SHARP $L^p - L^q$ ESTIMATES FOR SINGULAR FRACTIONAL INTEGRAL OPERATORS*

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1. Introduction

Let $Q = [-1, 1] \times [-1, 1]$, let $\varphi : Q \to R$ be a measurable function and let $\gamma_1, \gamma_2 > 0$; suppose $\mu$ is the measure on $R^3$ given by

$$
\mu(E) = \int_Q x \chi_E(x_1, x_2, \varphi(x_1, x_2)) |x_1|^\gamma_1 - 1 |x_2|^\gamma_2 - 1 dx_1 dx_2,
$$

where $dx_1 dx_2$ denotes the Lebesgue measure on $R^2$. Let $T_\mu$ be the convolution operator defined by $T_\mu f(x) = (\mu * f)(x)$ and let

$$
E_\mu = \left\{ \left( \frac{1}{p}, \frac{1}{q} \right) : \| T_\mu \|_{L^p, L^q} < \infty, 1 \leq p, q \leq \infty \right\}
$$

where the $L^p$-spaces are taken with respect to the Lebesgue measure on $R^3$. The set $E_\mu$ is known in several cases. For $\gamma_1 = \gamma_2 = 1$, if the graph of $\varphi$ has non zero Gaussian curvature at each point, a theorem of Littman implies that $E_\mu$ is the closed triangle with vertices $(0, 0), (1,1)$ and $(\frac{3}{4}, \frac{1}{4})$ (see [O]).

Now, if the curvature vanishes in some point, $E_\mu$ can be strictly contained in the above triangle. Related examples in a more general context can be found in [O], [C] and [R-S].

In this paper we study the set $E_\mu$ in the case $\varphi(x_1, x_2) = |x_1|^\alpha_1 + |x_2|^\alpha_2$, $\alpha_1, \alpha_2 > 1$ and $0 < \gamma_1, \gamma_2 \leq 1$. In [F-G-U] we obtain this characterization for $\gamma_1 = \gamma_2 = 1$.

Throughout this work, $c$ will denote a positive constant not necessarily the same at each occurrence and, without loss of generality we will assume that

$$
\frac{\alpha_1 + 2}{\gamma_1} \leq \frac{\alpha_2 + 2}{\gamma_2}.
$$

In section 2 we find a convex closed polygonal region $\Sigma$ such that $E_\mu \subset \Sigma$ and we obtain some estimates for the Fourier transform $\hat{\mu}$. In section 3 we study $L^p - L^p$ estimates for this kind of operators. In section 4 we prove,

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following a suitable extension of the ideas developed by M. Christ in [C], that, if \( \frac{1}{q} \leq \gamma_1, \gamma_2 \leq 1 \) then \( E_\mu = \Sigma \). Also we prove that, if \( 0 < \gamma_1, \gamma_2 \leq 1 \) then the interiors of \( E_\mu \) and \( \Sigma \) agree.

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2. Auxiliary results

Let \( Q, \varphi, \mu \) and \( E_\mu \) be as in the introduction. The Riesz Thorin theorem implies that \( E_\mu \) is a convex subset of the square \([0, 1] \times [0, 1]\). It is well known that if \( \left( \frac{1}{p}, \frac{1}{q} \right) \in E_\mu \) then \( p \leq q \). (See [S-W] p. 33). The above mentioned result due to Oberlin ([O]) implies that \( E_\mu \) is contained in the closed triangular region with vertices \((0, 0), (1, 1) \) and \((3/4, 1/4)\). In our particular case we can obtain a more precise statement.

**Lemma 2.1.** If \( \left( \frac{1}{p}, \frac{1}{q} \right) \in E_\mu \), then the following inequalities hold

\[
\frac{1}{q} \geq \frac{3}{p} - 2, \quad \frac{1}{q} \geq \frac{2\alpha_1 + 1}{\alpha_1 + 1} \quad \frac{1}{p} - \frac{\alpha_1 + \gamma_1}{\alpha_1 + 1}, \quad \frac{1}{q} \geq \frac{2\alpha_2 + 1}{\alpha_2 + 1} - \frac{\alpha_2 + \gamma_2}{\alpha_2 + 1},
\]

\[
\frac{1}{q} \geq \frac{1}{p} - \frac{\gamma_1 \alpha_2 + \gamma_2 \alpha_1}{\alpha_1 \alpha_2 + \alpha_1 + \alpha_2}, \quad \frac{1}{q} \geq \frac{1}{p} - \gamma_1, \quad \frac{1}{q} \geq \frac{1}{p} - \gamma_2.
\]

**Proof.** The assertion \( \frac{1}{q} \geq \frac{3}{p} - 2 \) follows from Theorem 1 in [O]. To see \( \frac{1}{q} \geq \frac{2\alpha_1 + 1}{\alpha_1 + 1} \quad \frac{1}{p} - \frac{\alpha_1 + \gamma_1}{\alpha_1 + 1} \) we take, for \( 0 < \delta < 1 \), \( f = \chi_{Q_\delta} \) where \( Q_\delta \) is given by

\[
Q_\delta = (-\delta^{\frac{1}{\alpha_1}}, \delta^{\frac{1}{\alpha_1}}) \times (-\delta, \delta) \times (-k\delta, k\delta)
\]

with \( k = 2^{\alpha_1 - 1}\alpha_1 + 2^{\alpha_1 - 1}\alpha_2 + 1 \) and we set \( A_\delta = \{(x_1, x_2, x_3) : |x_1| < \delta^{\frac{1}{\alpha_1}}, |x_2| < 1, |x_3 - \varphi(x_1, x_2)| < \delta\} \). It is easy to see that if \( x \in A_\delta \) implies \( \mu * f(x) \geq c\delta^{1+\frac{1}{\alpha_1}} \). Then

\[
\|\mu * f\|_p \geq \left( \int_{A_\delta} |\mu * f|_q \right)^\frac{1}{q} \geq \delta^{1+\frac{1}{\alpha_1}} |A_\delta|^\frac{1}{q} = c\delta^{1+\frac{1}{\alpha_1} + \left(1+\frac{1}{\alpha_1}\right)}.
\]

Now, \( \|\mu * f\|_q \leq c\|f\|_p = c\delta^{(2+\frac{1}{\alpha_1})} \) since these inequalities hold for all small enough \( \delta \), the second assertion of the lemma follows. The proof of the third is analogous. To prove the fourth let \( Q_\delta = (-\delta^{\frac{1}{\alpha_1}}, \delta^{\frac{1}{\alpha_1}}) \times (-\delta^{\frac{1}{\alpha_2}}, \delta^{\frac{1}{\alpha_2}}) \times (-k_1\delta, k_1\delta) \) and let

\[
A_\delta = \{(x_1, x_2, x_3) : |x_1| < \delta^{\frac{1}{\alpha_1}}, |x_2| < \delta^{\frac{1}{\alpha_2}}, |x_3 - \varphi(x_1, x_2)| < \delta\}.
\]

