ON THE CONVERGENCE OF SEQUENCES OF OPERATORS AND THE CONVERGENCE OF THE SEQUENCE OF THEIR SPECTRA

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Abstract

Let \( T_n \) be a sequence of bounded linear invertible operators on the Banach space \( E \) over \( \mathbb{C} \), and assume that the double sequence \( (T_n^k)_{n,k \in \mathbb{Z}} \) satisfies a certain growth condition. We prove the following theorem:

If the sequence \( \sigma(T_n) \) of the spectra \( \sigma(T_n) \) of \( T_n \) converges to the singleton \( \{1\} \) with respect to the Hausdorff metric on bounded subsets of \( \mathbb{C} \), then \( (T_n) \) converges uniformly to the identity operator. We also establish a generalization of this result.

1. Introduction

In the following let \( E \) be a fixed Banach space over \( \mathbb{C} \). Let \( (T_n)_{n \in \mathbb{N}} \) be a sequence of bounded linear operators on \( E \) which converges uniformly (i.e. with respect to the operator norm) to the identity \( I \) on \( E \). Then the sequence \( (\sigma(T_n))_{n \in \mathbb{N}} \) of the spectra \( \sigma(T_n) \) of \( T_n \) converges to \( \{1\} \) with respect to the Hausdorff metric which is defined for closed bounded sets \( A, B \) in two steps:

Set \( \operatorname{dist}(A,B) := \sup_{a \in A}(\inf_{b \in B}|a - b|) \), then

\[
\operatorname{d}(A,B) := \max(\operatorname{dist}(A,B), \operatorname{dist}(B,A)).
\]

It is the main aim of our paper to prove a certain converse to this result. To this end let us first of all consider the “stationary” case, i.e. let \( T \) be a bounded linear operator with \( \sigma(T) = \{1\} \). Then in general \( T \neq I \). But a very old and famous theorem of Gelfand [4] says that if in addition \( (T^k)_{k \in \mathbb{Z}} \) is norm bounded then \( T = I \).

This theorem was generalized by many authors; we cite some of them: E. Hille [7] proved the result under the condition that \( (\|T^k\|) = o(k) \). If one applies a result of Shilov [5, \S 41] together with the representation theory of...
Domar and Lindahl [3] then one gets the result under the conditions

\(|T^k| = 0(\|T\|^\alpha)\) for some \(\alpha \geq 0\) together with \(\liminf_{k \to \infty} \frac{\|T_k\|}{k} = 0\). Atzmon [1]

showed that the conditions (1) \(\sup\{\|T_k\| : k \geq 0\} < \infty\) and (2)

\[\lim_{k \to \infty} \frac{\log \|T_k\|}{\sqrt{k}} = 0\]

together imply \(T = I\). Other generalizations are to be found in [14] and [10]. The up to now most general case is treated in [9, §3]. We shall show that if the sequence \((T_n)\) satisfies uniformly one of the conditions mentioned above and if \((\sigma(T_n))\) converges to \(\{1\}\) then \((T_n)\) converges uniformly to \(I\).

Our paper is organized in the following manner: In the second section we recall some important facts on the Beurling algebras \(l^1_w(Z)\) which we need in the sequel, in the third section we state and prove our main theorem where as the fourth section is devoted to a generalization of it.

2. Preliminaries

We start by a recapitulation of what we need about the Beurling algebra \(l^1_w(Z)\) for a given weight \(w\) on \(Z\) (for notions not explained here we refer to [11]).

**Definition 2.1.** (a) A function \(w : Z \to [1, \infty[ = \{x \in \mathbb{R} : x \geq 1\}\) is called a weight if \(w(k + l) \leq w(k)w(l)\) holds for all \(k, l\).

(b) Such a weight is called nonquasianalytic (nqa for short) if

\[\sum_{k \in \mathbb{Z}} \frac{\log w(k)}{1 + k^2} < \infty\]

For \(w\) being a weight the space \(\{f \in C^Z : \sum_{k \in \mathbb{Z}} |f(k)|w(k) < \infty\} =: l^1_w(Z)\) is a subalgebra of \(l^1(Z)\) with respect to convolution, which is a Banach algebra when equipped with the norm

\[\|f\| = \sum_{k \in \mathbb{Z}} |f(k)|w(k)\]

If \(w\) is nonquasianalytic then it follows from the theory of Domar [2] on general Beurling algebras that \(l^1_w(Z)\) is a Wiener algebra with an approximate unit (see [11] for these notions).

