ON ADDITIVE K-THEORY WITH THE LODAY - QUILLEN *-PRODUCT

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Abstract

The *-product defined by Loday and Quillen [17] on the additive K-theory (the cyclic homology with shifted degrees) $K_*^c(A)$ for a commutative ring $A$ is naturally extended to a product (*-product) on the additive K-theory $K_*^c(\Omega)$ for a differential graded algebra $(\Omega, d)$ over a commutative ring. We prove that Connes' B-maps from the additive K-theory $K_*^c(\Omega)$ to the negative cyclic homology $HC_*^c(\Omega)$ and to the Hochschild homology $HH_*^c(\Omega)$ are morphisms of algebras under the *-product on $K_*^c(\Omega)$. Applications to topology of Connes' B-maps are also described.

§0. Introduction

Let $A$ be an algebra over a commutative ring. Let $HC_*^c(A)$ and $HH_*^c(A)$ denote the negative cyclic homology and the Hochschild homology of $A$, respectively. In the algebraic K-theory, C. Hood and J. D. S. Jones [11] have constructed the Chern character $ch_n : K_n(A) \to HC_*^c(A)$ which is a lift of the Dennis trace map $Dtr : K_n(A) \to HH_n(A)$ by modifying basic construction due to Connes [5] and Karoubi [12]. When the algebra $A$ is commutative, the usual pairing of $K_*(A)$ and the product on $HC_*^c(A)$ defined by Hood and Jones in [11] make the character $ch_n$ into a morphism of algebras. In consequence, we can have the following commutative diagram in the category of graded algebras:

\[
\begin{array}{ccc}
K_*(A) & \xrightarrow{ch} & HC_*^c(A) \\
\downarrow{Dtr} & & \downarrow{h} \\
& HH_*^c(A) &
\end{array}
\]

(0.1)

Here $h$ is the map induced from the natural projection to the Hochschild complex from the cyclic bar complex. The Chern character $ch : K_0(A) \to HC_0^c(A) = HC_0^{per}(A)$ is connected with the ordinary Chern character $K(X) \to H^\text{even}_{\text{deRham}}(X; \mathbb{C})$ when $A$ is the ring consisting of smooth
functions from a compact manifold $X$ to the complex number $\mathbb{C}$ (see, for example, [19, 6.2.9. Example]). Therefore, one may expect that the Chern character $\text{ch}_n : K_n(A) \to \text{HC}_n^-(A)$ or the Dennis trace map $\text{Dtr} : K_n(A) \to \text{HH}_n(A)$ becomes a map with value in the de Rham (singular) cohomology of some manifold (space) by replacing the algebra $A$ with an appropriate object concerning with the space.

Hochschild and (negative) cyclic homologies can be extended to functors defined on the category of commutative differential graded algebras (DGAs) over a commutative ring (see [8], [11], [4]). In particular, if we choose the de Rham complex $(\Omega(X), d)$ of a simply connected manifold $X$ as the DGA, the Hochschild and the negative cyclic homology of $\Omega(X)$ can be regarded as the real cohomology and the real $\mathbb{T}$-equivariant cohomology of the space of free loops on $X$ respectively (see [8]), where $\mathbb{T}$ denotes the circle group. However, in algebraic $K$-theory, we can not expect such an extension. What is “$K$-theory” which admits an extension to a functor on the category of DGAs and in which there is a commutative diagram corresponding to (0.1)? We can consider the additive $K$-theory $K^+_s(A)$ (see [6]) as “$K$-theory”, which is isomorphic to the positive cyclic homology group $\text{HC}_{s-1}(A)$. Let $\phi$ be the isomorphism form $K^+_s(A)$ to $\text{HC}_{s-1}(A)$ defined by Loday and Quillen in [17] and independently Tsygan in [21]. Tillmann’s commutative diagram [20, Theorem 1] connects the dual of the Dennis trace map with the Connes $B$-map by the dual of the isomorphism $\phi : K^+_s(A) \to \text{HC}_{s-1}(A)$ when $A$ is a Banach algebra. Therefore it is natural to choose the Connes $B$-map $B_{\text{HH}} : \text{HC}_{s-1}(A) \to \text{HH}_s(A)$ as a map in the additive $K$-theory corresponding to the Dennis trace map in algebraic $K$-theory. The Connes’ $B$-map $B_{\text{HH}} : K^+_s(A) \cong \text{HC}_{s-1}(A) \to \text{HH}_s(A)$ has a natural lift $B$, which is also called Connes’ $B$-map, to the negative cyclic homology $\text{HC}_{-s}^{-}(A)$. Moreover functors $\text{HC}_s$, $\text{HC}_{-s}^-$, $\text{HH}_s$ and the connecting maps can be extend on the category of DGAs by using the cyclic bar complex in [7] and [8]. In the consequence, we can obtain the following commutative diagram corresponding to (0.1) in the category of graded modules:

\[
\begin{array}{ccc}
K^+_s(\Omega) & \cong & \text{HC}_{s-1}(\Omega) \\
 & \overset{B}{\longrightarrow} & \text{HC}_{-s}^-(\Omega) \\
\downarrow \text{B}_{\text{HH}} & & \downarrow \text{B} \\
\text{HH}_s(\Omega) & & \text{HH}_s(\Omega),
\end{array}
\]

where $\Omega$ is a DGA. We propose a natural question that whether the diagram (0.2) is commutative in the category of graded algebras, as well as the diagram (0.1), under an appropriate product on $K^+_s(\Omega)$. To answer this question, we extend the $*$-product defined by Loday and Quillen [17] to a product on the additive $K$-theory (the cyclic homology with shifted degrees) of a
DGA, which is an explicit version of that of Hood and Jones [11, Theorem 2.6]. Since the product is defined at chain level, we can see that

**Theorem 0.1.** The diagram (0.2) is commutative in the category of graded algebras when the product of $K^+_\ast(\Omega)$ is given by the $\ast$-product.

Let $M$ be a simply connected manifold and $LM$ the space of all smooth maps from circle group $T$ to $M$. By using the Connes’ B-map $B_{\text{HH}}$, we consider the vanishing problem of string class of a loop group bundle $L\text{Spin}(n) \to LQ \to LM$. In the consequence, a generalization of the main theorem in [14] is obtained when the given manifold $M$ is formal (see Theorem 2.1).

