of $p$. Hence, we get that

\[
I(k, x) \leq 2 \sum_{d(y, z) \leq r_0} p_k(y, x)[p_k(y, x) - p_k(z, x)]m(y)p(y, z)e^{-d^2_{k(x,y)}k} + \sum_{d(y, z) \leq r_0} p_k(y, x)[p_k(y, x) - p_k(z, x)]m(y)p(y, z)[e^{-d^2_{k(x,y)}k} - e^{-d^2_{k(x,z)}k}]
\]

\[
= 2I_1(k, x) + I_2(k, x).
\]

Thanks to the preceding lemma, we can estimate $I_1$ and $I_2$. Indeed,

\[
I_1(k, x) = \sum_y p_k(y, x)e^{-d^2_{k(x,y)}k}m(y)\sum_z p(y, z)[p_k(y, x) - p_k(z, x)]
\]

\[
= \sum_y p_k(y, x)e^{-d^2_{k(x,y)}k}m(y)[p_k(y, x) - p_{k+1}(y, x)]
\]

hence

\[
|I_1(k, x)| \leq \frac{C_{\alpha}m(x)}{kV^2(x, \sqrt{k})} \sum_y \frac{p_k(y, x)}{V^2_{k}(y, \sqrt{k})}e^{-d^2_{k(x,y)}k}m(y)
\]

\[
\leq \frac{C_{\alpha}C_{\alpha}m^2(x)}{kV^2(x, \sqrt{k})} \sum_y \frac{1}{V^2_{k}(y, \sqrt{k})}e^{(\gamma - \alpha)d^2_{k(x,y)}k}m(y)
\]

\[
\leq \frac{C_{\alpha}C_{\gamma}m^2(x)}{kV^2(x, \sqrt{k})} \sum_y e^{(\gamma - \alpha)d^2_{k(x,y)}k}m(y)
\]

\[
\leq \frac{C_{\gamma}m^2(x)}{kV(x, \sqrt{k})}.
\]

The first line holds thanks to Lemma 5, the last one follows from Lemma 3. Note that, in this computation, it is possible to choose $\gamma' \in ]\gamma, \alpha[$ because $\gamma < \alpha$.

As for $I_2$, denote by $f$ the map defined by

\[
f(x) = e^{2x^2}.
\]

We may write, according to the mean-value theorem, that

\[
|f(b) - f(a)| \leq \frac{2\gamma}{k} |b - a| \sup(a, b)e^{2\sup(a, b)^2}.
\]

Applying this inequality with $a = d(x, y), b = d(x, z)$ when $d(y, z) \leq r_0$, so that $|d(x, y) - d(x, z)| \leq r_0$, we get, if we notice that $\sup(a, b) \leq a + r_0$, that
\[ |I_2(k, x)| \leq \frac{2\gamma}{K} \sum_{d(y, z) \leq r_0} m(y)p(y, z)p_k(y, x)|p_k(y, x)\]

\[-p_k(z, x)|d(x, y) + r_0|e^{\gamma[d(x, y)+r_0]}\]

\[= 2\sqrt{\frac{\gamma}{K}} \sum_{d(y, z) \leq r_0} m(y)p(y, z)p_k(y, x)|p_k(y, x)\]

\[-p_k(z, x)|d(x, y) + r_0|2e^{\gamma[d(x, y)+r_0]}\]

\[\leq 2\sqrt{\frac{\gamma}{K}} \sum_{d(y, z) \leq r_0} m(y)p(y, z)p_k(y, x)|p_k(y, x) - p_k(z, x)|e^{\gamma[d(x, y)+r_0]}\]

\[\leq 2\sqrt{\frac{\gamma}{K}} \left[ \sum_{d(y, z) \leq r_0} m(y)p(y, z)|p_k(y, x)|^2 e^{\gamma[d(x, y)+r_0]} \right]^{\frac{1}{2}}\]

\[\times \left[ \sum_{d(y, z) \leq r_0} m(y)p(y, z)|p_k(y, x) - p_k(z, x)|^2 e^{\gamma[d(x, y)+r_0]} \right]^{\frac{1}{2}}\]