It is easy to see that if \( k_1 = 1 + 2^{\alpha_1} + 2^{\alpha_2} \), then \( \mu * f(x) \geq c\delta^{\frac{1}{\alpha_1} + \frac{1}{\alpha_2}} \) for \( x \in A_\delta \). So, reasoning as above, we obtain the expected inequality. Finally, to see that \( \frac{1}{q} \geq \frac{1}{p} - \gamma_1 \) we choose \( Q_\delta = (-\delta, \delta) \times (-1, 1) \times (-3, 3) \), and
\[ A_\delta = \{(x_1, x_2, x_3) : |x_1| < \delta, |x_2| < 1, |x_3 - \varphi(x_1, x_2)| < 1\}. \]

We obtain \( \mu * f(x) \geq c \delta^{-n} \) for \( x \in A_\delta \). So as above the result follows. The proof of the last inequality is similar.

We denote by \( L, L_0, L_{\alpha_k, \gamma_k}, L_{\gamma_k}, (k = 1, 2) \), the lines (in the \((1/p, 1/q)\) plane) given by \( \frac{1}{q} = \frac{3}{p} - 2, \frac{1}{q} = \frac{1}{p} - \frac{\gamma_{\alpha_k + \gamma_{\gamma_k}}}{\alpha_k + \gamma_k + 2}, \frac{1}{q} = \frac{2\alpha_k + 1}{\alpha_k + 1} - \frac{\alpha_k + \gamma_k}{\alpha_k + 1}, \frac{1}{q} = \frac{1}{p} - \gamma_k \) respectively. Also we denote by \( B_{\alpha_k, \gamma_k}, B_{\alpha_k, \gamma_k}^0, k = 1, 2 \), the intersection of \( L_{\alpha_k, \gamma_k} \) with \( L, L_{\gamma_k} \) and \( L_0 \) respectively. We also set \( A, A_{\alpha_k, \gamma_k}, A_{\gamma_k} \) and \( A_0 \) the intersection of the non principal diagonal with \( L, L_{\alpha_k, \gamma_k}, L_{\gamma_k} \) and \( L_0 \) respectively.

A computation shows that \( A = (3/4, 1/4) \) and that, for \( k = 1, 2 \),

\[ A_{\alpha_k, \gamma_k} = \left( \frac{2\alpha_k + 1 + \gamma_k}{3\alpha_k + 2}, \frac{\alpha_k + 1 - \gamma_k}{3\alpha_k + 2} \right), \quad A_{\gamma_k} = \left( \frac{1 + \gamma_k}{2}, \frac{1 - \gamma_k}{2} \right) \]

and

\[ A_0 = \left( \frac{1}{2} + \frac{\gamma_2}{2(\alpha_1 + \alpha_2 + 1)\alpha_2}, \frac{1}{2} - \frac{\gamma_2}{2(\alpha_1 + \alpha_2 + 1)\alpha_2} \right). \]

Also

\[ B_{\alpha_k, \gamma_k} = \left( 1 - \frac{\gamma_k}{\alpha_k + 2}, 1 - \frac{3\gamma_k}{\alpha_k + 2} \right), \]

\[ B_{\alpha_k, \gamma_k}^0 = \left( 1 - \gamma_j + \frac{\gamma_k - \gamma_j}{\alpha_k}, 1 - 2\gamma_j + \frac{\gamma_k - \gamma_j}{\alpha_k} \right), \]

\[ B_{\alpha_k, \gamma_k}^0 = \left( 1 - \frac{\alpha_1 \gamma_2 + \gamma_2 - \gamma_1}{\alpha_1 + \alpha_2 + 1\alpha_2}, 1 - \frac{\alpha_2 \gamma_1 + 2\alpha_1 \gamma_2 - \gamma_1 + \gamma_2}{\alpha_1 + \alpha_2 + 1\alpha_2} \right), \]

and

\[ B_{\alpha_k, \gamma_k}^0 = \left( 1 - \frac{\alpha_2 \gamma_1 + \gamma_1 - \gamma_2}{\alpha_1 + \alpha_2 + 1\alpha_2}, 1 - \frac{\alpha_1 \gamma_2 + 2\alpha_2 \gamma_1 - \gamma_2 + \gamma_1}{\alpha_1 + \alpha_2 + 1\alpha_2} \right). \]

**Remark 2.2.** Lemma 2.1 holds for \( T^*_\mu \), taking in the proof -\( \varphi \) instead of \( \varphi \).

Let \( \Sigma_{\alpha_1, \alpha_2, \gamma_1, \gamma_2} \) be the closed convex polygonal region contained in \( Q \), given by the intersection of the lower half space determined by the principal diagonal with all the upper half spaces determined by the lines \( L, L_0, L_{\alpha_k, \gamma_k}, L_{\gamma_k}, (k = 1, 2) \), and all the upper half spaces determined by their symmetric lines with respect to the non principal diagonal. Lemma 2.1, Remark 2.2 and a duality argument say that \( E_\mu \subset \Sigma_{\alpha_1, \alpha_2, \gamma_1, \gamma_2} \). Now we give a more precise description of \( \Sigma_{\alpha_1, \alpha_2, \gamma_1, \gamma_2} \). Since \( \frac{\alpha_1 + 2}{\gamma_1} \leq \frac{\alpha_2 + 2}{\gamma_2} \), \( \Sigma_{\alpha_1, \alpha_2, \gamma_1, \gamma_2} \) is determined only by \( L, L_0, L_{\alpha_2, \gamma_2}, L_{\gamma_2} \). Indeed, \( B_{\alpha_2, \gamma_2} \) is closer to \((1,1)\) than \( A_{\alpha_1, \gamma_1} \) and if the intersec-
tion of $L_{a_1,\gamma_1}$ with $L_{a_2,\gamma_2}$ belongs to $Q$ then it is discarded either by $L$ or by $L_0$. Moreover $L_0$ lies below $L_{\gamma_1}$ if and only if $\gamma_1(\alpha_1 + 1) < \gamma_2$, in this case, from $\frac{\alpha_2 + 2}{\gamma_2} \leq \frac{\alpha_1 + 2}{\gamma_1}$, we also obtain $\gamma_2(\alpha_1 + 1) < \gamma_1$; adding both inequalities we get a contradiction.

Let us consider the points $A$, $A_{\alpha_2,\gamma_2}$, $A_0$, on the non principal diagonal. We distinguish the following cases

Case I. $A$ is the highest of these points. This occurs if and only if $\frac{\alpha_1 + 2}{\gamma_1} \leq 4$.