We also show that the algebra structure of $\text{HC}^+_\ast(\Omega)$ can be described with the $\ast$-product on $K^+_\ast$ via Connes’ B-map $B_{\text{HH}}$:

**Theorem 0.2.** Let $X$ be a formal simply connected manifold. Then

$$H^+_T(LX; \mathbb{R}) \cong \{H^*(LX; \mathbb{R})/\text{Im}(B_{\text{HH}} \circ 1)\}^{s+1} \oplus \mathbb{R}[u]$$

as an algebra, where $I : H^*(LX; \mathbb{R}) = \text{HH}_{-\ast}(\Omega(X)) \to K^+_{\ast}(\Omega(X))$ is the map in Connes’ exact sequence (1,1) mentioned in §1 for the de Rham complex $\Omega(X)$ with negative degrees and $\mathbb{R}[u]$ is the polynomial algebra over $u$ with degree 2. The multiplication of the algebra on the right hand side is given as follows: $w \ast u' = 0$ and $w \ast w' = w \cdot Bw'$, where $\cdot$ is the cup product on $H^*(LX; \mathbb{R})$.

In particular,

(i) if $H^*(X; \mathbb{R}) \cong \mathbb{R}[x]/(x^{s+1})$ and $s > 1$, then

$$H^+_T(LX; \mathbb{R}) \cong \oplus_{k \geq 0, 1 \leq j \leq s} \mathbb{R}\{\beta(j, k)\} \oplus \mathbb{R}[u]$$

as an algebra, where $\deg \beta(j, k) = j \deg x + k((s+1)\deg x - 2) - 1$, $\beta(j, k) \ast \beta(j', k') = 0$ and $\beta(j, k) \ast u = 0$ for any $j, k, j', k'$, and

(ii) if $H^*(X; \mathbb{R}) \cong \Lambda(y)$, then

$$H^+_T(LX; \mathbb{R}) \cong \oplus_{k \geq 0} \mathbb{R}\{\beta(k)\} \oplus \mathbb{R}[u]$$

as an algebra, where $\deg \beta(k) = (k+1)(\deg y - 1)$, $\beta(k) \ast \beta(j) = \beta(k+j+1)$ and $\beta(k) \ast u = 0$ for any $j, k$.

As for the algebra structure of $H^+_T(LX; \mathbb{R})$, the above results cover [13, Theorem 2.4].

This paper is set out as follows. In Section 1, we define the additive K-Theory $K^+_\ast(\Omega)$ of a DGA $(\Omega, d)$ over a commutative ring and a product ($\ast$-product) on $K^+_\ast(\Omega)$. Some properties of the $\ast$-product will also be studied.
In Section 2, we will describe the applications of Connes' B-maps $B$ and $B_{\text{HH}}$ which are mentioned above.

§1. The *-product on $K^+_n$

Let $(\Omega, d)$ be a commutative differential graded algebra (DGA) over a commutative ring $k$, $\Omega = \bigoplus_{i \leq 0} \Omega_i$, with unit 1 in $\Omega_0$, endowed with a differential $d$ of degree $-1$ satisfying $d(1) = 0$. We assume that differential graded algebras are non-positively graded algebras with the above properties unless otherwise stated. We recall the cyclic bar complex defined in [7] and [8]. The complex $(C(\Omega)[u^{-1}], b + uB)$ is defined as follows:

$$C(\Omega) = \sum_{k=0}^{\infty} \Omega \otimes \Omega^k,$$

$$b(\omega_0, \ldots, \omega_k) = -\sum_{i=0}^{k} (-1)^{i-1}(\omega_0, \ldots, \omega_{i-1}, d\omega_i, \omega_{i+1}, \ldots, \omega_k)$$

$$-\sum_{i=0}^{k-1} (-1)^i(\omega_0, \ldots, \omega_{i-1}, \omega_i\omega_{i+1}, \omega_{i+2}, \ldots, \omega_k)$$

$$+(-1)^{\deg \omega_k-1} \varepsilon_{k-1}(\omega_k\omega_0, \ldots, \omega_{k-1}), \quad b(u^{-1}) = 0$$

and

$$B(\omega_0, \ldots, \omega_k) = \sum_{i=0}^{k} (-1)(\varepsilon_{i-1} + 1)(\varepsilon_k - \varepsilon_{i-1})(1, \omega_0, \ldots, \omega_k, \omega_0, \ldots, \omega_{i-1}), \quad B(u^{-1}) = 0,$$

where $\overline{\Omega} = \Omega/k$, $\deg(\omega_0, \ldots, \omega_k) = \deg \omega_0 + \cdots + \deg \omega_k + k$, for $(\omega_0, \ldots, \omega_k)$ in $C(\Omega)$, $\varepsilon_i = \deg \omega_0 + \cdots + \deg \omega_i - i$ and $\deg u = -2$. Note that the formulas $bB + Bb = 0$ and $b^2 = B^2 = 0$ hold, see [7]. The negative cyclic homology $\text{HC}_{\varepsilon}^-(\Omega)$, the periodic cyclic homology $\text{HC}^\text{per}^-(\Omega)$ and the Hochschild homology $\text{HH}_n(\Omega)$ of a DGA $(\Omega, d)$ are defined as the homology of the complexes $(C(\Omega)[[u]], b + uB)$, $(C(\Omega)[[u, u^{-1}]], b + uB)$ and $(C(\Omega), b)$, respectively. Since a DGA in our case has negative degrees, the power series algebra $C(\Omega)[[u]]$ agrees with the polynomial algebra $C(\Omega)[u]$, similarly, $C(\Omega)[[u, u^{-1}]] = C(\Omega)[u, u^{-1}]$.

We define the $n$th additive $K$-theory $K_n^+(\Omega, d)$ of a DGA $(\Omega, d)$ to be the $(n-1)$-th cyclic homology $\text{HC}_{n-1}^-(\Omega, d)$ which is the $(n-1)$-th homology of the cyclic bar complex $(C(\Omega)[u^{-1}], b + uB)$:

$$K_n^+(\Omega) = \text{HC}_{n-1}^-(\Omega) = H_{n-1}(C(\Omega)[u^{-1}], b + uB).$$
Unless we note the differential $d$ of a DGA in particular, $K_n^+(\Omega,d)$ will be denoted by $K_n^+(\Omega)$. We define a product ($\ast$-product) on the complex $(C(\Omega)[u^{-1}], b + uB)$ as follows:

\[
\sum_{i=0}^{n} x_i u^{-i} \ast \sum_{j=0}^{m} y_j u^{-j} = \sum_{i=0}^{n} x_i \cdot B y_0 u^{-i},
\]

where $\ast$ is the shuffle product on $C(\Omega)$.

**Proposition 1.1.** (i) The $\ast$-product induces a degree $+1$ map of complexes $C(\Omega)[u^{-1}] \otimes C(\Omega)[u^{-1}] \to C(\Omega)[u^{-1}]$ which is associative.

(ii) The $\ast$-product on the cyclic bar complex defines an associative graded commutative algebra structure on $K^+_n(\Omega)$.