But

\[\frac{\gamma}{K} |d(x, y) + r_0|^2 = \frac{\gamma}{K} \left[ d^2(x, y) + 2r_0d(x, y) + r_0^2 \right]\]

and, in the sums which define \(I(k, x)\), we may assume that \(d(x, y) \leq (k+1)r_0\) and that \(d(x, z) \leq (k+1)r_0\). If it was not the case, since \(d(y, z) \leq r_0\), we would obtain \(d(x, y) > kr_0\) and \(d(x, z) > kr_0\), so \(p_k(y, x) = p_k(z, x) = 0\) according to (5). Finally,

\[\frac{\gamma}{K} |d(x, y) + r_0|^2 \leq \frac{\gamma}{K} d^2(x, y) + C\]

and we can write that

\[|I_2(k, x)| \leq C\sqrt{\frac{\gamma}{K}} \left[ \sum_{y} |p_k(y, x)|^2 e^{\gamma m^2(x, y)}m(y) \right]^{\frac{1}{2}} \sqrt{I(k, x)}\]

\[\leq \frac{C\gamma}{\sqrt{K}} \frac{m(x)}{V(x, \sqrt{K})} \sqrt{I(k, x)}.\]

If we use simultaneously the estimates about \(I_1\) and \(I_2\), we find that
\[ I \leq \frac{C_{*}m^2(x)}{kV(x, \sqrt{k})} + \frac{C_{*}m(x)}{kV(x, \sqrt{k})} \sqrt{I} \]

from which we get the right estimate for \( I \), hence Lemma 7 by (17).

Thanks to Lemma 7 and Lemma 3, we can finally show Lemma 2. Indeed, we just have to write that, if \( \beta < \frac{4}{k} \),

\[
\sum_{y, d(y,x) \geq \sqrt{I}} |\nabla_y p_k(x,y)| m(y) \leq \left[ \sum_{y, d(y,x) \geq \sqrt{I}} |\nabla_y p_k(x,y)|^2 e^{4/\beta^2(x,y)} m(y) \right]^\frac{1}{2} \times \left[ \sum_{d(y,x) \geq \sqrt{I}} e^{-4/\beta^2(x,y)} m(y) \right]^\frac{1}{2} \leq \frac{Cm(x)}{V(x, \sqrt{k})} e^{-d} \sqrt{\frac{V(x, \sqrt{k}) e^{-d}}{m(x)}}
\]

which is Lemma 2.

3. Proof of Theorem 1

The proof follows closely [6]. The \( L^2 \) boundedness of \( T \) is obvious. Indeed, if \( f \in L^2(\Gamma) \),

\[
\nabla f(x) = \left[ \frac{1}{2} \sum_y p(x,y) |f(y) - f(x)|^2 \right]^\frac{1}{2}
\]

so that

\[
\|\nabla f\|_2^2 \leq C \sum_{y,x} p(x,y) |f(x) - f(y)|^2 m(x)
\]

\[
= 2C \left\langle (I - P)f, f \right\rangle
\]

\[
= 2C \left\| (I - P)^{\frac{1}{2}} f \right\|_2^2.
\]

If we show that \( T \) is weak \((1, 1)\), the Marcinkiewicz interpolation theorem will give the \( L^p \) boundedness for \( 1 < p < 2 \). Therefore, we are going to show that \( T \) is weak \((1, 1)\).

Let \( f \in L^1(\Gamma) \cap L^2(\Gamma) \). Our aim is to show that, if \( \lambda > 0 \),

\[
m(\{x | |Tf(x)| > \lambda\}) \leq \frac{C}{\lambda} \|f\|_1.
\]
Write the Calderon-Zygmund decomposition of $f$:

$$f = g + b = g + \sum_i b_i.$$ 

One has

$$m(\{x : |Tf(x)| > \lambda \}) \leq m(\{x : |Tg(x)| > \lambda/2 \}) + m(\{x : |Tb(x)| > \lambda/2 \})$$

because $T$ is subadditive. Since $T$ is $L^2$ bounded, we may write that

$$m(\{x : |Tg(x)| > \lambda/2 \}) \leq \frac{4}{\lambda^2} \|Tg\|_2^2$$

$$\leq \frac{C}{\lambda^2} \|g\|_2^2$$

$$\leq \frac{C}{\lambda^2} \lambda \|g\|_1$$

$$\leq \frac{C}{\lambda} \|f\|_1.$$ 

The last but one line is true because of the property $a)$ of the decomposition. Therefore, what remains to be proved is the fact that

$$m(\{x : |Tb(x)| > \lambda \}) \leq \frac{C}{\lambda} \|f\|_1.$$ 

To this purpose, we write that

$$b = \sum_i b_i = \sum_i P^k_i b_i + \sum_i (I - P^k_i) b_i$$

where $k_i = r_i^2$ if $b_i$ is supported in $B_i = B(x_i, r_i)$. First, we prove that

$$\left\| \sum_i P^k_i b_i \right\|_2^2 \leq C\lambda \|f\|_1.$$ 

One has

$$|P^k_i b_i(x)| \leq \sum_y |p_{k_i}(x, y)| |b_i(y)|$$

$$\leq C \frac{1}{V(x, r_i)} \sum_y \exp\left(-\alpha \frac{d^2(x, y)}{k_i} \right) |b_i(y)| m(y).$$