Let \(\Gamma\) be the Gelfand space of \(l^1_w(Z)\). If \(w\) is nonquasianalytic then \(\Gamma\) can be identified with \(T := \{\xi \in \mathbb{C} : |\xi| = 1\}\) by setting \(\varphi_\xi(f) = \sum f(k)\xi^k =: \hat{f}(\xi)\).

If \(J\) is an ideal of \(\mathcal{A} = l^1_w(Z)\) then \(h(J) = \{\xi \in \Gamma : \hat{f}(\xi) = 0\text{ for all } f \in J\}\) is called the hull of \(J\). It is always a closed subset of \(\Gamma\). If conversely \(\Delta \subset \Gamma\) is a closed set then we set \(k(\Delta) = \{f \in \mathcal{A} : \hat{f}(\Delta) = \{0\}\}\) and \(m(\Delta) = \)
{f ∈ A : f vanishes on an open set containing Δ}. m(Δ) is the smallest closed ideal J satisfying h(J) = Δ, and k(Δ) is the largest one.

**Definition 2.2** (cf. [9, Definition 3.3]). The nqa-weight w is called a weight of uniqueness (U-weight for short) if m({ξ}) = k({ξ}) for every ξ ∈ I.

**Remark 2.3.** A closed ideal J is called primary if the hull h(J) is a singleton. w is an U-weight if every primary ideal is maximal.

Concerning concrete examples we mention the following ones:

**Theorem 2.4.** (a) Let \( w(k) = 0(\vert k\vert^\alpha) \) for some \( \alpha \geq 0 \) and in addition let \( \liminf_{k→±\infty} \frac{w(k)}{k} = 0 \). Then w is an U-weight.

(b) Assume that
\[
\lim_{n→∞} \left( \frac{w(n)}{n^\alpha} + \frac{\log(n)}{\sqrt{n}} \right) = 0 \text{ for some } \alpha \geq 0.
\]
If \( 0 ≤ \alpha < 1 \) or \( \liminf_{n→±∞} w(n) < ∞ \) then w is an U-weight.

**Remark 2.5.** Part (a) is Shilov’s theorem mentioned in the introduction. The corresponding assertion for \( \alpha < 1 \) in part (b) was announced in [6, Theorem 8.1 (i)], and a complete proof of it was given in [1]. The other assertion of part (b) is due to the first author [9, Prop. 3.12].

We shall make heavy use of results of Domar and Lindal [3] specialized to our situation of representations of \( l^1_e(Z) \).

Let \( T \) be a bounded linear invertible operator on the Banach space E. Then \( w(k) := \max\{1, \|T^k\|\} \) defines a weight. Let \( Uf := \sum f(k)T^k \) for \( f ∈ A := l^1_e(Z) \). Then U is a contractive representation of A in \( L(E) \). We denote by \( \ker(U) \) its kernel \( U^{-1}(\{0\}) \).

**Theorem 2.6.** (cf. [3, Theorem 6.7], [13, Prop. 3.6], [8, Prop. 1.3.8]) Let w be nonquasianalytic. Then \( h(\ker(U)) = σ(T) \) where \( σ(T) \) denotes the spectrum of T.

This theorem gives us back our results in the first section concerning the “\( T = I \)” problem.

**Corollary 2.7** (cf. [9, Theorem 3.10]). Let w be an U-weight. Then \( σ(T) = \{1\} \) implies \( T = I \).

**Proof.** Since w is an U-weight and \( σ(T) = \{1\} \), the kernel of U has codimension 1. It follows that \( A/ker(U) = C1 \) and the assertion follows from [13], Remark 3.7 (2) to Proposition 3.6.
3. The main result

Theorem 3.1. Let \((T_n)_{n \in \mathbb{N}}\) be a sequence of bounded linear invertible operators on the Banach space \(E\). Assume that there exists a \(U\)-weight \(w\) on \(\mathbb{Z}\) such that 
\[
\|T_n^k\| \leq w(k) \quad \text{for all } n \in \mathbb{N} \text{ and } k \in \mathbb{Z}.
\]
If the sequence \((\sigma(T_n))\) of the spectra \(\sigma(T_n)\) of \(T_n\) converges to \(\{1\}\) (with respect to the Hausdorff metric) then \((T_n)\) converges uniformly to the identity.

In order to prove this theorem and related results we use the theory of ultraproducts. (Compare the following paragraph with the introduction of section 2 of [12])

Let \(\mathcal{U} \subset \mathcal{P}(\mathbb{N})\) be a free ultrafilter and denote by \(m_\mathcal{U}\) the finitely additive \(\{0,1\}\)-valued measure on \(\mathbb{N}\), given by \(m_\mathcal{U}(A) = 1\) iff \(A \in \mathcal{U}\). Note that for such a measure the intersection of finitely many sets of measure 1 has also measure 1.