In [7], to give an $A_{\infty}$-algebra structure to the graded $k$-module $C(\Omega)[u]$, E. Getzler and J. D. S. Jones have defined a sequence of operators $B_k: C(\Omega)^{\otimes k} \to C(\Omega)$ of degree $k$ and have clarified relation of $B_k$, $B_{k-1}$ and the shuffle products on $C(\Omega)$. In particular, in order to prove Proposition 1.1, we need the following formula representing the relation of the operator $B_2$, Connes’ $B$-operator $B: C(\Omega) \to C(\Omega)$ and the shuffle products.

**Lemma 1.2.** ([7, Lemma 4.3]) There exists an operator $B_2: C(\Omega)^{\otimes 2} \to C(\Omega)$ of rank 2 satisfying

\[
(1.2.1) \quad (-1)^{|\alpha|+1} b B_2(\alpha, \beta) + B(\alpha \cdot \beta) = (-1)^{|b\alpha|+1} B_2(b\alpha, \beta) + (B\alpha) \cdot \beta
\]

\[+ (-1)^{|\alpha|} \{ \alpha \cdot B\beta + (-1)^{|\alpha|+1} B_2(\alpha, b\beta) \}.\]

The definitions of $B_2$ (see [7, page 280]) and $B$ enable us to deduce that, for any elements $z$ and $z'$ in $C(\Omega)$,

\[
(1.2.2) \quad B_2(z, Bz') = B_2(Bz, z') = 0.
\]

**Proof of Proposition 1.1.** (i) From the formulas (1.2.1) and (1.2.2), by replacing the element $\beta$ with $B\gamma$, it follows that $B(\alpha \cdot B\gamma) = B\alpha \cdot B\gamma$. For any elements $x = \sum x_i u^{-i}$, $y = \sum y_j u^{-j}$ and $z = \sum z_k u^{-k}$ in $C(\Omega)[u^{-1}]$, we see that, on $C(\Omega)[u^{-1}]$, $x \ast (y \ast z) = x \ast (\sum y_j \cdot Bz_0)u^{-j} = \sum x_i \cdot B(y_0 \cdot Bz_0)u^{-i} = (x \ast y) \ast z$. We will prove that $\ast$-product is a map of complexes. Since the differential $b$ is a derivation under the shuffle product on $C(\Omega)[u^{-1}]$, by the formula $b \circ B + B \circ b = 0$, we have
On the other hand, by the formula $B_0$, we have

$$(b + uB)(x * y) = (b + uB)\left(\sum_{i \geq 0} (x_i \cdot By_0)u^{-i}\right)$$

$$= \sum_{i \geq 0} (bx_i) \cdot By_0u^{-i} + \sum_{i \geq 0} (-1)^{|x_i|} x_i \cdot bBy_0u^{-i} + \sum_{i \geq 0} B(x_i \cdot By_0)u^{-i+1}$$

$$= \sum_{i \geq 0} (bx_i) \cdot By_0u^{-i} + \sum_{i \geq 0} (-1)^{|x_i|+1} x_i \cdot Bby_0u^{-i} + \sum_{i \geq 0} Bx_{i+1} \cdot By_0u^{-i}.$$ 

Thus we can conclude that $(b + uB)(x * y) = (b + uB)x * y + (-1)^{|x|+1} x * (b + uB)y$ for any $i$. 

(ii) To prove that the $*$-product defines a graded commutative algebra structure on $K^+ / C_{10}$, it suffices to prove that, for any elements $x = \sum x_i u^{-i}$ and $y = \sum y_j u^{-j}$ in $\text{Ker}(b + uB)$, there exists an element $\omega = \sum_{k \geq 0} \omega_k u^{-k}$ such that

$$x_k \cdot By_0 = (-1)^{|x|+1} (|y|+1) y_k \cdot Bx_0 = b\omega_{k-1} + B\omega_k$$ 

for any $k \geq 0$. We will verify that

$$\omega_k = (-1)^{|y|+1} |x| \left(\sum_{i+j=k} y_i \cdot x_j - \sum_{i+j=k+1} (-1)^{|y_i|} B_2(y_i, x_j)\right)$$

for $k \geq 0$ and

$$\omega_{-1} = (-1)^{|x|+1} |y| + 1 B_2(y_0, x_0)$$

are factors of the required element. Since equalities $by_i = -By_{i+1}$ and $bx_j = -B_{j+1}x_{j+1}$ hold, it follows that, if $k \geq 0$, 

$$\omega_k = (-1)^{|y|+1} |x| \left(\sum_{i+j=k} y_i \cdot x_j - \sum_{i+j=k+1} (-1)^{|y_i|} B_2(y_i, x_j)\right)$$
where maps B from the formulas (1.2.1) and (1.2.2). We can check that equality

\[ \sum_{i+j=k} B y_i \cdot x_j = \sum_{i+j=k} \sum_{i,j \geq 1} (-1)^{|y_i|} y_i \cdot B x_j \]

from the formulas (1.2.1) and (1.2.2). We can check that equality $b \omega_{-1} + B \omega_0 = x_0 \cdot B y_0 - (-1)^{|y|+1} \omega_{-1} \cdot B x_0$ holds in a similar way.

We define Connes’ B-maps $B_{HH} : K_n^+(\Omega) \to \text{HH}_n(\Omega)$ and $B : K_n^+(\Omega) \to \text{HC}_n^-(\Omega)$ by $B_{HH}(\sum_{i \geq 0} x_i u^{-i}) = B x_0$ and $B(\sum_{i \geq 0} x_i u^{-i}) = B x_0$. Note that the maps $B_{HH}$ and $B$ are connecting maps in Connes’ exact sequences ([16, Theorem 2.2.1 and Proposition 5.1.5])

\[ \cdots \to \text{HH}_{n+1}(\Omega) \xrightarrow{I} K_{n+2}^+(\Omega) \xrightarrow{S} K_n^+(\Omega) \xrightarrow{B_{HH}} \text{HH}_n(\Omega) \to \cdots \]

and

\[ \cdots \to \text{HC}_{n+1}^-(\Omega) \xrightarrow{\times u} \text{HC}_{n+1}^\text{per}(\Omega) \xrightarrow{B} K_n^+(\Omega) \to \cdots \]

respectively.

**Proof of Theorem 0.1.** The product structure $m_2$ on $C(\Omega)[u]$ defined by $m_2(\alpha_1, \alpha_2) = \alpha_1 \cdot \alpha_2 + (-1)^{|\alpha_1|+1} u B_2(\alpha_1, \alpha_2)$ induces the algebra structure of HC$_n^+(\Omega)$. From (1.2.2), we see that the product $m_2$ agrees with the shuffle product if $\alpha_1$ or $\alpha_2$ belongs to the image of the operator $B : C(\Omega) \to C(\Omega)$. Therefore the formula $B(\alpha \cdot B \gamma) = B \alpha \cdot B \gamma$ implies that the map $B : K_n^+(\Omega) \to \text{HC}_n^+(\Omega)$ is a morphism of algebras.