In the last sum, we can assume that $y \in B(x_i, r_i)$ because of the support of $b_i$. But, if $d(y, x_i) < r_i$, then $d(x, x_i) \leq d(x, y) + r_i$, hence $d^2(x, x_i) \leq 2d^2(x, y) +$
2\eta_i^2. It follows that
\[ |p^{k_i}b_i(x)| \leq \frac{C}{V(x, r_i)} \exp \left( -\alpha \frac{d^2(x, x_i)}{2k_i} \right) \|b_i\|_1 \]
\[ \leq \frac{C}{V(x, r_i)} \exp \left( -\alpha \frac{d^2(x, x_i)}{2k_i} \right) \lambda m(B_i) \]
\[ \leq C\lambda \sum_y \frac{1}{V(x, r_i)} \exp \left( -\frac{\alpha}{2k_i} d^2(x, x_i) \right) 1_{B_i}(y)m(y) \]
\[ \leq C\lambda \sum_y \frac{1}{V(x, r_i)} \exp \left( -\frac{\alpha}{4k_i} d^2(x, y) \right) 1_{B_i}(y)m(y). \]

The second line follows from property b) of the decomposition. This inequality implies that
\[
\left\| \sum_i p^{k_i}b_i \right\|_2 \leq \sum_i C\lambda \sum_y \frac{1}{V(x, r_i)} \exp \left( -\frac{\alpha}{k_i} d^2(x, y) \right) 1_{B_i}(y)m(y) \right\|_2 \]
\[ \leq C\lambda \left\| \sum_i \sum_y \frac{1}{V(x, r_i)} \exp \left( -\frac{\alpha}{k_i} d^2(x, y) \right) 1_{B_i}(y)m(y) \right\|_2 \]
\[ = C\lambda \sup_{\|f\|_2=1} \left[ \sum_{x} \sum_i \sum_y \frac{1}{V(x, r_i)} \right. \]
\[ \times \exp \left( -\frac{\alpha}{k_i} d^2(x, y) \right) 1_{B_i}(y)m(y)f(x)m(x) \].

But, because of the doubling property,
\[ V(y, r_i) \leq \left[ 1 + \frac{d(x, y)}{r_i} \right]^D V(x, r_i) \]
(actually, \( B(y, r_i) \subset B(x, r_i + d(x, y)) = B(x, \left[ 1 + \frac{d(x, y)}{r_i} \right] r_i \) and one may apply (3)). Thus,
Riesz transforms on graphs for $1 \leq p \leq 2$

$$\sum_x \frac{1}{V(x,r_i)} \exp\left( -\alpha \frac{d^2(x,y)}{k_i} \right) |f(x)|m(x) \leq \frac{1}{V(y,r_i)} \sum_x \left[ 1 + \frac{d(x,y)}{r_i} \right]^\beta$$

$$\times \exp\left( -\alpha \frac{d^2(x,y)}{k_i} \right) |f(x)|m(x) \leq \frac{1}{V(y,r_i)} \sum_x \exp\left( -\alpha' \frac{d^2(x,y)}{k_i} \right) |f(x)|m(x)$$

$$\leq \frac{1}{V(y,r_i)} \left[ \sum_{l=0}^{\infty} \sum \exp\left( -\alpha \frac{d^2(x,y)}{k_i} \right) |f(x)|m(x) \right]$$

$$+ \frac{1}{V(y,r_i)} \sum \exp\left( -\alpha \frac{d^2(x,y)}{k_i} \right) |f(x)|m(x) \leq \frac{1}{V(y,r_i)} \sum_{l=0}^{\infty} e^{-\alpha 2^l}$$

$$\times \sum_{x \in B(y,2^{l+1}r_i)} |f(x)|m(x) + \frac{1}{V(y,r_i)} \sum_{d(x,y)<r_i} |f(x)|m(x).$$

In the second line, we used the fact that $k_i = r_i^2$.