In this case $\Sigma^{a_1,a_2,\gamma_1,\gamma_2}$ is the triangle with vertices $(0,0)$, $(1,1)$ and $A$.

Case II. $A_{a_2,\gamma_2}$ is the highest of these points and $A_{a_2,\gamma_2} \neq A$. This occurs if and only if $\gamma_1 \geq \frac{1}{3}$, $\frac{a_1 + 2}{\gamma_2} > 4$ and $\alpha_1 \alpha_2 + \alpha_1 + \alpha_2 \leq \gamma_1(3\alpha_2 + 2) + \gamma_2(\alpha_1 - 2)$.

Here $\Sigma^{a_1,a_2,\gamma_1,\gamma_2}$ is the pentagon with vertices $(1,1)$, $B_{a_0,\gamma_2}$, $A_{a_2,\gamma_2}$ and their symmetric points with respect to the non principal diagonal.

Case III. $A_{a_2,\gamma_2}$ is the highest of these points and $A_{a_2,\gamma_2} \neq A, A_{a_2,\gamma_2} \neq A_{a_2,\gamma_2}$. This occurs if and only if $\gamma_2 < \frac{1}{3}$, and $\gamma_1(\alpha_1 + 1) \leq \gamma_1$. Here $\Sigma^{a_1,a_2,\gamma_1,\gamma_2}$ is the hexagon with vertices $(1,1)$, $B_{a_0,\gamma_2}$, $A_{a_2,\gamma_2}$ and their symmetric points with respect to the non principal diagonal.

Case IV. $A_0$ is the highest of these points, $A_0$ different from the others and $B_{a_0,\gamma_2} = B_{a_0,\gamma_2}$. This happens if and only if $\frac{a_2 + 2}{\gamma_2} > 4$. Here $\Sigma^{a_1,a_2,\gamma_1,\gamma_2}$ is the trapezoid with vertices $(1,1)$, $B_{a_0,\gamma_2}$ and their symmetric points with respect to the non principal diagonal.

Case V. $A_0$ is the highest of these points, $A_0$ different from the others and $B_{a_0,\gamma_2} \neq B_{a_0,\gamma_2}$. This happens if and only if $\frac{a_1 + 2}{\gamma_2} > 4$, $\gamma_2(\alpha_1 + 1) > \gamma_1$ and $\alpha_1 \alpha_2 + \alpha_1 + \alpha_2 > \gamma_1(3\alpha_2 + 2) + \gamma_2(\alpha_1 - 2)$. Now $\Sigma^{a_1,a_2,\gamma_1,\gamma_2}$ is the hexagon with vertices $(1,1)$, $B_{a_0,\gamma_2}$, $B_{a_0,\gamma_2}$ and their symmetric points with respect to the non principal diagonal.

In order to obtain some estimate for $\hat{\mu}$, we will need the following lemma, similar to Lemma 2.2 in [R-S].

**Lemma 2.3.** Suppose $\alpha > 1$, $0 < \text{Re}(\gamma)$, $\xi, \eta \in \mathbb{R}$.

(i) If $\text{Re}(\gamma)/\alpha \leq 1/2$ then

$$\left| \int_0^1 e^{-i(\xi x + \eta y) \gamma \gamma^{-1} x} dx \right| \leq \frac{c_\alpha(1 + |\text{Im}(\gamma)|)}{\text{Re}(\gamma)(1 + |\eta|)^{\text{Re}(\gamma)/\alpha}}$$

where $c_\alpha$ is independent of $\xi, \eta, \gamma$.

(ii) If $\text{Re}(\gamma) < 1/2$ then

$$\left| \gamma \int_0^1 e^{-i(\xi x + \eta y) \gamma \gamma^{-1} x} dx \right| \leq \frac{d_\alpha(1 + |\text{Im}(\gamma)|)^2}{(1 + |\eta|)^{\text{Re}(\gamma)/\alpha}}$$

where $d_\alpha$ is independent of $\xi, \eta, \gamma$. 
(iii) If $\Re(\gamma)/\alpha > 1/2$ then
\[ \left| \int_0^1 e^{-i(\xi + x^\alpha)} x^{1-1} dx \right| \leq \frac{e_{\gamma,\alpha}(1 + |\Im(\gamma)|)}{(1 + |\eta|)^{1/2}} \]

where $e_{\gamma,\alpha}$ depends only on $\alpha$ and $\Re(\gamma)$.

Proof. We can assume that $\eta > 0$. To prove (i) we note that the change of variable $x = \eta^{-\frac{1}{\alpha}}t^\frac{1}{\alpha}$ gives
\[ \int_0^1 e^{-i(\xi + x^\alpha)} x^{1-1} dx = \frac{\eta^{-\frac{\Im(\gamma)}{\Re(\gamma)}}}{\Re(\gamma) \eta} \int_0^{\Re(\gamma)/\eta} \eta^{-\frac{\alpha}{\Re(\gamma)}} e^{-i\left(\frac{\eta^{\alpha} - 1}{\Re(\gamma)} + \frac{\Im(\gamma)}{\Re(\gamma)} \ln(t)\right)} dt. \]

It is enough to prove that, for $a, b \in \mathbb{R}, a > 1$
\[ \left| \int_1^a e^{-i\left(\frac{\eta^{\alpha}}{\Re(\gamma)} + b \frac{\Im(\gamma)}{\Re(\gamma)} \ln(t)\right)} dt \right| \leq c_\alpha (1 + |\Im(\gamma)|) \]

with $c_\alpha$ independent of $a, b$ and $\gamma$. Let $s_0 = \max\left\{1, \left(\frac{2|\Im(\gamma)||2\Re(\gamma) - 1|}{\alpha (\alpha - 2\Re(\gamma) + 1)}\right)^{\Re(\gamma)}\right\}$. If $a \leq s_0$, then the integral on $[1, a]$ is bounded by $\left(\frac{2|\Im(\gamma)||2\Re(\gamma) - 1|}{\alpha (\alpha - 2\Re(\gamma) + 1)}\right)^{\Re(\gamma)}$. If $s_0 \leq a$ the integral on $[1, s_0]$ has the same bound, so it only remains to study
\[ \left| \int_0^{s_0} \int e^{-i\left(\frac{\eta^{\alpha}}{\Re(\gamma)} + b \frac{\Im(\gamma)}{\Re(\gamma)} \ln(t)\right)} dt \right|. \]