In study of the cyclic homology theory, it is often useful to consider the reduced theory. To prove some theorems below, we will use the reduced additive K-theory $\tilde{K}_n(\Omega)$ defined by $\tilde{K}_n(\Omega) = \text{Coker}(\iota : K_n^+(\Omega) \to K_n^+(\Omega)),$ where $\iota : \text{k} \to \Omega$ is the unit. The reduced additive K-theory $\tilde{K}_n(\Omega)$ is a direct summand of $K_n^+(\Omega)$ because the exact sequence $0 \to C(\text{k})[u^{-1}] \to C(\Omega)[u^{-1}] \to C(\Omega)[u^{-1}] \to 0$ of cyclic chain complexes is a split sequence.
More precisely, $K^+_c(\Omega)$ is isomorphic to $\tilde{K}^+_c(\Omega) \oplus k[u^{-1}]$ as a graded $HC_c(k) = k[u^{-1}]$-module. When one notices the direct summand $k[u^{-1}]$ of $K^+_c(\Omega)$, by definition of *-product, it follows that $k[u^{-1}]$ is included in the annihilator ideal of $K^+_c(\Omega)$. Therefore we can also conclude that the algebra $(K^+_c(\Omega), *)$ does not have an unit.

Let us consider a relation of the *-product on $K^+_c(\Omega)$ to the suspension map $S: K^+_c(\Omega) \to K^+_{c-2}(\Omega)$ in Connes’ exact sequence (1.1). Since the suspension map $S$ is defined by $S(\sum_{i \geq 0} x_i u^{-i}) = \sum_{i \geq 0} x_{i+1} u^{-i}$, it follows that $Sx \cdot y = S(x \cdot y)$ on $C(\Omega)[u^{-1}]$. From this fact and commutativity of the *-product, we have

**Proposition 1.3.** For any elements $\omega$ and $\eta$ in $K^+_c(\Omega)$,

$$S\omega \cdot \eta = S(\omega \cdot \eta) = \omega \cdot S\eta.$$  

For the rest of this paper, unless otherwise mentioned, we will assume that any DGA $(\Omega, d)$ is a commutative algebra over a field $k$ of characteristic zero, connected and simply connected, that is, $\Omega = \bigoplus_{i \leq 0} \Omega_i$, $\Omega_0 = k$, $H_1(\Omega) = 0$ and $d(1) = 0$. A DGA $(\Omega, d)$ is said to be *formal* if there exists a DGA-morphism from the minimal model $\mathbb{M}$ of $\Omega$ to the DGA $(H^*(\Omega, d), 0)$ which induces a isomorphism between their homologies (see [10]).

For any DGA $(\Omega, d)$ with the trivial differential, M. Vigué-Poirrier has given a decomposition of the negative cyclic homology $HC^-_c(\Omega, 0) = \sum_{q \geq 0} H(\mathcal{C}^q_{\infty}[u], b + uB)$, and has shown that the $S$-action on $HC^-_c(\Omega, 0)$ is trivial, see [22, Proposition 5], where $\mathcal{C}^q_{\infty} = \{(a_0, \ldots, a_p) | \sum \deg a_i = -q, -q + p = -n\}$. This fact implies that the $S$-action on $HC^-_c(\Omega, d)$ for any formal DGA $(\Omega, d)$ is trivial ([22, Théorème A]). The proof of [22, Proposition 5] is based on Goodwillie’s result [9, Corollary III.4.4], which is led from the following proposition.

**Proposition 1.4.** [9]. Let $(\Omega, d)$ be a DGA over a commutative ring and $D$ a derivation on $\Omega$ with degree $|D|$ satisfying that $D(ab) = (Da)b + (-1)^{|D||a|}a(Db)$ and $[D, d] = 0$. Then there exist chain maps $e_D: C(\Omega) \to C(\Omega)$ of degree $|D| - 1$, $E_D: C(\Omega) \to C(\Omega)$ of degree $|D| + 1$ and an operator $L_D: C(\Omega) \to C(\Omega)$ of degree $|D|$ such that $[u^{-1}e_D + E_D, b + uB] = L_D$ in $C(\Omega)[u, u^{-1}]$, where $[a, b] = ab - (-1)^{|a||b|}ba$ for any operators $a$ and $b$.

We can obtain a lemma by using Proposition 1.4 and the idea of the decomposition of cyclic homology due to Vigué-Poirrier [22].

**Lemma 1.5.** Let $(\Omega, 0)$ be a DGA with the trivial differential. For any ele-
ment $\omega$ in $K_+^+(\Omega, 0) = \widehat{H}_{-1}(\Omega, 0)$, there exists an element $\eta_0$ in $C(\Omega) \cap \ker b$ such that $\omega = [\eta_0]$ in $K_+^+(\Omega, 0)$.

**Proof.** According to Vigué-Poirrier [22], we define a derivation $D$ on $\Omega$ by $D(a) = (\deg a)a$. Consider a decomposition $K_+^+(\Omega, 0) = \sum_{q \geq 0} K_+^+(\Omega, 0)^q$ defined by $K_+^+(\Omega, 0)^q = H_{-1}(\mathbb{K}^*_q[u^{-1}], b + uB)$. Since $K_+^+(\Omega)$ is isomorphic to $\sum_{q \geq 1} K_+^+(\Omega)^q$, in order to prove Lemma 1.5, it suffices to show that there exists an element $\eta_0$ with the property in Lemma 1.5 for any element $\omega$ in $K_+^+(\Omega)^q$ ($q \geq 1$). Since the operation $L_D$ on $C(\Omega)$ is defined by $L_D(a_0, \ldots, a_p) = \sum_{0 \leq j \leq p} (a_0, \ldots, D_{a_j}, \ldots, a_p)$, it follows that the operator $L_D$ on $K_+^+(\Omega)^q$ is given by $L_D(\omega) = -q\omega$ in our case. On the other hand, for any element $\omega$ in $K_+^+(\Omega)^q$ which is represented by $\sum_{i \geq 0} \omega_i u^{-i}$ in $C(\Omega)[u^{-1}]$, we have that $[u^{-1}e_D + E_D, b + uB]\omega = e_0 B_{\omega_0} + uE_D b_{\omega_0} - (b + uB)(u^{-1}e_D + E_D)\omega$ in $C(\Omega)[u^{-1}]$. By virtue of Proposition 1.4, we can conclude that $e_0 B_{\omega_0} - (b + uB)(u^{-1}e_D + E_D)\omega = -q\omega$ in $C(\Omega)[u^{-1}]$. Thus, we see that $-\frac{1}{q} e_0 B_{\omega_0}$ is the required element $\eta_0$.