Hence, if we denote by

$$Mf(x) = \sup_{x \in B} \frac{1}{|B|} \sum_{z \in B} |f(z)|m(z)$$

the Hardy-Littlewood maximal function of $f$, we get

$$\sum_x \frac{1}{V(x,r_i)} \exp\left( -\alpha \frac{d^2(x,y)}{r_i} \right) |f(x)|m(x) \leq \sum_{l=0}^{\infty} e^{-\alpha 2^l} \frac{V(y,2^{l+1}r_i)}{V(y,r_i)} \frac{1}{V(y,2^{l+1}r_i)}$$

$$\times \sum_{x \in B(y,2^{l+1}r_i)} |f(x)|m(x)$$

$$+ \frac{1}{V(y,r_i)} \sum_{x \in B(y,r_i)} |f(x)|m(x)$$

$$\leq Mf(y) + \sum_{l=0}^{\infty} e^{-\alpha 2^l} 2^l Mf(y)$$

$$= KMf(y).$$

Returning to (19), we find that
\[
\left\| \sum_i p_i b_i \right\|_2 \leq C \lambda \sup_{\|f\|_2=1} \sum_y Mf(y) \sum_i 1_{B_i}(y)m(y)
\]

\[
\leq C \lambda \sup_{\|f\|_2=1} \left\| Mf \right\|_2 \left\| \sum_i 1_{B_i} \right\|_2.
\]

But it is well-known that \(\|Mf\|_2 \leq \|f\|_2\) (see [10] for a proof in the setting of homogeneous spaces), so that we obtain

\[
\left\| \sum_i p_i b_i \right\|_2^2 \leq C \lambda^2 \left\| \sum_i 1_{B_i} \right\|_2^2
\]

\[
\leq C \lambda^2 \left\| \sum_i 1_{B_i} \right\|_\infty \left\| \sum_i 1_{B_i} \right\|_1.
\]

We notice that, because of property d), \(\sum_i 1_{B_i}(z) = |\{i \in B_i\}| \leq k\). Moreover,

\[
\left\| \sum_i 1_{B_i} \right\|_1 \leq \sum_i \|1_{B_i}\|_1
\]

\[
\leq \sum_i V(B_i)
\]

\[
\leq \frac{C}{\lambda} \|f\|_1.
\]

Property c) of the decomposition implies the last line.

We get

\[
\left\| \sum_i p_i b_i \right\|_2^2 \leq C \lambda \|f\|_1.
\]

Finally, one has

\[
m\left( \left\{ x \mid T \left[ \sum_i p_i b_i \right](x) > \frac{\lambda}{2} \right\} \right) \leq \frac{4}{\lambda^2} \left\| T \left[ \sum_i p_i b_i \right] \right\|_2^2
\]

\[
\leq \frac{C}{\lambda^2} \left\| \sum_i p_i b_i \right\|_2^2
\]

\[
\leq \frac{C}{\lambda} \|f\|_1.
\]
where the $L^2$ boundedness of $T$ is used in the second line.

We are now going to estimate $m\left(\left\{ x \mid T\left[\sum_i (I - P^k_i)b_i\right](x) > \frac{\lambda}{2}\right\}\right)$. One has

(20) $m\left(\left\{ x \mid T\left[\sum_i (I - P^k_i)b_i\right](x) > \frac{\lambda}{2}\right\}\right) \leq m\left(\left\{ x \in \bigcup_i 2B_i \mid T\left[\sum_i (I - P^k_i)b_i\right](x) > \frac{\lambda}{2}\right\}\right)$

$+ m\left(\left\{ x \notin \bigcup_i 2B_i \mid T\left[\sum_i (I - P^k_i)b_i\right](x) > \frac{\lambda}{2}\right\}\right)$

$\leq m\left(\bigcup_i 2B_i\right) + m\left(\left\{ x \notin \bigcup_i 2B_i \mid T\left[\sum_i (I - P^k_i)b_i\right](x) > \frac{\lambda}{2}\right\}\right)$.

As for the first term of (20),

$m\left(\bigcup_i 2B_i\right) \leq \sum_i m(2B_i)$

$\leq C \sum_i m(B_i)$

$\leq \frac{C}{\lambda} \|f\|_1$.