Denote by \(E_\infty\) the Banach space of all bounded sequences \(x = (x_n)_{n \in \mathbb{N}}\) with \(x_n \in E\) for all \(n\), equipped with the norm \(\|x\| = \sup_n \|x_n\|\). Then 
\(E_0 := \{x : \lim_\mathcal{U} \|x_n\| = 0\}\) is a closed subspace of \(E_\infty\). The quotient \(\hat{E} := E_\infty/E_0\) is called the ultrapower of \(E\) with respect to \(\mathcal{U}\). The norm on \(\hat{E}\) is given by \(\|\hat{y}\| = \|\hat{y} + E_0\| = \lim_\mathcal{U} \|y_n\|\). \(E\) is isometrically embedded into \(\hat{E}\) by means of \(x \mapsto (x,x,x,\ldots) + E_0\).

Every bounded sequence \((S_n)\) of operators \(S_n\) on \(E\) defines an operator \(\hat{S}\) on \(E_\infty\) by \(\hat{S}x = (S_nx_n)_{n \in \mathbb{N}}\), with norm given by \(\|\hat{S}\| = \sup_n \|S_n\|\). So \(E_0\) is invariant for \(\hat{S}\), and we obtain a uniquely defined operator \(\hat{S}\) on \(\hat{E}\) given by \(\hat{S}\hat{y} = \hat{S}\hat{y} + E_0\). Moreover, \(\|\hat{S}\| = \lim_\mathcal{U} \|S_n\|\). If \((R_n)\) is another bounded sequence of operators \(R_n\) on \(E_n\) such that \(R_n = S_n \quad m_\mathcal{U}\)-a.e., then \(\hat{R} = \hat{S}\). Thus each subfamily \((S_n)_{n \in M}\) with \(m_\mathcal{U}(M) = 1\) defines in a canonical way an operator on \(\hat{E}\) which coincides with \(\hat{S}\). It should cause no confusion if we denote the operator on \(\hat{E}\) induced by \((S_n)_{n \in M}\) also by \(\hat{S} = (S_n)\). If \(T\) is a bounded operator on \(E\) then the constant sequence \((T,T,\ldots)\) defines the extension \(\hat{T}\) of \(T\) on \(\hat{E}\). The mapping \(T \mapsto \hat{T}\) is an isometric embedding of the algebra \(\mathcal{L}(E)\) into \(\mathcal{L}(\hat{E})\).

It will become clear from the context whether \(\hat{S}\) denotes an operator coming from a constant sequence or from an arbitrary sequence. But one should keep in mind the following two different notations: For \((S_n)_{n \in \mathbb{N}}\) being a bounded sequence of operators we consider not only \(\hat{S} = (S_n)\) but also each individual operator \(\hat{S}_n\), defined through the constant sequence \((S_n,S_n,S_n,\ldots)\).

Proposition 3.2. Let \(w\) be a nonquasianalytic weight on \(\mathbb{Z}\). Let \((T_n)\) be a sequence of bounded linear invertible operators on \(E\) satisfying \(\|T_n^k\| \leq w(k)\)
for all \( n \in \mathbb{N} \) and \( k \in \mathbb{Z} \). Let finally \( A \subset T \) be closed. Then \( \lim_{n \to \infty} d(\sigma(T_n), A) = 0 \) implies \( \sigma(\hat{T}) = A \).

**Proof.** (I) The assumption \( \|T_n^k\| \leq w(k) \) for all \( k \) and \( n \) implies \( \|\hat{T}^k\| \leq w(k) \). Consider the representation \( U : \mathcal{A} := l^1_w(\mathbb{Z}) \) in \( L(\hat{E}) \) given by \( Uf := \sum f(k)\hat{T}^k \). Then Theorem 2.6 implies that \( \sigma(\hat{T}) = h(\ker(U)) \).

(II) For fixed \( n \in \mathbb{N} \) we also consider \( U_n : f \in \mathcal{A} \to U_nf := \sum f(k)\hat{T}^k_n \).

Then Theorem 2.6 combined with [13, Theorem 3.4] yields that \( \sigma(T_n) = \sigma(\hat{T}_n) = h(\ker(U_n)) \).

(III) Now let \( \lambda \in T \setminus A \) be given and set \( \varepsilon := \text{dist}(\lambda, A)/2 \). Then there exists \( n_0 \) such that for \( n \geq n_0 \) the spectrum \( \sigma(T_n) \) is contained in \( C := \{ \xi : \text{dist}(\xi, A) \leq \varepsilon \} \).