We will consider the algebra structure of $K_+^+(\Omega)$ by using a minimal model of $(\Omega, d)$. Let $\varphi : (\mathcal{A}_d, d_M) \longrightarrow (\Omega, d)$ be a minimal model of a DGA $(\Omega, d)$. Then $\varphi$ induces an isomorphism of algebras $K(\varphi) : K_+^+(\mathcal{A}_d) \longrightarrow K_+^+(\Omega)$. Therefore if a DGA $(\Omega, d)$ is formal, then there exist isomorphisms $K_+^+(\Omega, d) \simeq K_+^+(\mathcal{A}_d, d_M) \simeq K_+^+(H(\Omega), 0)$. It follows immediately that the isomorphisms are compatible with the S-action. Since Lemma 1.5 asserts that any element of $K_+^+(\Omega, 0)$ can be represented by an element with column degree 0, from the definition of S-action, we can get

**Proposition 1.6.** If a DGA $(\Omega, d)$ is formal, then the suspension map $S : K_+^+(\Omega) \longrightarrow K_+^{+2}(\Omega)$ is trivial.

Let $(\mathcal{A}_d, d_M)$ be a free commutative differential graded algebra $(AV, d)$ over $k$. We denote by $(\mathcal{E}(\mathcal{A}_d), \delta, \beta)$ the double complex defined in [4, Example 2] by D. Burghelena and M. Vigué-Poirrier. Namely, $\mathcal{E}(\mathcal{A}_d) = \Lambda(V \oplus \hat{V})$, $\beta$ is the unique derivation of degree +1 defined by $\beta v = \hat{v}$ and $\delta$ is the unique derivation of degree −1 which satisfies $\delta v = d$ and $\delta \beta + \beta \delta = 0$. Here $\hat{V}$ is the vector space with $\hat{V}_{n+1} = V_n$. We here mention that the double complex induces the complex $(\mathcal{E}(\mathcal{A}_d)[u^{-1}], \delta + u\beta)$ with a product defined by $\sum \omega_i u^{-i} \circ \sum \eta_j u^{-j} = \sum \omega_i \eta_j u^{-i-j}$. By [4, Theorem 2.4 (i)], we see that the map $\Theta : C(\mathcal{A}_d) \rightarrow \mathcal{E}(\mathcal{A}_d)$ defined by $\Theta(a_0, a_1, \ldots, a_p) = 1/p! a_0 \beta a_1 \cdots \beta a_p$ is a chain map between the double complexes $(C(\mathcal{A}_d), b, B)$ and $(\mathcal{E}(\mathcal{A}_d), \delta, \beta)$. [4, Theorem 2.4 (iii)] shows that the induced map $K(\Theta)$ from $K_+^+(\mathcal{A}_d)$ to $H_{-1}(\mathcal{E}(\mathcal{A}_d)[u^{-1}], \delta + u\beta)$ is an isomorphism of graded vector spaces. Moreover we have
Proposition 1.7. The map $K(\Theta) : K^+_*(\mathcal{M}) \to H_{*-1}(\delta(\mathcal{M})[u^{-1}], \delta + u\beta)$ is an isomorphism of algebras.

The following lemma will be needed to prove that $K(\Theta)$ is a morphism of algebras.

Lemma 1.8. Let $(\mathcal{M}, d_{\mathcal{M}})$ be a free commutative DGA.

(i) The chain map $\Theta : C(\mathcal{M}) \to \delta(\mathcal{M})$ is compatible with $B : C(\Omega) \to C(\Omega)$ and $\beta : \delta(\mathcal{M}) \to \delta(\mathcal{M}) : \beta \Theta = \Theta B$.

(ii) Let $W$ be a subspace of $C(\mathcal{M})$ consisting of the elements whose first factor have even degree. $W = \{ \sum_i (a_0, \ldots, a_k(i)) \in C(\mathcal{M}) \mid \deg a_0 \text{ is even} \}$. Then $\Theta(\omega \cdot \omega') = \Theta \omega \cdot \Theta \omega'$ for any element $\omega'$ in $W$ and any element $\omega$ in $C(\mathcal{M})$, where $\ast$ in the left hand side and right hand side are the shuffle product on $C(\mathcal{M})$ and the natural product on $\delta(\mathcal{M})$ respectively.

Proof. It is straightforward to check that identities $\beta \Theta = \Theta B$ and $\Theta(\omega \cdot \omega') = \Theta \omega \cdot \Theta \omega'$ hold.

Proof of Proposition 1.7. From the definition of Connes' B-operator, it follows that Im $B$ is a subspace of $W$ in Lemma 1.8. By virtue of Lemma 1.8, we see that $\Theta(\omega \cdot B\omega') = \Theta \omega \cdot \beta \Theta \omega'$ for any element $\omega$ and $\omega'$ in $C(\mathcal{M})$. Thus we can conclude that $K(\Theta)$ is a morphism of graded algebras.

By virtue of Proposition 1.7, we can determine $K^+_*(\Omega)$ explicitly as an algebra when the homology of $(\Omega, d)$ is generated with one generator.

Theorem 1.9. For any formal DGA $(\Omega, d)$,$$
K^+_*(\Omega) \cong \widetilde{\operatorname{HH}}_{*-1}(\Omega) / \operatorname{Im}(B_{\operatorname{HH}} \circ I : \widetilde{\operatorname{HH}}_{*-2}(\Omega)
\to \widetilde{\operatorname{HH}}_{*-1}(\Omega)) \oplus k\{1, u^{-1}, u^{-2}, \ldots\}
$$as an algebra, where $\deg u^{-k} = 2k + 1$, $\omega \ast \omega' = \omega \cdot B\omega'$, $\omega \ast u^{-k} = 0$ for any elements $\omega$ and $\omega'$ in $\widetilde{\operatorname{HH}}_{*}(\Omega) / \operatorname{Im}(B_{\operatorname{HH}} \circ I)$ and $u^{-1} \ast u^{-1} = 0$. In particular,

(i) when $\deg x$ is even,

$$
K^+_*(k[x]/(x^{s+1})) \cong \oplus_{k \geq 0, j \leq 2} k\{\beta(j, k)\} \oplus k\{1, u^{-1}, u^{-2}, \ldots\},
$$

where $\deg \beta(j, k) = j \deg x + k((s + 1)\deg x + 1) + 1$, $\omega \ast \omega' = 0$ for any elements $\omega$ and $\omega'$ in $K^+_*(k[x]/(x^{s+1}))$, and

(ii) when $\deg y$ is odd,

$$
K^+_*(k[y]/(y^{s+1})) \cong \oplus_{k \geq 0} k\{\beta(k)\} \oplus k\{1, u^{-1}, u^{-2}, \ldots\},
$$

where $\deg \beta(k) = \deg y + k((s + 1)\deg y + 1)$, $\beta(k) \ast \beta(j) = \beta(k + j + 1)$, $1 \ast \beta(k) = 0$ and $\beta(k) \ast u^{-1} = 0$.

