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ANALYTIC PERTURBATION PRESERVES DETERMINACY OF INFINITE INDEX

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In this note we give an answer to a question of Christian Berg. To formulate the question and our result we need some definitions and notation.

Let μ be a positive measure on the real axis having moments of every order. A measure μ is determinate if no other measure has the same moments as those of μ, otherwise μ is indeterminate. By $\mathscr{M}_0$ we denote the set of measures having a finite number of real points as support. Following [2–5] (see also [8]), we say that a measure μ has an infinite index of determinacy if, for any measure $\mu_0 \in \mathscr{M}_0$, the measure $\mu + \mu_0$ is determinate. The question posed by Berg is the following. Suppose that the measure μ has infinite index of determinacy and the measure ν has a compact support. Is it true that the measure $\mu + \nu$ is determinate?

Here, we will give a positive answer to this question. Moreover, we will prove a stronger result.

One of the most known sufficient determinacy condition on a measure ν is the following [6, Theorem 5.2]

$$(1) \qquad \int_{-\infty}^{\infty} e^{\epsilon|t|}\, d\nu < \infty, \quad \text{for some } \epsilon > 0.$$

In this case, the Fourier transform of the measure ν

$$\phi(z) = \int_{-\infty}^{\infty} e^{itz}\, d\nu,$$

is analytic in the strip $|\mathrm{Im} z| < \epsilon$. The main result of this note is the following:

Theorem. *Suppose that μ has infinite index of determinacy and ν satisfies condition* (1). *Then the measure $\mu + \nu$ is determinate.*

A counterpart of this theorem in the setting of the Bernstein approximation problem was recently proved by Sodin [8]. However, his proof is based

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on the de Branges criterion of density of polynomials in weighted spaces with uniform norm which apparently does not work in the case of L^2-norm. As usual, the question in L^2-norm can be treated by relatively simpler tools. Our proof is based on the following lemma.

LEMMA. *Let $\tilde{\mu}$ be an indeterminate measure and let ν satisfy* (1). *Let* $H = \operatorname{clos}_{L^2_{d\tilde{\mu}}} \mathscr{P}$ *be the closure of the polynomials in the space $L^2_{d\tilde{\mu}}$. Then the quadratic form $\int_{-\infty}^{\infty} \overline{Q} P \, d\nu$ defines a (bounded positive) operator A in H*

$$\int_{-\infty}^{\infty} \overline{Q} P \, d\nu = \langle AP, Q \rangle$$

of the trace class: i.e. $\operatorname{tr} A < \infty$.

PROOF. Let $\{P_n\}$ be the system of orthonormal polynomials with respect to the measure $\tilde{\mu}$. They form an orthonormal basis of the space H. The matrix elements of the operator A with respect to this basis are of the form

$$a_{m,n} = \langle AP_n, P_m \rangle = \int_{-\infty}^{\infty} \overline{P_m} P_n \, d\nu.$$

In particular,

$$a_{n,n} = \int_{-\infty}^{\infty} P_n^2 \, d\nu,$$

and we are going to prove that

$$\sum_{n=0}^{\infty} a_{n,n} = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} P_n^2 \, d\nu < \infty.$$

It follows that the matrix $(a_{m,n})$ determines a positive bounded operator A of trace class in H.

A well known theorem of M. Riesz (see, for example, [1, Ch. II, §4]) says that in the case of indeterminacy the series $\sum_{n=0}^{\infty} P_n^2$ converges and satisfies the following estimate

$$\sum_{n=0}^{\infty} P_n^2(t) \le C(\delta) e^{\delta |t|}, \quad \text{for any } \delta > 0.$$

Making use of (1), we obtain

$$\sum_{n=0}^{\infty} a_{n,n} = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} P_n^2 \, d\nu = \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} P_n^2 \, d\nu \le C(\delta) \int_{-\infty}^{\infty} e^{\delta |t|} \, d\nu < \infty,$$

completing the proof.

The following proposition yields the Theorem.

Proposition. *Let μ be a positive measure, let ν satisfy condition* (1) *and assume that $\tilde{\mu} = \mu + \nu$ is an indeterminate measure. Then there exists a measure $\mu_0 \in \mathscr{M}_0$ such that $\mu + \mu_0$ is indeterminate.*

Proof. Let A be the operator in the space $H = \operatorname{clos}_{L^2_{d\tilde{\mu}}} \mathscr{P}$ associated to the measure ν as in the previous lemma. Since

$$\langle (I - A)P, P\rangle = \int_{-\infty}^{\infty} |P|^2 \, d(\tilde{\mu} - \nu) = \int_{-\infty}^{\infty} |P|^2 \, d\mu,$$

we have $(0 \leq)A \leq I$. Suppose that $||A|| < 1$. In this case the operator $I - A$ is invertible and we have

$$\begin{aligned} \int_{-\infty}^{\infty} |P|^2 \, d\mu &= \langle (I - A)P, P\rangle \\ &\geq ||(I - A)^{-1}||^{-1} \langle P, P\rangle = ||(I - A)^{-1}||^{-1} \int_{-\infty}^{\infty} |P|^2 \, d\tilde{\mu}\,. \end{aligned} \tag{2}$$

Let us use the criterion which follows from [1, Ch. II]: the measure $\tilde{\mu}$ is indeterminate if and only if the point–evaluation functional $P \mapsto P(z)$ is bounded in $L^2_{d\tilde{\mu}}$: i.e.

$$|P(z)|^2 \leq C \int_{-\infty}^{\infty} |P|^2 \, d\tilde{\mu}. \tag{3}$$

As it follows from (2) and (3),

$$|P(z)|^2 \leq C \int_{-\infty}^{\infty} |P|^2 \, d\tilde{\mu} \leq C||(I - A)^{-1}|| \int_{-\infty}^{\infty} |P|^2 \, d\mu. \tag{4}$$

And hence, due to this criterion, the measure μ is itself indeterminate.