We define $\Phi : \mathbb{R} \times (1, +\infty) \to \mathbb{R}$ by $\Phi(b, t) = t^{\frac{\eta^{\alpha}}{\Re(\gamma)}} + b t^{\frac{\Im(\gamma)}{\Re(\gamma)}} - \frac{\Im(\gamma)}{\Re(\gamma)} \ln(t)$. Also we set $g_1, g_2 : (1, +\infty) \to \mathbb{R}$ given by $g_1(t) = t^{\frac{\eta^{\alpha}}{\Re(\gamma)}}$ and $g_2(t) = t^{\frac{\eta^{\alpha}}{\Re(\gamma)}} - \frac{\Im(\gamma)}{\Re(\gamma)} \ln(t)$, then $\Phi(b, t) = bg_1(t) + g_2(t)$. We note that
\[
(2.4) \quad \left[ \frac{\partial}{\partial t} \Phi(b, t) \right]^2 + \left[ \frac{\partial^2}{\partial t^2} \Phi(b, t) \right]^2 + \left[ \frac{\partial^3}{\partial t^3} \Phi(b, t) \right]^2 \neq 0
\]

for all $b \in \mathbb{R}, t > 1$. Otherwise there exist $t_0 > 1$ and $b \in \mathbb{R}$ such that $\frac{\partial}{\partial t} \Phi(b, t_0) = \frac{\partial^2}{\partial t^2} \Phi(b, t_0) = \frac{\partial^3}{\partial t^3} \Phi(b, t_0) = 0$. Thus $\frac{\partial}{\partial t} \Phi(b, t) = \frac{\partial}{\partial t} \Phi(b, t_0)$, $\left[t \frac{\partial}{\partial t} \left[t \frac{\partial}{\partial t} \Phi(b, t) \right] \right] = 0$, then
\[
\left( \frac{\alpha}{\Re(\gamma)} \right)^2 t_0^{\frac{\alpha}{\Re(\gamma)}} + b \left( \frac{1}{\Re(\gamma)} \right) t_0^{\frac{1}{\Re(\gamma)}} = 0,
\]
\[
\left( \frac{\alpha}{\Re(\gamma)} \right)^3 t_0^{\frac{\alpha}{\Re(\gamma)}} + b \left( \frac{1}{\Re(\gamma)} \right)^3 t_0^{\frac{1}{\Re(\gamma)}} = 0.
\]
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Since the only solution of this homogeneous linear system in $t_0^{\frac{n}{2}+}, b t_0^{\frac{n}{2}+}$ is the trivial one, we obtain (2.4). For a fixed $t > 1$, $[\frac{\partial}{\partial t}\Phi(b, t)]^2 + [\frac{\partial^3}{\partial t^3}\Phi(b, t)]^2$ is a quadratic expression on $b$ with a minimum $m_t$. By (2.4), $m_t \neq 0$. A computation shows that

$$m_t = \left[ \frac{(g'_1 g''_2 - g''_1 g'_2)^2 + (g''_1 g''_2 - g''_2 g'_1)^2 + (g''_1 g''_2 - g''_2 g'_1)^2}{(g'_1)^2 + (g''_1)^2 + (g''_2)^2} \right](t).$$

We note that

$$(g'_1(t))^2 + (g''_1(t))^2 + (g''_2(t))^2 =$$

$$= \frac{t^{2-2\text{Re}(\gamma)}}{\text{Re}(\gamma)^6} P_1(\text{Re}(\gamma)) + \frac{t^{2-4\text{Re}(\gamma)}}{\text{Re}(\gamma)^6} P_2(\text{Re}(\gamma)) + \frac{t^{2-4\text{Re}(\gamma)}}{\text{Re}(\gamma)^6} P_3(\text{Re}(\gamma))$$

where $P_j(\text{Re}(\gamma))$, $j = 1, 2, 3$ are polynomials in $\text{Re}(\gamma)$ with $\deg P_j = 4$. Thus there exists $c > 0$, $c$ independent of $\gamma$, such that the last expression is bounded, for all $t > s_0$, by $ct^{2-2\text{Re}(\gamma)}(1+\text{Re}(\gamma))^4$. On the other hand

$$(g'_1(t)g''_2(t) - g''_1(t)g'_2(t))^2 =$$

$$= \frac{(\text{Re}(\gamma))^{-6}}{t^{2-2\text{Re}(\gamma)}} \alpha(\alpha - 2\text{Re}(\gamma) + 1) + \text{Im}(\gamma)(2\text{Re}(\gamma) - 1) \geq$$

$$\geq \frac{(\text{Re}(\gamma))^{-6}}{t^{2-2\text{Re}(\gamma)}} \left[ \frac{\text{Re}(\gamma)}{4} \right]^2$$

So, if $t \geq s_0$, then $m_t \geq A_{\gamma, \alpha}$ where $A_{\gamma, \alpha} = \frac{1}{4} \alpha^2 \frac{(\alpha - 2\text{Re}(\gamma) + 1)^2}{(1+\text{Re}(\gamma))}$. We note that

$$\frac{4\alpha^2}{(2+\alpha)} \leq A_{\gamma, \alpha} \leq \frac{\alpha^2(n+1)^2}{4}. \quad \text{Now, let } U_{j,b} = \left\{ t > s_0 : \left| \frac{\partial^2}{\partial t^2} \Phi(b, t) \right| > \frac{A_{\gamma, \alpha}}{4} \right\}, j = 1, 2, 3.$$

Then $U_{j,b} = \bigcup_{k \in K_j} I_{j,b,k}$ for some family $\{I_{j,b,k}\}_{k \in K_j}$ of disjoint open intervals.

Moreover $\frac{\partial}{\partial t} \Phi(b, t) = \pm \sqrt{A_{\gamma, \alpha}}$ if $t \in \partial(I_{j,b,k})$. Suppose that the equation

$$\frac{\partial}{\partial t} \Phi(b, t) = \pm \sqrt{A_{\gamma, \alpha}}$$

has $N$ solutions $t_1, ..., t_N$ in $(1, +\infty)$, then the equation

$$\alpha \left( \frac{\alpha}{\text{Re}(\gamma)} - 1 \right) \left[ \frac{\alpha}{\text{Re}(\gamma)} - 1 \right] + \frac{1}{\text{Re}(\gamma)} \left( \frac{1}{\text{Re}(\gamma)} - 1 \right) \frac{\text{Re}(\gamma)}{\text{Re}(\gamma)} + \text{Im}(\gamma) = 0$$

has at least $N - 1$ solutions in $(1, +\infty)$. Indeed, since the left side agrees with $\frac{\partial^2}{\partial t^2} \Phi(b, t)$, this assertion follows from Rolle Theorem. So
\[
\frac{\alpha}{\text{Re}(\gamma)} \left( \frac{\alpha}{\text{Re}(\gamma)} - 1 \right) s^\alpha + b \frac{1}{\text{Re}(\gamma)} \left( \frac{1}{\text{Re}(\gamma)} - 1 \right) s + \text{Im}(\gamma) \frac{1}{\text{Re}(\gamma)} = 0
\]

has at least \( N - 1 \) solutions \( s_1, \ldots, s_{N-1} \). Then

\[
\frac{\alpha^2}{\text{Re}(\gamma)} \left( \frac{\alpha}{\text{Re}(\gamma)} - 1 \right) s^{\alpha-1} + b \frac{1}{\text{Re}(\gamma)} \left( \frac{1}{\text{Re}(\gamma)} - 1 \right) = 0
\]

has at least \( N - 2 \) solutions. Thus \( N \leq 3 \). Similarly the equations

\[
\frac{\partial \phi}{\partial t}(b, t) = -\frac{i}{2} \phi, \quad j = 1, 2, 3;
\]

have at most 3 solutions on \((1, +\infty)\). Then each \( U_{j,b} \) is a union of at most 4 open intervals. Assertion (i) follows from the Van der Corput lemma applied to each \( I_{j,b,k} \).