Proof. By Proposition 1.6, the suspension map $S : \tilde{K}^+_*(\Omega) \to \tilde{K}^+_{*-2}(\Omega)$ is
trivial. From this fact and Connes’ exact sequence (1.1) obtained by using \( \Omega \) instead of a DGA \( \Omega \), it follows that the map \( I: \overline{\text{HH}}_{n-1}(\Omega) \to \overline{\text{K}}_{n}^+(\Omega) \) is surjective and that the kernel of \( I \) is the image of \( B_{\text{HH}} \circ I: \overline{\text{HH}}_{n-2}(\Omega) \to \overline{\text{HH}}_{n-1}(\Omega) \). Thus we can conclude that \( \overline{\text{K}}_{n}^+(\Omega) \cong \overline{\text{K}}_{n}^+(\Omega) \oplus \mathbb{k}\{1,u^{-1},u^{-2},\ldots\} \cong \overline{\text{HH}}_{n-1}(\Omega)/\text{Im}(B_{\text{HH}} \circ I) \oplus \mathbb{k}\{1,u^{-1},u^{-2},\ldots\} \) as algebras. From Proposition 1.7 and the explicit formulas of the Hochschild homology of \( \mathbb{k}[x]/(x^{r+1}) \) and \( A(y) \) in [15], we can get (i) and (ii).

**Remark.** In Theorem 1.9, the elements \( \beta(j,k) \) and \( \beta(k) \) correspond to the elements \( x^j \omega^k \) and \( y \overline{\tau}^k \) in [15, Proposition 1.1(iii)], respectively.

As mentioned before Proposition 1.3, the algebra \( \overline{\text{K}}_{n}^+(\Omega) \) does not have an unit. Since \( \overline{\text{K}}_{n}^+(\Omega) \) is non zero in general, the algebra \( \overline{\text{K}}_{n}^+(\Omega) \) may be have an unit. However, the results of Theorem 1.9 (i) and (ii) enable us to conjecture that the reduced additive K-theory \( \overline{\text{K}}_{n}^+(\Omega) \) does not have an unit for any DGA either. The first assertion in the following proposition is an answer to the conjecture.

**Proposition 1.10.** (i) Let \( (\Omega,d) \) be a DGA. Assume that \( \overline{\text{K}}_{n}^+(\Omega) \neq 0 \). Then the algebra \( \overline{\text{K}}_{n}^+(\Omega) \) does not have an unit.

(ii) If \( \deg QH(\Omega,d) \geq n \), then there exist \( n \) elements \( x_1, x_2, \ldots, x_n \) in \( \overline{\text{K}}_{n}^+(\Omega) \) such that \( x_1 \ast x_2 \ast \cdots \ast x_n \neq 0 \), where \( QH(\Omega,d) \) denotes the space of indecomposable elements in the graded algebra \( H(\Omega,d) \).

**Proof.** From the usual argument on a minimal model of \( \Omega \), we can assume that \( \Omega \) is free.

(i) Suppose that there exists an element \( e \) in \( \overline{\text{K}}_{n}^+(\Omega) \) such that \( e \ast x = x \) for any \( x \) in \( \overline{\text{K}}_{n}^+(\Omega) \). Let us consider the Hodge decomposition of Hochschild homology ([3], [4, Theorem 3.1]): \( \overline{\text{HH}}_n(\Omega) = \bigoplus_{i=0}^\infty \overline{\text{HH}}^i_n(\Omega) \). Since \( B_{\text{HH}} : \overline{\text{K}}_{n}^+(\Omega) \to \overline{\text{HH}}_n(\Omega) \) is a morphism of algebras by Theorem 0.1, it follows that \( B_{\text{HH}}(e) \ast B_{\text{HH}}(x) = B_{\text{HH}}(x) \). We see that \( B_{\text{HH}}(e) \) belongs to \( \overline{\text{HH}}^0_n(\Omega) \) because \( \deg B_{\text{HH}}(e) = 0 \). The definition of the Hodge decomposition and Lemma 1.8 (i) enables us to deduce that \( \text{Im} B_{\text{HH}} \) is included in \( \bigoplus_{i \geq 1} \overline{\text{HH}}^i_n(\Omega) \). Thus we have \( B_{\text{HH}}(e) = 0 \). On the other hand, we see that \( S^N e = 0 \) for some sufficient large integer \( N \). If \( B_{\text{HH}}(x) = 0 \) for all \( x \in \overline{\text{K}}_{n}^+(\Omega) \), then the map \( S : \overline{\text{K}}_{n+2}^+(\Omega) \to \overline{\text{K}}_{n}^+(\Omega) \) is epimorphism. Therefore, for any \( x \in \overline{\text{K}}_{n}^+(\Omega) \), there is an element \( x' \in \overline{\text{K}}_{n}^+(\Omega) \) such that \( S^N x' = x \). It follows from Proposition 1.3 that \( x = e \ast x = e \ast S^N x' = S^N e \ast x' = 0 \) for any \( x \), which a contradiction. Thus \( B_{\text{HH}}(x) \neq 0 \) for some \( x \in \overline{\text{K}}_{n}^+(\Omega) \). However, \( B_{\text{HH}}(x) = B_{\text{HH}}(e) \cdot B_{\text{HH}}(x) = 0 \). The result now follows.

(ii) We can choose \( n \) elements of \( \Omega \) corresponding to \( x_i \) in \( \overline{\text{K}}_{n}^+(\Omega) \) which are part of generators of \( \Omega \). We represent the elements with the same notation \( x_1, \ldots, x_n \), respectively. Under the isomorphism \( H(\Theta) \) in [4, Theorem 2.4}
(ii)] \( B_{HH}(x_1 \cdots x_n) = B_{HH}x_1 \cdots B_{HH}x_n = \tilde{x}_1 \cdots \tilde{x}_n \) in \( H_*(\Tot \delta(\Omega), \delta) \). Since \( \text{Im} \delta \) consists of elements whose factors have an element in \( \Omega \), it follows that \( \tilde{x}_1 \cdots \tilde{x}_n \neq 0 \) in \( HH_*(\Omega) \cong H_*(\Tot \delta(\Omega), \delta) \). By virtue of Proposition 1.7, we can see that \( x_1 \cdots x_n \neq 0 \) in \( K_+^*(\Omega) \).

From Proposition 1.10 (ii), Theorem 1.9 (i) and (ii), we can conclude that \( K_+^*(\Omega) \) has trivial algebra structure if and only if the homology of \( (\Omega, d) \) is generated by one element with even degree.