Let now $||A|| = 1$. Since $\operatorname{tr} A < \infty$, the value $\lambda = 1$ is just an isolated eigenvalue of finite multiplicity. Let us split the space H into the orthogonal sum

$$H = H_0 \oplus H_1,$$

where $H_0 = \ker(I - A)$. The operator A has the following decomposition

$$A = \begin{bmatrix} I_{H_0} & 0 \\ 0 & A_1 \end{bmatrix} : H_0 \oplus H_1 \to H_0 \oplus H_1,$$

and, what is essential for us, $\dim H_0 = n < \infty$ and

$$||A_1|| < 1. \tag{5}$$

Let $\{f_1, \ldots, f_n\}$ be a system of entire functions which forms an orthonormal basis in $H_0 \subset H$ (condition (3) readily yields that the elements of H are entire functions). We state that the measure we need is any measure $\mu_0 \in \mathscr{M}_0$ with support $\operatorname{supp}(\mu_0) = \{t_1, \ldots, t_n\}$ possessing the property

$$\det\begin{vmatrix} f_1(t_1) & \dots & f_n(t_1) \\ \vdots & & \vdots \\ f_1(t_n) & \dots & f_n(t_n) \end{vmatrix} \neq 0. \tag{6}$$

Linear independence of $f_1, ..., f_n$ guaranties that such a choice is always possible.

Let μ_0 be a measure of this type. Introduce an operator B from H to the n-dimensional space C^n of the form

$$Bf = \begin{bmatrix} \sqrt{\mu_0(t_1)}f(t_1) \\ \vdots \\ \sqrt{\mu_0(t_n)}f(t_n) \end{bmatrix}, \quad f \in H.$$

In this case,

$$\begin{aligned} \int_{-\infty}^{\infty} |P|^2 \, d(\mu + \mu_0) &= \int_{-\infty}^{\infty} |P|^2 \, d\tilde{\mu} - \int_{-\infty}^{\infty} |P|^2 \, d\nu + \int_{-\infty}^{\infty} |P|^2 \, d\mu_0 \\ &= \langle (I - A + B^* B)P, P \rangle. \end{aligned} \tag{7}$$

As before, we just need to prove that the operator $I - A + B^*B$ is invertible. Let us consider the block-decomposition of this operator:

$$\begin{aligned} I - A + B^* B &= \begin{bmatrix} I_{H_0} & 0 \\ 0 & I_{H_1} \end{bmatrix} - \begin{bmatrix} I_{H_0} & 0 \\ 0 & A_1 \end{bmatrix} + \begin{bmatrix} B_0^* \\ B_1^* \end{bmatrix} [B_0 \quad B_1] \\ &= \begin{bmatrix} B_0^* B_0 & B_0^* B_1 \\ B_1^* B_0 & I_{H_1} - A_1 + B_1^* B_1 \end{bmatrix}, \end{aligned}$$

where $[B_0 \quad B_1]$ is the block decomposition of the operator B.

A positive block–operator

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

is invertible if and only if the operators C_{11} and $C_{22} - C_{21}C_{11}^{-1}C_{12}$ are invertible (see e.g. [7]):

$$C^{-1} = \begin{bmatrix} I_{H_0} & -C_{11}^{-1}C_{12} \\ 0 & I_{H_1} \end{bmatrix} \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} - C_{21}C_{11}^{-1}C_{12} \end{bmatrix}^{-1} \begin{bmatrix} I_{H_0} & 0 \\ -C_{21}C_{11}^{-1} & I_{H_1} \end{bmatrix}.$$

In our case, we need invertibility of the operators $B_0^* B_0$ and

$$I_{H_1} - A_1 + B_1^* B_1 - B_1^* B_0 (B_0^* B_0)^{-1} B_0^* B_1 = I_{H_1} - A_1 \,.$$

In the basis $f_1, \dots, f_n$, the matrix of the operator B_0 has the form

$$\begin{bmatrix} \sqrt{\mu_0(t_1)} & & \\ & \ddots & \\ & & \sqrt{\mu_0(t_n)} \end{bmatrix} \begin{bmatrix} f_1(t_1) & \dots & f_n(t_1) \\ \vdots & & \vdots \\ f_1(t_n) & \dots & f_n(t_n) \end{bmatrix},$$

hence invertibility of B_0 is equivalent to (6). Invertibility of $I_{H_1} - A_1$ follows from (5). So, exactly as in (4), we have

$$|P(z)|^2 \le C||(I - A + B^* B)^{-1}|| \int_{-\infty}^{\infty} |P|^2 \, d(\mu + \mu_0)$$

and this completes the proof.

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