To prove (ii) we first show that

\[
(2.5) \quad \left| \text{Im}(\gamma) \int_0^1 e^{-i(x^\alpha + x^\eta)} x^{\gamma-1} dx \right| \leq C_\alpha \frac{1 + |\text{Im}(\gamma)|}{|\eta|^{\text{Re}(\gamma)/\alpha}},
\]

where \( C_\alpha \) is independent of \( \xi, \eta \) and \( \gamma \). We can assume that \( \text{Im}(\gamma) \neq 0 \). Now

\[
\int_0^1 e^{-i(x^\alpha + x^\eta)} x^{\gamma-1} dx = \frac{1}{\eta^{\text{Re}(\gamma)/\alpha}} \int_0^{\eta^{1/\alpha}} e^{-i \left( \frac{x^\alpha}{\eta^{1/\alpha}} + \frac{x^\eta}{\eta^{1/\alpha}} \right)} x^{\gamma-1} dx.
\]

If \( \eta \geq 1 \), we decompose this last integral as \( \int_0^1 + \int_1^\eta \). Now

\[
\left| \int_0^{\eta^{1/\alpha}} e^{-i \left( \frac{x^\alpha}{\eta^{1/\alpha}} + \frac{x^\eta}{\eta^{1/\alpha}} \right)} x^{\gamma-1} dx \right| =
\]

\[
= \left| \int_{-\infty}^0 e^{-i \left( e^{\frac{x^\alpha}{\eta^{1/\alpha}}} - \text{Im}(\gamma)t \right)} e^{\text{Re}(\gamma)t - i\alpha t} dt \right| = \left| \int_{-\infty}^0 e^{-i\phi(t)} \psi(t) dt \right|,
\]

where \( \phi(t) = e^{t \frac{x^\alpha}{\eta^{1/\alpha}}} - \text{Im}(\gamma)t \), \( \psi(t) = e^{\text{Re}(\gamma)t - i\alpha t} \). We use corollary of proposition 2 in ([St], p. 334]) obtaining that if \( \frac{|\text{Im}(\gamma)|^{1/\alpha}}{2|\xi|} \geq 1 \) then

\[
\left| \int_{-\infty}^0 e^{-i\phi(t)} \psi(t) dt \right| \leq \frac{c}{|\text{Im}(\gamma)|}, \quad \text{for some positive constant independent of } \xi, \eta, \gamma.
\]

If \( \frac{|\text{Im}(\gamma)|^{1/\alpha}}{2|\xi|} < 1 \), we decompose the integral over \((-\infty, 0)\) in the sum of the integrals over \((-\infty, M)\) and \((M, 0)\) where \( M = \log \left( \frac{|\text{Im}(\gamma)|^{1/\alpha}}{2|\xi|} \right) \). The same corollary gives us now

\[
\left| \int_{-\infty}^M e^{-i\phi(t)} \psi(t) dt \right| \leq \frac{c}{|\text{Im}(\gamma)|^{1/\alpha}} \leq \frac{c(1 + |\text{Im}(\gamma)|)}{|\text{Im}(\gamma)|}. \quad \text{The same considerations yields to (ii) in the case } \eta < 1.
\]
It remains to study \( R_1 = 1 \varepsilon^{-\gamma} x^{\gamma-1} \) in the case \( \eta > 1 \). We write this integral as \( \int_1^{\eta^{1/\alpha}} e^{-i\phi(x)} \psi(x) dx \) where \( \phi(x) = x^\alpha + \frac{\varepsilon}{\eta^{1/\alpha}} x \) and \( \psi(x) = x^{-\gamma} \). If \( \alpha \geq 2 \), we apply corollary p.334 in [St] with the second derivative to obtain

\[
\left| \int_1^{\eta^{1/\alpha}} e^{-i\phi(x)} \psi(x) dx \right| \leq c(1 + |\text{Im}(\gamma)|).
\]

If \( \alpha < 2 \), and \( \left| \frac{\varepsilon}{\eta^{1/\alpha}} \right| < \frac{a}{2} \), the same corollary, applied with the first derivative, gives us the same bound. If \( \left| \frac{\varepsilon}{\eta^{1/\alpha}} \right| \geq \frac{a}{2} \), let \( J_1 = (-\infty, \left| \frac{\varepsilon}{\eta^{1/\alpha}} \right|) \), \( J_2 = \left( \left| \frac{\varepsilon}{\eta^{1/\alpha}} \right|, \frac{2\varepsilon}{\eta^{1/\alpha}} \right) \) and \( J_3 = \left( \frac{2\varepsilon}{\eta^{1/\alpha}}, +\infty \right) \) and let \( I_j = J_j \cap \left[ 1, \eta^{1/\alpha} \right], \) \( j = 1, 2, 3 \). We decompose

\[
\int_1^{\eta^{1/\alpha}} e^{-i\phi(x)} \psi(x) dx = \int_{I_1} + \int_{I_2} + \int_{I_3}.
\]

To estimate these integrals we note that \( |\phi'(x)| = |\alpha x^\alpha + \frac{\varepsilon}{\eta^{1/\alpha}}| \geq \frac{a}{4} \) for \( x \in J_1 \), \( |\phi'(x)| \geq \left| \frac{\varepsilon}{\eta^{1/\alpha}} \right| \geq \frac{a}{2} \) for \( x \in J_3 \) and \( |\phi''(x)| \geq |\alpha(\alpha - 1) x^{\alpha - 2}| \geq \alpha(\alpha - 1) \left| \frac{2\varepsilon}{\eta^{1/\alpha}} \right| \) for \( x \in J_2 \). We also have

\[
\nu\left( \left| \frac{2\varepsilon}{\alpha \eta^{1/\alpha}} \right| \right) + \int_{J_2} |\psi'(x)| dx \leq c(1 + |\text{Im}(\gamma)|) \left| \frac{\varepsilon}{\eta^{1/\alpha}} \right|^{\frac{\beta(\gamma - 1)}{\alpha - 1}} \leq c'(1 + |\text{Im}(\gamma)|) \left| \frac{\varepsilon}{\eta^{1/\alpha}} \right|^{\frac{\beta(\gamma - 1)}{\alpha - 1}}.
\]