\section{Applications of Connes’ B-maps \( B_{HH} \) and \( B \)}

Let \( M \) be a simply connected manifold and \( LM \) the space of \( C^\infty \)-free loops on \( M \). When an \( SO(n) \)-bundle \( P \to M \) over \( M \) has a spin structure \( Q \to M \), the string class \( \mu(Q) \), which belongs to \( H^3(LM; \mathbb{Z}) \), is defined as an obstruction to lift the structure group \( LSpin(n) \) of \( LQ \to LM \) to \( LSpin(n) \), for details see [18]. Here \( LSpin(n) \) is the universal central extension of \( LSpin(n) \) by the circle. One of important properties for the string class \( \mu(Q) \) is the fact that the class \( \mu(Q) \) is the image of \( \frac{1}{2} p_1 \) by the map \( \int_{S^1} \circ \text{ev}^*: H^*(M; \mathbb{Z}) \to H^*(LM \times S^1; \mathbb{Z}) \to H^{*-1}(LM; \mathbb{Z}) \), where \( p_1 \) is the first Pontrjagin class of the bundle \( P \to M \), \( \text{ev}: LM \times S^1 \to M \) is the evaluation map and \( \int_{S^1} \) is the integration along \( S^1 \). Let \( G \) be a linear Lie group and \( \xi: Q \to M \) a \( G \)-bundle over \( M \). Let \( \text{Ch}^{p+1}(\xi) \) be the Chern character of the bundle \( \xi \). The higher string classes \( \tilde{C}^p(L\xi) \) (\( p \geq 1 \)) (see [2]) in \( H^{2p+1}(LM; \mathbb{C}) \) defined for the \( LG \)-bundle \( L\xi: LQ \to LM \) has a similar property to the ordinary string class \( \mu(Q) \). Indeed, the \( p \)th string class \( \tilde{C}^p(L\xi) \) is the image of \( -(2\pi \sqrt{-1})^{p+1} p! \text{Ch}^{p+1}(\xi) \) by the map \( \int_{S^1} \circ \text{ev}^* \). As mentioned in the introduction, in the study of the problem of whether the map \( \int_{S^1} \circ \text{ev}^* \) is injective, the Connes’ B-map \( B_{HH}: K_+^*(\Omega(M)) \to HH_*(\Omega(M)) \cong H^*(LM; \mathbb{R}) \) plays an important role. We will have the following theorem which is a generalization of [14, Theorem 2]. We may call a simply connected manifold \textit{formal} if its de Rham complex is formal (see [10]).

\textbf{Theorem 2.1.} Let \( M \) be a simply connected manifold and formal.

(i) For any \( SO(n) \)-bundle \( P \to M \) with a spin structure \( Q \to M \), if \( H^3(M; \mathbb{Z}) \) is torsion free, then the string class \( \mu(Q) \) vanishes if and only if \( \frac{1}{2} p_1 \) vanishes.

(ii) Let \( G \) be a linear Lie algebra and \( \xi: Q \to M \) a \( G \)-bundle. The string class \( \tilde{C}^p(L\xi) \) vanishes if and only if the Chern character \( \text{Ch}^{p+1}(\xi) \) of the bundle \( \xi \) vanishes.

By virtue of [14, Proposition 2.1], we can regard the map \( \int_{S^1} \circ \text{ev}^*: H^*(M; \mathbb{R}) \to H^{*-1}(LM; \mathbb{R}) \) as the map \( \alpha: H^*(M; \mathbb{R}) \to HH_*(\Omega(M), d) \) defined by \( \alpha(x) = (1, x) \) under the identification by the iterated integral map \( \sigma: HH_*(\Omega(M)) \to H^*(LM; \mathbb{R}) \) ([8]), where \( \Omega_i(M) \) is the \( i \)th de Rham
complex $\Omega^i_{\text{de Rham}}(M)$ and the differential $d : \Omega^i(M) \to \Omega^{i-1}(M)$ is the exterior differential on the de Rham complex $\Omega_{\text{de Rham}}^*(M)$. Thus, to prove Theorem 2.1, it suffices to show that the map $\alpha$ is injective when $M$ is formal. Note that, for any DGA $(\Omega, d)$, we can define the map $\alpha : H_*(\Omega) \to \text{HH}_*(\Omega)$ by $\alpha(x) = (1, x)$. The definition of the map $\alpha$ allows us to deduce that $\alpha$ factors through Connes’ B-map $B_{\text{HH}}$ as follows: $\alpha = B_{\text{HH}} \circ i \circ i$, where $i : H_*(\Omega) \to \text{HH}_*(\Omega)$ and $I : \text{HH}_*(\Omega) \to K^+_*(\Omega)$ are the homomorphisms induced by the natural inclusions $\Omega \to C(\Omega)$ and $C(\Omega) \to C(\Omega)[u-1]$ respectively. For any DGA $(\Omega, d)$, we have

**Lemma 2.2.** The map $H_{-*}(\Omega) \xrightarrow{i} \text{HH}_{-*}(\Omega) \xrightarrow{i} K^+_{-*}(\Omega)$ is injective.

**Proof.** It suffices to prove that Lemma 2.2 holds when $\Omega$ is free. In this case, we can identify $K^+_*(\Omega)$ with the homology of the complex $(\mathcal{E}(\Omega)[u^{-1}], \delta + u\beta)$ by Proposition 1.7. Since $\text{Im}(\delta + u\beta) \cap \Omega$ is contained in $\text{Im} d$ which is a subspace of $\Omega$, it follows that if $II(x)$ is zero in $K^+_*(\Omega)$, then so is $x$ in $H_*(\Omega)$.

**Proof of Theorem 2.1.** The reduced additive K-theory $\check{K}^+_*(\Omega)$ includes $\text{Im}(I \circ i : \text{HC}^+_*(\Omega) \to \text{HC}^+_*(\Omega))$ for $* < 1$. By Proposition 1.6, Connes’ B-map $B_{\text{HH}} : \check{K}^+_*(\Omega) \to \text{HH}_*(\Omega)$ is injective. Therefore we can have Theorem 2.1 by virtue of Lemma 2.2.

In general case, we can show that $II(\text{Ker } \alpha)(= \text{Im}(I \circ i) \cap \text{Ker } B_{\text{HH}})$ is contained in the space of annihilators of $K^+_*(\Omega)$.

**Proposition 2.3.** For any DGA $(\Omega, d)$, $K^+_*(\Omega) * \{\text{Im}(I \circ i) \cap \text{Ker } B_{\text{HH}}\} = 0$.

**Proof.** For any element $\omega$ in $\text{Im}(I \circ i) \cap \text{Ker } B_{\text{HH}}$, we can write $\beta \omega = \delta \eta$ for some element $\eta$ in $\mathcal{E}(\Omega)$. For any element $\omega'$ in $\text{Ker}(u\beta + \delta)$ which is the subspace of $\mathcal{E}(\Omega)[u^{-1}]$,

$$(u\beta + \delta)(\omega' \cdot \eta) = (-1)^{\deg \omega'} \omega' \cdot (u\beta + \delta) \eta$$

$$= (-1)^{\deg \omega'} \omega' \cdot (0 + \beta \omega)$$

$$= (-1)^{\deg \omega'} \omega' * \omega$$

Note that $\beta \eta = 0$ in $\mathcal{E}(\Omega)[u^{-1}]$. Thus we see that $\omega' \ast \omega = 0$ in $K^+_*(\Omega)$.