So, we shall deal with the second term of (20).
Let us prove that, for every \( i \),
\[
\sum_{y \notin \bigcup B_j} |T(I - P^k) b_i(y)| m(y) \leq C \|b_i\|_1.
\]
If we denote by \( q_{ki} \), the kernel of \( T(I - P^k) \), we have
\[
\sum_{y \notin \bigcup B_j} |T(I - P^k) b_i(y)| m(y) \leq \sum_{y \notin \bigcup B_j} \sum_{x \in B_i} |q_{ki}(y, x)||b_i(x)| m(y)
\]
\[= \sum_{x \in B_i} |b_i(x)| \sum_{y \notin \bigcup B_j} |q_{ki}(y, x)| m(y).
\]
But, when \( y \notin \bigcup B_j, y \notin \bigcup B_i \), so that \( d(y, x) \geq r_i = \sqrt{k_i} \). This implies that
\[
\sum_{y \notin \bigcup B_j} |T(I - P^k) b_i(y)| m(y) \leq \sum_{x \in B_i} |b_i(x)| \sum_{d(y, x) \geq \sqrt{k_i}} |q_{ki}(y, x)| m(y)
\]
and we just have to show that
\[
(22) \sum_{d(y, x) \geq \sqrt{k_i}} |q_{ki}(y, x)| m(y) \leq C m(x)
\]
where \( C > 0 \) is independent of \( x \), for, if this is proved, we will get
\[
(23) \sum_{y \notin \bigcup B_j} |T(I - P^k) b_i(y)| m(y) \leq C \|b_i\|_1
\]
as claimed.

Let us compute \( q_k \), the kernel of \( \nabla (I - P)^{-\frac{1}{2}} (I - P^k) \). If \( (a_p)_{p \geq 0} \) is the sequence of real numbers defined by
\[(1-x)^{-\frac{1}{2}} = \sum_{p=0}^{\infty} a_p x^p\]

we see that

\[
(I - P)^{-\frac{1}{2}}(I - P^k) = \sum_{p=0}^{\infty} a_p P^p (I - P^k) = \sum_{p=0}^{\infty} a_p P^p - \sum_{p=0}^{\infty} a_p P^{p+k} = \sum_{p=0}^{\infty} (a_p - b_p) P^p
\]

where

\[
b_p = \begin{cases} 
  a_{p-k} & \text{if } p \geq k \\
  0 & \text{if } p < k 
\end{cases}
\]

Hence,

\[
T(I - P^k) = \nabla \left[ \sum_{p=0}^{\infty} (a_p - b_p) P^p \right]
\]

so that

\[
|q_k(x, y)| \leq \sum_{p=0}^{\infty} |a_p - b_p| \nabla x p^p(x, y).
\]

We find that

\[
(24) \sum_{d(y, x) \geq \sqrt{k}} |q_k(x, y)| m(x) \leq \sum_{p=0}^{\infty} |a_p - b_p| \sum_{d(y, x) \geq \sqrt{k}} \nabla x p^p(x, y) m(x)
\]

\[
\leq \left[ C + C \sum_{p=1}^{k-1} |a_p - b_p| e^{-\frac{p}{2} \sqrt{d_{x}}} \right] m(y)
\]

\[
\leq \left[ C + C \sum_{p=1}^{k-1} |a_p| e^{-\frac{p}{2} \sqrt{d_{x}}} + \sum_{p=k}^{\infty} |a_p - b_p| p^{-\frac{1}{2}} \right] m(y).
\]

The upper bound given by Lemma 2 shows the second line.
It is well-known (by the Stirling formula) that
Hence, we get that
\[ \sum_{p=1}^{k-1} a_p e^{-\frac{ap}{p}} p^{\frac{1}{2}} \leq C \sum_{p=1}^{k-1} \frac{e^{-\frac{ap}{p}}}{p}. \]
If we denote by \( f \) the function
\[ f(x) = \frac{e^{-\frac{x}{x}}}{x}, \]
we observe, considering \( \log f \), that \( f \) is nondecreasing on \([1, \beta k]\) and non-increasing on \([\beta k, k - 1]\), so that
\[ \sum_{p=1}^{E(\beta k)+1} \frac{e^{-\frac{ap}{p}}}{p} \leq \int_{1}^{E(\beta k)+1} \frac{e^{-\frac{x}{x}}}{x} dx \]
and
\[ \sum_{p=E(\beta k)+1}^{k-1} \frac{e^{-\frac{ap}{p}}}{p} \leq \int_{E(\beta k)}^{k-1} \frac{e^{-\frac{x}{x}}}{x} dx. \]
We get
\[ \sum_{p=1}^{k-1} \frac{e^{-\frac{ap}{p}}}{p} \leq 2 \int_{0}^{k} \frac{e^{-\frac{x}{x}}}{x} dx = 2 \int_{0}^{1} \frac{e^{-\frac{u}{u}}}{u} du = Cste. \]
As for the second term of (24), we write that
\[ a_p - a_{p+1} = 2 \frac{(2p)!((p+1))}{4^p+1((p+1))}. \]
The Stirling formula shows that
\[ a_p - a_{p+1} \sim \frac{C}{p^2}, \]
So, there exists a constant \( K \) such that
\[ \forall p \in \mathbb{N}^*, \ a_p - a_{p+1} \leq \frac{K}{p^2}. \]
For \( p \geq k + 2 \), \( |a_p - b_p| = |a_p - a_{p-k}| = a_{p-k} - a_p \), hence
\[ |a_p - b_p| = \sum_{l=p-k}^{p-1} (a_l - a_{l+1}) \]