Now we apply the corollary in [St], p. 334, to obtain (2.6) in the case \( \alpha < 2 \). So (2.5) holds. From (2.5) and (i) we obtain (ii). To prove (iii), we first assume that \( \text{Re}(\gamma) \neq 1 \). We have

\[
\left| \int_0^{1} e^{-i(\xi + x^{\eta})} x^{\gamma-1} dx \right| \leq \sum_{r=0}^{\infty} \int_{2^{-r-1}}^{2^{-r}} e^{-i(\xi + x^{\eta})} x^{\gamma-1} dx.
\]

We apply again the same corollary in [St] to write
Now, since \( \frac{1-2^{-\text{Re}(\gamma)}}{\text{Re}(\gamma)-1} \) tends to \( \ln(2) \) as \( \text{Re}(\gamma) \) tends to 1, we obtain the case \( \text{Re}(\gamma) = 1 \) from the above, by a limit argument.

3. \( L^p - L^q \) estimates

**Theorem 3.1.** The following statements hold

(i) If the case I occurs, then \( A \in E^\mu_1 \).

(ii) If the case II occurs, then \( A_{\alpha_2, \gamma_2} \in E^\mu_1 \).

(iii) If either the case IV or the case V occurs, then \( A_0 \in E^\mu_1 \).

(iv) If the case III occurs, then \( A_{\alpha_2, \gamma_2} \in E^\mu_1 \).

**Proof.** Let \( \gamma_j(z) = 1 - (1 - \gamma_j)(1 - z) \), \( j = 1, 2 \). We set, for \( \text{Re}(\gamma_j(z)) > 0 \),

\[
\mu_z(E) = \int_Q \chi_E(x_1, x_2, \varphi(x_1, x_2))|x_1|^{\gamma_j(z)-1}|x_2|^{\gamma_j(z)-1}dx_1dx_2.
\]

For \( z \in C \), we consider the analytic family of distributions \( I_z \), that, for \( \text{Re}(z) > 0 \), are given by \( I_z(t) = \frac{2\pi i}{t^z} |t|^{z-1} \). We set \( J_z = \delta \otimes \delta \otimes I_z \), hence \( (J_z)^{\wedge} = 1 \otimes 1 \otimes J_{1-z} \). We define the analytic family of operators given by \( T_z f = \mu_z * J_z * f, f \in S(\mathbb{R}^3) \).

To prove (i), we note that, since \( \gamma_j > 1/2 \), \( \text{Re}\gamma_j(z) > 0 \) for \( \text{Re}(z) \in [-1, 1] \). It is easy to show that, if \( \text{Re}(z) = 1 \) then \( \|T_z\|_{1, \infty} = \|\mu_z * J_z\|_{\infty} \leq c \). We also observe that if \( \text{Re}(z) = -1 \), then \( \frac{\text{Re}\gamma_j(z)}{\alpha_j} \geq \frac{1}{2}, j = 1, 2 \). Then Lemma 2.3, (iii), implies \( \|(\mu_z)^{\wedge}(y_1, y_2, y_3)\| \leq c(z)(1 + |y_3|)^{-1} \). Then \( \|T_z\|_{1, 2} \leq \|(\mu_z)^{\wedge}(J_z)^{\wedge}\|_{\infty} \leq c(z) \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2})} \). It is easy to see that \( \{T_z : -1 \leq \text{Re}(z) \leq 1 \} \) satisfies the hypothesis of the complex interpolation theorem as stated in [S-W], p. 205. Since \( T_0 = cT_{\mu_1} \), (i) follows.

To prove (ii) we study first the case \( \gamma_2 > \frac{1}{3} \). Let \( \tau = \frac{1}{\alpha_2 + \frac{2}{\alpha_2} \gamma_2} \), so \( \tau > 0 \). For \( -\tau \leq \text{Re}(z) \leq 1 \) we have \( 0 < \text{Re}\gamma_j(z), j = 1, 2 \). As in (i), if \( \text{Re}(z) = 1 \) then \( \|T_z\|_{1, \infty} \leq c \). If \( \text{Re}(z) = -\tau \), then \( \text{Re}(\gamma_1(z)) > 0 \) and \( \text{Re}(\gamma_2(z)) > 0 \), moreover \( \frac{\text{Re}\gamma_1(z)}{\alpha_1} \geq \frac{1}{2} \) and \( \frac{\text{Re}\gamma_2(z)}{\alpha_2} < \frac{1}{2} \), so Lemma 2.3 implies...
where \( c(z) \) has at most a polynomial growth in \( |\text{Im}(z)| \) along the line \( \text{Re}(z) = -\tau \). As before, by complex interpolation, (ii) follows in this case.

If \( \gamma_2 = \frac{1}{3} \) and \( \alpha_1 \alpha_2 + \alpha_1 + \alpha_2 \leq \gamma_1 (3\alpha_2 + 2) + \gamma_2 (\alpha_1 - 2) \), then for \( \varepsilon > 0 \) small enough, we think in the pair \( (\gamma_1, \gamma_2, \alpha_1, \alpha_2) = (\gamma_1 + \varepsilon, \gamma_2 + \varepsilon) \) instead of \( (\gamma_1, \gamma_2) \) with \( \varepsilon \geq 0 \) such that \( \alpha_1 \alpha_2 + \alpha_1 + \alpha_2 \leq \gamma_1 (3\alpha_2 + 2) + \gamma_2 (\alpha_1 - 2) \).

We define as above the corresponding \( \hat{\gamma}_j(z) \) and \( \hat{T}_{\tau_r} \) and the strip \( \Re(z) \leq -\tau \). We think in the pair (ii) and (iii) of Lemma 2.3, (ii) imply that \( \hat{\gamma}_j(z), \mu_{\alpha_2}, \) and \( T_{\tau_r} \). We take now \( \tau_r = \frac{1}{2} \alpha_2 + 2\gamma_0 \) and we consider the analytic family of operators \( \gamma_0(z)T_{\tau_r} \) on the strip \( -\tau \leq \Re(z) \leq 1 \). As above, but now taking account of (ii) and (iii) of Lemma 2.3, we have that \( \|\gamma_0(z)T_{\tau_r}\|_{2,2} \leq \alpha(z) \Re(z) = -\tau_r \) and for all positive and small enough \( \varepsilon \). Now (ii) follows by complex interpolation and a limit argument.