We will describe some applications of Connes’ B-map $B : K^+_*(\Omega) \to \text{HC}^+_*(\Omega)$.

**Proposition 2.4.** The following diagram is commutative:
The map $B : \hat{K}_s^+(\Omega) \rightarrow \hat{HC}_s^+(\Omega) \rightarrow \hat{HC}_s^+(\Omega)$ induced by Connes’ B-map is an isomorphism of algebras.

The S-action on $\hat{K}_s^+(\Omega)$ is trivial if and only if so is the S-action on $\hat{HC}_s^+(\Omega)$.

If the S-action on $\hat{HC}_s^+(\Omega)$ is trivial, then $\hat{HC}_s^+(\Omega)$ is an algebra. By the assertions (i) and (ii), we see that $\hat{HC}_s^+(\Omega)$ is an algebra. From the proof of Theorem 1.9, we deduce the results of (iii).

We can now prove Theorem 0.2.

Proof of Theorem 0.2. If $H^*(X; R)$ is isomorphic to the algebra $\mathbb{R}[x]/(x^{r+1})$ or $\mathbb{A}(y)$, then $X$ is a formal. By virtue of Theorem 2.5 (iii), we see that $H_*(LX; R) \cong HC_{s-1}(\Omega(X)) \cong HC_{s-1}(H_*(X; R)) \cong R[u] \oplus \hat{K}_s^+(H_*(X; R))$. Therefore, Theorem 1.9 yields Theorem 0.2. In particular, we deduce (i) and (ii) by virtue of Theorem 1.9 (i) and (ii).

Let $M$ be a simply connected manifold (simplicial complex) and $\Omega^*(M)$ its de Rham algebra of differential forms (simplicial differential forms) with coefficient in $k = \mathbb{R}, \mathbb{C}$ (or $\mathbb{R}, \mathbb{C}, \mathbb{Q}$). Then the isomorphism $B : \hat{K}_s^+(\Omega) \rightarrow \hat{HC}_s^+(\Omega)$ in Theorem 2.5 (i) agrees with the isomorphism $b_M : HC_{s-1}(\Omega(M)) \rightarrow H_*(LM; k)$ in [3, Theorem B]. Therefore, if we regard $\hat{K}_s^+(\Omega)$ as a graded algebra with the *-product, the isomorphism $b_M$ becomes a morphism of algebras.
Let $(\Omega, d)$ and $(\Omega', d')$ be DGAs over a field $k$ of characteristic zero. If one wants to know about the $k$-module structure of the negative cyclic homology $\mathrm{HC}_-^-(\Omega \otimes \Omega')$, the use of the Künneth theorem [11, Theorem 3.1 (a)] for negative cyclic homology theory may be effective, because the exact sequence

$$0 \to \mathrm{HC}_-^-(\Omega) \otimes_{k[u]} \mathrm{HC}_-^-(\Omega') \to \mathrm{HC}_-^-(\Omega \otimes \Omega') \to \mathrm{Tor}_{k[u]}(\mathrm{HC}_-^-(\Omega), \mathrm{HC}_-^-(\Omega'))_{s-1} \to 0$$

is split. However, it is not easy to determine the algebra structure of $\mathrm{HC}_-^-(\Omega \otimes \Omega')$ from the exact sequence even if $\Omega$ and $\Omega'$ are formal. Theorem 2.5 (ii) enables us to represent the graded algebra structure of $\mathrm{HC}_-^-(\Omega \otimes \Omega')$ with the Hochschild homologies $\mathrm{HH}_-(\Omega)$, $\mathrm{HH}_-(\Omega')$ and the *-product when $\Omega$ and $\Omega'$ are formal. In term of spaces, we also assert that the $\mathcal{T}$-equivariant cohomology of the space of loops on the product space $M \times M'$ can be represented with the cohomologies of the spaces of loops on $M$ and $M'$, Connes' $\mathcal{B}$-map $B_{\mathrm{HH}}$ and *-product.

**Corollary 2.6.** Let $M$ and $M'$ be formal simply connected manifolds. Then

$$H_*^T(L(M \times M') ; R) \cong \{(H^*(LM; R) \otimes H^*(LM'; R) / \text{Im}(B \circ I \otimes 1 \pm 1 \otimes B \circ I))^{s+1} \otimes R[u]\}$$

as an algebra, where $\deg u = 2$. Here the multiplication $*$ of the algebra on the right hand side is given as follows: $\omega \otimes \omega' * u = 0$, $\omega \otimes \omega' * \eta \otimes \eta' = \omega \otimes \omega' \cdot (B I \eta \otimes \eta' + (-1)^{|\eta|} \eta \otimes B I \eta')$ for any $\omega \otimes \omega'$ and $\eta \otimes \eta'$ in $H^*(LM; R) \otimes H^*(LM'; R) / \text{Im}(B \circ I \otimes 1 \pm 1 \otimes B \circ I)$, where $\cdot$ is the cup product on $H^*(LM; R) \otimes H^*(LM'; R)$.

**Proof.** Let $(\mathcal{M}, d)$ and $(\mathcal{M}', d)$ be minimal models of de Rham complexes $(\Omega(M), d)$ and $(\Omega(M'), d)$ respectively. We know that $\mathrm{HH}_-(\mathcal{M}) \cong H^*(LM; R)$ and $\mathrm{HC}_-^-(\mathcal{M}) \cong H_*^T(LM; R)$ as algebras ([8]). By virtue of [22, Proposition 5], the $S$-action on $\mathrm{HC}_-^-(\mathcal{M})$ is trivial. Therefore, it follows from Theorem 2.5 (ii) that, as algebras,
$H^+_1(L(M \times M'); R) \cong HC^{-s}_-(\mathcal{M} \otimes \mathcal{M'})$

$\cong HH_{-s-1}(\mathcal{M} \otimes \mathcal{M})/\text{Im}(B_{HH} \circ I) \oplus R[u]$

$\cong H_{-s-1}(\mathcal{M} \otimes \mathcal{M}')/\text{Im}(B_{HH} \circ I) \oplus R[u]$

$\cong \{H_*(\mathcal{M}) \otimes H_*(\mathcal{M}')\}/$

$\text{Im}(\beta \circ I \otimes 1 \pm 1 \otimes \beta \circ I)\}_{-s-1} \oplus R[u]$

$\cong \{H^*(LM; R) \otimes H^*(LM'; R)/$

$\text{Im}(B \circ I \otimes 1 \pm 1 \otimes B \circ I)\}_{s+1} \oplus R[u].$

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