Since \( m(C) \) is the smallest closed ideal \( J \) in \( \mathcal{A} \) with \( h(J) = C \), and since \( \sigma(T_n) \subset C \) we obtain \( m(C) \subset \ker(U)_n =: J_n \) by (II).

Set
\[
\hat{f} = \lambda \delta_0 - \delta_1 \quad \text{where} \quad \delta_k(r) = \begin{cases} 1, & r = k, \\ 0, & r \in \mathbb{Z} \setminus \{k\} \end{cases}
\]

Then \( \hat{f}(\xi) = 0 \) implies \( \lambda = \xi \), hence \( \hat{f} \) does not vanish on \( C \). Since \( \mathcal{A} \) is a Wiener algebra, in particular regular, there exists \( g \in \mathcal{A} \) such that \( \hat{f}g = 1 \) on \( C \). This in turn implies that \( \hat{f} := f + m(C) \) is invertible in the quotient algebra \( \mathcal{A}/m(C) \). Let \( \alpha := \|g + m(C)\| \) be the quotient norm of \( \hat{g} := g + m(C) \).

(IV) Since \( m(C) \subset J_n \), \( g + J_n \) is the inverse of \( f + J_n \) in \( \mathcal{A}/J_n \) and \( \|g + J_n\| \leq \alpha \). Since \( U_n \) is obviously contractive, \( U_ng \) is the inverse of \( (\lambda - T_n) \) and \( \|U_ng\| \leq \alpha \), or in other words \( \lambda \in \rho(T_n) \) and \( \|\lambda - T_n^{-1}\| \leq \alpha \). This holds for all \( n \geq n_0 \) (see (III) above). But then \( \lambda \in \rho(\hat{T}) \) by [12, Lemma 2.1].

(V) Let conversely \( \lambda \in A \) be arbitrary. Then by assumption to every \( n \) there exists \( \lambda_n \in \sigma(T_n) \) such that \( \lim_{n \to \infty} \lambda_n = \lambda \) holds. Assume now that \( \lambda \in \rho(\hat{T}) \). Then by Lemma 2.1 of [12] there exists \( \delta > 0 \) such that \( \lambda \in \rho(T_n) \) and \( \|\lambda - T_n^{-1}\| < \delta \) holds \( m_U \)-a.e. To this \( \delta \) there exists \( n_0 \in \mathbb{N} \) such that \( |\lambda_n - \lambda| < 1/\delta \) for all \( n \geq n_0 \). But this implies \( |\lambda_n - \lambda| < \frac{1}{\|\lambda - T_n^{-1}\|} \) \( m_U \)-a.e. From the equation
\[
(\lambda_n - T_n) = (\lambda - T_n)(I - (\lambda - \lambda_n)(\lambda - T_n)^{-1})
\]
we obtain that \( \lambda_n \in \rho(T_n) \) holds \( m_U \)-a.e., a contradiction.

**Proof of 3.1:** If the assertion fails there exists a free ultrafilter \( \mathcal{U} \) such that \( \lim_{\mathcal{U}} \|T_n - I\| > 0 \), or in other words, such that \( \hat{T} \neq I \) holds. But \( \lim_{n \to \infty} d(\sigma(T_n), \{1\}) = 0 \) implies \( \sigma(\hat{T}) = \{1\} \) by Proposition 3.3. Now from \( \|\hat{T}^k_n\| \leq w(k) \) for all \( n \in \mathbb{N} \) and \( k \in \mathbb{Z} \) we obtain \( \|\hat{T}^k\| \leq w(k) \) for all \( k \in \mathbb{Z} \) (see part (I) of the proof of Prop. 3.3). Now \( w \) is an \( U \)-weight by hypothesis, so \( \hat{T} = I \), a contradiction.
4. A generalization

In this section we consider the convergence of \( \{T_n\} \) to a finite set \( \{\lambda_1, \ldots, \lambda_r\} = A \). For the formulation of our result we need the following definition:

**Definition 4.1.** A decomposition of the identity operator \( I \) on the Banach space \( E \) is a finite set \( \{P_1, \ldots, P_r\} \) of pairwise disjoint nonvanishing projections which add up to \( I \), i.e., \( \sum_{k=1}^r P_k = I \).