(iii) Let \( \tau = \frac{\alpha_1}{\alpha_2 + 1} \). Let \( \gamma_j(z), j = 1, 2, \mu_{\alpha_2}, \) and \( T_{\tau_r} \) be defined as above, but now on the strip \( -\tau \leq \Re(z) \leq 1 \). We can check that \( \Re(\gamma_j(z)) > 0 \), \( j = 1, 2 \) on this strip and that \( \Re(\gamma_j(z))/\alpha_j < 1/2, j = 1, 2 \) if \( \Re(z) = -\tau \). For these \( z \), Lemma 2.3 gives us

\[
\|\mu_{\alpha_2}(\gamma_j(z))\|_{2,2} \leq \alpha(z) \Re(z) = -\tau_r \]

for some positive constant \( c(z) \). Then \( \|T_{\tau_r}\|_{1,\infty} \leq c \) for \( \Re(z) = 1 \), (iii) follows by complex interpolation.

(iv) To see that \( A_{\gamma_2} \in E_\mu \), we set \( \gamma_j(z), j = 1, 2, \mu_{\alpha_2}, \) and \( T_{\tau_r} \) be defined as above for \( -\tau \leq \Re(z) \leq 1 \), where \( \tau = \frac{\alpha_1}{\alpha_2} \). We note that \( \gamma_1(-\tau) > 0 \) and \( \gamma_2(-\tau) = 0 \). For \( \varepsilon > 0 \) small enough, we set \( \tau_r = \frac{\gamma_1(-\tau) - \varepsilon}{1 - \gamma_2(\tau)} \), so for \( \Re(z) = -\tau_r \), \( \Re(\gamma_1(z)) \) and \( \Re(\gamma_2(z)) \) are positive. We consider the analytic family of operators \( \gamma_2(z)T_{\tau_r} \) on the strip \( -\tau \leq \Re(z) \leq 1 \). As before, \( \|\gamma_2(z)T_{\tau_r}\|_{1,\infty} \leq c \|\gamma_2(z)\| \) if \( \Re(z) = 1 \).

We now consider \( \Re(z) = -\tau_r \). We write \( \mu_{\alpha_2}(\gamma_j(z)) = \int_1 e^{-i(x_1y_1 + |x_1|y_3)}|x_1|\gamma_1(z)-1 dx_1 \int_1 e^{-i(x_2y_2 + |x_2|y_3)}|x_2|\gamma_2(z)-1 dx_2 = \mathcal{F}_1 \mathcal{F}_2 \).

Lemma 2.3, (ii) imply that \( \|\gamma_2(z)\|_2 \leq c(1 + |\text{Im}(z)|)(1 + |y_3|)^{-\frac{\gamma_1(-\tau)}{\alpha_1}} \). If \( \frac{\gamma_1(-\tau)}{\alpha_1} \geq \frac{1}{2} \), then \( \frac{\gamma_1(-\tau)}{\alpha_1} > \frac{1}{2} \), so Lemma 2.3, (iii) imply that \( \|\mathcal{F}_1\| \leq c(1 + |\text{Im}(z)|)^2(1 + |y_3|)^{-\frac{1}{2}} \). If \( \frac{\gamma_1(-\tau)}{\alpha_1} \leq \frac{1}{2} \), then \( \frac{\gamma_1(-\tau)}{\alpha_1} < \frac{1}{2} \), so by Lemma 2.3 (i), \( \|\mathcal{F}_1\| \leq c(1 + |\text{Im}(z)|)(1 + |y_3|)^{-\frac{\gamma_1(-\tau)}{\alpha_1}} \). Moreover in these estimates we can choose \( c \) independent of \( \varepsilon \).

Since case III occurs, we have for \( \varepsilon \) small enough, \( \frac{\gamma_1(-\tau)}{\alpha_1} > \tau_r \) and \( \frac{1}{2} > \tau_r \). Now, \( \|\mathcal{F}_1\| \leq c \) with \( c \) independent of \( \varepsilon \), so \( \|\mathcal{F}_1\| \leq c(1 + |\text{Im}(z)|)^2 \).
(1 + |y_3|)^{-\gamma_2 + \frac{2(a_2 - \gamma_2)}{\gamma_1}}, \text{ from this we obtain } |\gamma_2(z)(\mu_z)^\Lambda(y_1, y_2, y_3)| \leq c(1 + |\text{Im}(z)|)^2(1 + |y_3|)^{-\gamma_2} \text{ with } c \text{ independent of } \epsilon. \text{ Now (iv) follows by complex interpolation and a limit argument.}

For \( j = 1, 2 \) we consider an even function \( \Phi_j \in C_0^\infty(\mathbb{R}) \), such that \( \text{supp} \Phi_j \subset \{ t \in \mathbb{R} : 2^j \leq |t| \leq 2^{j+1} \} \), \( 0 \leq \Phi_j \leq 1 \) and \( \sum_{j \in \mathbb{Z}} \Phi_j(2^j t) = 1 \) if \( t \neq 0 \). For \( r_1, r_2 \in N \), and a Borel set \( E \), we set \( \nu_{r_1, r_2}(E) = \int \chi_{E}(x_1, x_2, \varphi(x_1, x_2))\Phi_1\left(2^{\frac{j_1}{2}}x_1\right)\Phi_2\left(2^{\frac{j_2}{2}}x_2\right)|x_1|^{\gamma_1-1}|x_2|^{\gamma_2-1}dx_1dx_2. \)

For \( f \in S(\mathbb{R}^3) \), let \( T_{\nu_{r_1, r_2}}f = \nu_{r_1, r_2}*f \). We observe that \( \mu \leq \nu = \sum_{r_1, r_2 \in N} \nu_{r_1, r_2} \).

**Lemma 3.2.** There exists a positive constant \( c \) such that
\[
\left\| T_{\nu_{r_1, r_2}} \right\|_{L^p} \leq c2^{\frac{\alpha_2(a_2-\gamma_2+2)}{2\gamma_1}+\frac{\alpha_1(a_2-\gamma_2+2)}{2\gamma_2}}, r_1, r_2 \in N.
\]

**Proof.** We observe that \((\nu_{r_1, r_2})^\Lambda(y_1, y_2, y_3) = \mathcal{J}_{1,r_1}(y_1, y_3), \mathcal{J}_{2,r_2}(y_2, y_3)\), where
\[
\mathcal{J}_{j,r_j}(y_j, y_3) = \int e^{-i(x_j y_j+|y_j|^2)}\Phi_j\left(2^{\frac{j_1}{2}x_1}\right)|x_j|^{\gamma_j-1}dx_j,
\]
Corollary of the proposition 2 [St. p. 334] gives us
\[
|\mathcal{J}_{j,r_j}(y_j, y_3)| \leq c2^{\frac{\alpha_1(a_1-1)}{2\gamma_1}}|y_3|^{-1}
\]
and so
\[
(3.3) \quad \left| \nu_{r_1, r_2}(y_1, y_2, y_3) \right| \leq c2^{\frac{\alpha_1(a_1-1)}{2\gamma_1}}+\frac{\alpha_2(a_2-1)}{2\gamma_2} |y_3|^{-1}.
\]