\[ \leq K \sum_{l=p-k}^{p-1} \frac{1}{l^2} \]

\[ \leq K \int_{p-k-1}^{p-1} \frac{dx}{x^2} \]

\[ = K \left[ \frac{1}{\sqrt{p-k-1}} - \frac{1}{\sqrt{p-1}} \right]. \]

This implies that

\[ \sum_{p=k+2}^{\infty} |a_p - b_p| p^{-\frac{1}{2}} \leq K \sum_{p=k+2}^{\infty} \left[ \frac{1}{\sqrt{p-k-1}} - \frac{1}{\sqrt{p-1}} \right] \frac{1}{\sqrt{p}} \]

\[ \leq K \sum_{p=k+2}^{\infty} \left[ \frac{k}{\sqrt{p-k-1} \sqrt{p-1} \left[ \sqrt{p-k-1} + \sqrt{p-1} \right]} \right] \frac{1}{\sqrt{p}} \]

\[ \leq K \int_{k+1}^{\infty} \frac{k}{\sqrt{x-k-1} \sqrt{x-1} \left[ \sqrt{x-k-1} + \sqrt{x-1} \right]} \frac{dx}{\sqrt{x}} \]

\[ = K \int_{0}^{\infty} \frac{k}{\sqrt{y} \sqrt{y+k} \left[ \sqrt{y} + \sqrt{y+k} \right]} \frac{dy}{\sqrt{y+k+1}} \]

\[ = K \int_{0}^{\infty} \frac{dz}{\sqrt{z} \sqrt{z+1} \left[ \sqrt{z} + \sqrt{z+1} \right]} \sqrt{z+1}. \]

What is left to estimate is \( |a_k - a_0| k^{-\frac{1}{2}} + |a_{k+1} - a_1| (k+1)^{-\frac{1}{2}} \), which is obviously bounded, because so are \( a_k \) and \( a_{k+1} \). We have shown (22), therefore (23). According to (21),

\[ m \left( \left\{ y \not\in \bigcup_j 2B_j \bigg| \sum_i |T(I-P^k)b_i(y)| > \frac{\lambda}{2} \right\} \right) \leq \frac{1}{\lambda} C \sum_i \| b_i \|_1 \]

\[ \leq \frac{C}{\lambda} \| f \|_1. \]

Theorem 1 is completely proved.
4. A counterexample for $p > 2$

We intend to explain that Theorem 1 is false for $p > 2$. For $n \geq 2$, let $\Gamma_n$ be the graph formed by two copies of $\mathbb{Z}^n$ linked between with an edge. Define, for any $x \in \Gamma_n$, $m(x)$ as being the number of neighbours of $x$. Actually, $m(x)$ is equal to $2n$, except for the two points linked by an edge for which $m(x) = 2n + 1$. Equip $\Gamma$ with the measure $m$. On this graph, there exists a constant $C > 0$ such that, for any $x \in \Gamma_n$ and any $r > 0$,

$$C^{-1}r^n \leq V(x, r) \leq Cr^n.$$  

For any $p > n$ and any function $f$ finitely supported in $\Gamma_n$, one has, for any $x, y \in \Gamma$,

$$|f(x) - f(y)| \leq Cd(x, y)^{1 - \frac{2}{p}}\|\nabla f\|_p$$

Moreover, for any function $f$ finitely supported in $\Gamma_n$, the following Nash inequality holds:

$$\|f\|_2^{1 + \frac{2}{p}} \leq C\|f\|^{\frac{2}{p}}\|\nabla f\|_2.$$  

For $n \geq 3$, this Nash inequality is equivalent to the Sobolev inequality (see [3]):

$$\|f\|_p \leq C\|\nabla f\|_2.$$  

Define $p$ as being the kernel of the standard random walk on $\Gamma_n$ (see [7], p.148). Then, $p$ is a Markov kernel, reversible with respect to $m$, and satisfies (5) and (6). The Nash inequality (26) shows that, for any $x \in \Gamma$ and any $k \in \mathbb{N}^*$,

$$p_k(x, x) \leq Ck^{-\frac{2}{p}},$$

(see [4]). Theorem 1.1 in [8] shows that, for any $x, y \in \Gamma_n$ and any $k \in \mathbb{N}^*$,

$$p_k(x, y) \leq Ck^{-\frac{2}{p}}\exp\left[-\frac{d^2(x, y)}{Ck}\right].$$

(29)

One also has, for any $x \in \Gamma$ and any $k \in \mathbb{N}^*$,

$$p_k(x, x) \geq Ck^{-\frac{2}{p}}.$$  

(30)

This result follows from Theorem 4.6 of [7].

Assume that $p > n$ and that the Riesz transform on $\Gamma$ is $L^p$-bounded. Then, for any $f$ finitely supported,

$$\|\nabla f\|_p \leq C_p\|(I - P)^{\frac{1}{2}}f\|_p.$$  

(31)
Fix \( z \in \Gamma \) and apply (25) and (31) to \( f(x) = p^k(x, z) \). One gets
\[
|p_k(x, z) - p_k(y, z)| \leq C_\rho d(x, y)^{1 - \frac{\gamma}{p}}(I - P)^{\frac{1}{2}}p^k(., z)\|_p.
\]
Choose an integer \( l \sim \frac{k}{2} \) and write that
\[
(I - P)^{\frac{1}{2}}p_k(., z) = (I - P)^{\frac{1}{2}}P^l(., z),
\]
which implies that
\[
\| (I - P)^{\frac{1}{2}}p_k(., z) \|_p \leq \| (I - P)^{\frac{1}{2}}P^l(., z) \|_p.
\]
But the analyticity of \( P \) implies that
\[
\| (I - P)^{\frac{1}{2}}P^l \|_{p-p} \leq \frac{C}{F}.
\]
(Recall that \( P \) is analytic on \( L^p \) for any \( p \in ]1, +\infty[ \), because \( P \) is sub-markovian and analytic on \( L^2 \), cf [9], p. 426). Moreover,
\[
\| p_l(., z) \|_p \leq \| p_l(., z) \|_2^\frac{1}{2} \| p_l(., z) \|_\infty^{1 - \frac{\gamma}{2}}
\]
\[
\leq p_l(z, z)^{\frac{1}{2}} \| p_l(., z) \|_\infty^{1 - \frac{\gamma}{2}}
\]
\[
\leq C\| p_l(., z) \|_\infty^{1 - \frac{\gamma}{2}}.
\]
In the last line, we used (29). Finally,
\[
(32) \quad |p_k(x, z) - p_k(y, z)| \leq C_\rho d(x, y)^{1 - \frac{\gamma}{2}}k^{-\frac{1}{2}(1 - \frac{\gamma}{2})}.
\]
Thanks to (30), one gets, applying (32),
\[
(33) \quad |p_k(x, z) - p_k(y, z)| \leq C_\rho \left[ \frac{d(x, y)}{\sqrt{k}} \right]^{1 - \frac{\gamma}{2}}p_k(z, z).
\]
Choosing \( z = x \) in (33) yields
\[
|p_k(x, x) - p_k(y, x)| \leq C_\rho \left[ \frac{d(x, y)}{\sqrt{k}} \right]^{1 - \frac{\gamma}{2}}p_k(x, x).
\]
As a consequence, for \( d(x, y) < a\sqrt{k} \) where \( a > 0 \) is small enough,
\[
|p_k(x, x) - p_k(y, x)| \leq \frac{1}{2}p_k(x, x),
\]
which implies that
\[
p_k(y, x) \geq c k^{-\frac{\gamma}{2}}
\]
whenever \(d(x, y) < a \sqrt{k}\). Therefore, by a chaining argument analogous to [13], p. 688, one gets that, whenever \(d(x, y) < ak\),

\[
p_k(x, y) \geq c k^{-\frac{3}{2}} \exp \left[ -\frac{d^2(x, y)}{C k} \right].
\]

This estimate, joined to (29), implies that the Harnack inequality, and therefore the Poincaré inequality holds on \(\Gamma\) (see [11], Theorem 1.7), which is false. Thus, the Riesz transform on \(\Gamma\) is not bounded on \(L^p\) for \(p > n\).