**Theorem 4.2.** Let \( w \) be an \( U \)-weight on \( \mathbb{Z} \) and let \( \{T_n\} \) be a sequence of bounded linear invertible operators such that \( \|T_n^k\| \leq w(k) \) for all \( n \in \mathbb{N} \) and \( k \in \mathbb{Z} \). Assume that \( \lim_{n \to \infty} d(\sigma(T_n), A) = 0 \) for some finite set \( A = \{\lambda_1, \ldots, \lambda_r\} \). Then there exists \( n_0 \) such that for each \( n \geq n_0 \) there exists a partition \( \{P_{1n}, \ldots, P_{rn}\} \) of \( I \) satisfying \( \lim_{n \to \infty} \|T_n - \sum_{k=1}^r \lambda_k P_{kn}\| = 0 \).

**Remark 4.3.** Note that the sequence \( \sum_{k=1}^r \lambda_k P_{kn} \) itself needs not converge as easy examples show.

**Proof.** (I) Let \( \delta := \inf\{|\lambda_i - \lambda_j| : i \neq j\}/8 \) and let \( n_0 \in \mathbb{N} \) be chosen so that \( d(\sigma(T_n), A) < \delta \) for all \( n \geq n_0 \). Then, we have for \( D_{k,n} := \{\lambda \in \sigma(T_n) : |\lambda - \lambda_k| < \delta\} \) that

(i) \( D_{k,n} \cap D_{l,n} = \emptyset \) for \( k \neq l \); and

(ii) \( \sigma(T_n) = \bigcup_{k=1}^r D_{k,n} \).

(II) Let \( P_{kn} \) be the spectral projection corresponding to \( D_{k,n} \), i.e.,

\[
P_{kn} = \frac{1}{2\pi i} \int_{|\xi - \lambda_k| = 2\delta} (\xi - T_n)^{-1} d\xi.
\]

The set \( \{P_{kn} : k = 1, \ldots, r\} \) is a partition of \( I \), and we want to prove that \( (T_n - \sum_{k=1}^r \lambda_k P_{kn}) \) converges to 0.

(III) We use an ultrapower \( \tilde{E} \) of \( E \) with respect to an arbitrary free ultrafilter. Proposition 3.3 yields \( \sigma(\tilde{T}) = \{\lambda_1, \ldots, \lambda_r\} \). Let \( Q_j \) be the spectral projection corresponding to \( \lambda_j \); moreover set \( F_j := Q_j(\tilde{E}) \). Then \( \lambda_j \tilde{T}|_{F_j} = I \) by Theorem 2.6 since \( \|(\lambda_j \tilde{T})^k\| \leq w(k) \) (cf. the proof of 3.1). Hence \( \tilde{T} = \sum \lambda_j Q_j \).

(IV) Set \( C_k := \{\xi : |\xi - \lambda_k| \leq \delta\} \) and \( C := \bigcup_{k=1}^r C_k \).
As in the proof of Proposition 3.3 we have $m(C) \subset \ker(U_n) =: J_n$ at least for $n \geq n_0$. The function $\xi \to g_\xi := \xi \delta_0 - \delta_1$ has the property that $g_\xi$ does not vanish on $C$, hence $\tilde{g}_\xi =: g_\xi + m(C)$ is invertible and uniformly continuous (as a function of $\xi$). Since $m(C) \subset J_n$ for $n \geq n_0$ and since $U_n : l^1_n(\mathbb{Z}) \to \mathcal{L}(E)$ $(U_nf := \sum_{-\infty}^{\infty} f(k) T_n^k)$ is a contraction we obtain for the induced representation $\tilde{U}_n : l^1_n(\mathbb{Z})/J_n \to \mathcal{L}(E)$ that $(\xi - T_n)^{-1} = \tilde{U}_n((g_\xi + J_n)^{-1})$ is uniformly equicontinuous. Thus for each $\varepsilon > 0$ there exists a partition $\{\xi_1, \ldots, \xi_m\}$ of the circle $C_k$ such that
\[ \left\| P_{k,n} \frac{1}{2\pi} \sum (\xi_j - T_n)^{-1}(\xi_{j+1} - \xi_j) \right\| < \varepsilon/2 \text{ uniformly in } n \geq n_0. \]
Passing to $\tilde{E}$ we obtain $\|Q_k - \frac{1}{2\pi} \sum (\xi_j - \tilde{T})^{-1}(\xi_{j+1} - \xi_j)\| \leq \varepsilon/2$ as well as $\|\tilde{P}_k - \frac{1}{2\pi} \sum (\xi_j - \tilde{T})^{-1}(\xi_{j+1} - \xi_j)\| \leq \varepsilon/2$. Since $\varepsilon > 0$ was arbitrary we get $Q_k = \tilde{P}_k$, hence $(\tilde{T} - \sum \lambda_j P_k) = 0$. Since the free ultrafilter $\mathcal{U}$ was chosen arbitrarily the assertion follows.

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