5. Appendix: proof of a density result on semigroups

Let us show Lemma 1. The proof is made up of three steps. We start with a general result about contracting semigroups on \(L^2\).

**Lemma 8.** Let \((T_t)_{t \geq 0}\) be a continuous semigroup of self-adjoint contracting operators on \(L^2(X)\), \(A\) its infinitesimal generator. Then

\[
\lim_{t \to +\infty} T_t f = I(f), \quad \forall f \in L^2(X),
\]

where \(I(f)\) is the orthogonal projection of \(f\) on \(\ker(A)\).

Here is the proof of this lemma. Write

\[
L^2(X) = \ker(A) \oplus \ker(A)^\perp.
\]

This is true because \(A\) is closed, so \(\ker(A)\) is closed. Let \(f\) belong to \(L^2(X)\). We decompose

\[
f = I(f) + g \quad \text{where} \quad g \in \ker(A)^\perp.
\]

and we notice that, for every \(t \geq 0\),

\[
T_t I(f) - I(f) = \int_0^t A T_s I(f) ds \\
= \int_0^t T_s A I(f) ds \\
= 0.
\]

Therefore, what is left to be shown is the fact that

\[
\lim_{t \to +\infty} T_t g = 0.
\]

We may regard \(A\) as an operator from \(\ker(A)^\perp\) to \(\ker(A)^\perp\). Considered in this way, \(A\) is one-to-one. Since \(A\) is normal, there exists a resolution of the identity \(E\) on \(\sigma(A)\) such that
\[ A = \int_{\sigma(A)} \lambda dE(\lambda). \]

Since \( T_t \) is contracting, \( \sigma(A) \subset [-\infty, 0] \). Moreover, \( A \) is one-to-one, so that \( E(\{0\}) = 0 \). Therefore, the dominated convergence theorem shows that

\[ \lim_{t \to +\infty} \int_{\sigma(A)} e^{2\lambda t} dE_{g,g}(\lambda) = 0. \]

The lemma is proved.

We now state the corresponding result for \( L^p(X) \):

**Lemma 9.** Let \( (T_t) \) be as in Lemma 8. Assume also that \( T_t \) contracts \( L^1(X) \) and \( L^\infty(X) \). Then one has

\[ \lim_{t \to +\infty} T_t f = I(f) \quad \text{in the } L^p \text{ norm, } \forall p \in ]1, +\infty[, \ \forall f \in L^1(X) \cap L^\infty(X). \]

First, we show that \( I(f) \in L^1(X) \cap L^\infty(X) \). Let \( (t_n) \) be a sequence of positive real numbers which converges to 0. One has \( T_{t_n} f \to I(f) \) in \( L^2(X) \), so that there exists a subsequence of \( (t_n) \), which we still call \( (t_n) \), such that \( T_{t_n} f \to f \) almost everywhere. Therefore, \( I(f) \in L^\infty(X) \) and Fatou’s lemma shows that \( I(f) \in L^1(X) \). Hence, we just have to write that \( T_t f - I(f) = T_t (f - I(f)) \), use Hölder’s inequality, Lemma 8 and the fact that \( \|T_t(f - I(f))\|_1 \leq \|f - I(f)\|_1 \) when \( 1 < p < 2 \) (resp. \( \|T_t(f - I(f))\|_\infty \leq \|f - I(f)\|_\infty \)) to prove Lemma 9.

We are now ready to prove Lemma 1. Let \( p \in ]1, +\infty[ \). We consider \( (T_t)_{t \geq 0} \) as a continuous semi-group on \( L^p(X) \), and write that

\[ T_t f - f = A(f_t) \quad \text{for } f \in L^p(X) \]

where

\[ f_t = \int_0^t T_sfds. \]

When \( t \to +\infty \), if \( f \in L^1_0(X) \cap L^\infty(X) \), \( T_t f \to 0 \) in the \( L^p \) norm. Moreover, \( f_t \in L^p(X) \cap D_p(A) \). Lemma 1 is proved.

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