In a similar way as in theorem 3.1, we define, for \( \text{Re}(z) \in [-1, 1] \), the analytic family of operators \( \{ T_z \} \) given by \( T_z f = e^{z^2}f * \nu_{r_1, r_2} * J_z \). For \( \text{Re}(z) = 1 \),
\[
\| T_z \|_{1, \infty} \leq c2^{\frac{\alpha_1(a_1-1)}{2\gamma_1}}+\frac{\alpha_2(a_2-1)}{2\gamma_2} \cdot \text{On the other hand, (3.3) implies that if } \text{Re}(z) = -1, \text{ then } \| T_z \|_{2, 2} \leq c2^{\frac{\alpha_1(a_1-1)}{2\gamma_1}}+\frac{\alpha_2(a_2-1)}{2\gamma_2}. \text{ The lemma follows by complex interpolation.}

We denote with \( \nu_{r_1} = r_1 \sum \nu_{r_1, r_2} \) and with \( \nu_{r_1} = r_2 \sum \nu_{r_1, r_2} \).

**Lemma 3.4.** (i) If \( \gamma_2 \geq 1/3 \) and \( \frac{\alpha_1+\gamma_2}{\gamma_2} > 4 \), then
\[
\left\| T_{\nu_{r_1}} \right\|_{L^p} \leq c2^{\frac{\alpha_1(a_1-1)}{2\gamma_1}+\frac{\alpha_2(a_2-1)}{2\gamma_2}}.
\]
(ii) If \( \gamma_1 \geq 1/3 \) and \( \frac{\alpha_1+\gamma_1}{\gamma_1} > 4 \), then
\[ \left\| T_{\phi_j} \right\|_{A_{\gamma_1}} \leq c^{\frac{\gamma_2 (\alpha_2 - 2)(\alpha_1 + 1 - \gamma_2) + (\gamma_2 - 1)(\alpha_1 + 1)}{\alpha_2 + 1 - \gamma_2}}. \]

**Proof.** To see (i), we define an analytic family of operators, on the strip $-\frac{1}{2} \frac{\alpha_2 + 2\gamma_2}{\alpha_2 + 1 - \gamma_2} \leq \text{Re}(z) \leq 1$, in the following way. We set $\nu_{\gamma_1}(E) = \int_{[-1,1]} \chi_E(x_1, x_2, \varphi(x_1, x_2)) \Phi_1 \left( \frac{2^{\gamma_1} x_1}{x_1} \right) |x_1|^{\gamma_1(z) - 1} |x_2|^{\gamma_2(z) - 1} dx_1 dx_2$ with $\gamma_j(z)$ as in theorem 3.1 and $T_z f = e^{zf} * \nu_{\gamma_1} * J_z$. Now it is easy to show that, if $\text{Re}(z) = 1$ then $\| T_z \|_{1, \infty} \leq c$. To study $\| T_z \|_{2, 2}$, for $\text{Re}(z) = -\frac{1}{2} \frac{\alpha_2 + 2\gamma_2}{\alpha_2 + 1 - \gamma_2}$, we observe that

\[ (\nu_{\gamma_1})^{\gamma}(y_1, y_2, y_3) = \int e^{-i (x_1 y_1 + x_1 y_2)} \Phi_1 \left( \frac{2^{\gamma_1} x_1}{x_1} \right) |x_1|^{\gamma_1(z) - 1} dx_1 \times \int_{[-1,1]} e^{-i (x_2 y_2 + x_2 y_3)} |x_2|^{\gamma_2(z) - 1} dx_2. \]

Since

\[ \left| \int e^{-i (x_1 y_1 + x_1 y_2)} \Phi_1 \left( \frac{2^{\gamma_1} x_1}{x_1} \right) |x_1|^{\gamma_1(z) - 1} dx_1 \right| \leq c^{\frac{\alpha_2}{\alpha_2 + 1 - \text{Re}(\gamma_1(z))}} |y_3|^{-\frac{1}{2}} \]

the assertion (i) of the lemma follows as in (ii), Theorem 3.1. Part (ii) follows in a similar way.

**4. Endpoint bounds**

In this section we will characterize $E_{\mu}$ in the case $\frac{1}{3} \leq \gamma_1, \gamma_2 \leq 1$. The use of the Littlewood Paley theory at this point, goes back to [C]. We will also describe the interior of $E_{\mu}$ in the case $0 < \gamma_1, \gamma_2 \leq 1$.

For $U \in S'(\mathbb{R}^2)$ and a test function $g$ we set $U^\vee(g) = U(g^\vee)$, where $g^\vee(y) = g(-y)$. For $g_1 : \mathbb{R}^2 \rightarrow C$ and $g_2 : \mathbb{R} \rightarrow C$ we define

\[ (g_1 \otimes_1 g_2)(\xi_1, \xi_2, \xi_3) = g_1(\xi_1, \xi_3) g_2(\xi_2) \]

and

\[ (g_1 \otimes_2 g_2)(\xi_1, \xi_2, \xi_3) = g_1(\xi_2, \xi_3) g_2(\xi_1). \]

Also for $U \in S'(\mathbb{R}^2)$ and $V \in S'(\mathbb{R})$ and $k = 1, 2$, we define

\[ (U \otimes_k V)(g_1 \otimes_k g_2) = U(g_1) V(g_2). \]

For $1 \leq j \leq 2$, we introduce a $C^\infty$ partition of unity $\{m_{j,r}\}_{r \in \mathbb{Z}}$ in $\mathbb{R}^2$ minus the coordinate axes, with $m_{j,r}$ homogeneous of degree zero (with respect to
the Euclidean dilations on \(\mathbb{R}^2\) such that \(m_{j,0}(t_1, t_2) = m_{j,0}(2^{-\frac{r}{\gamma}}t_1, 2^{-r}t_2)\) and \(\text{supp } m_{j,r} \subset \left\{ (t_1, t_2) : 2^{-\frac{r}{\gamma}}|t_1| \leq 2^{-r}|t_2| \leq 2^{\frac{r}{\gamma}}|t_1| \right\}\). We also define 
\(M_{j,r}(\xi_1, \xi_2, \xi_3) = m_{j,r}(\xi_1, \xi_3)\). We put, for \(s > 0\) and \(\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3\), \(s \cdot \xi = (s^{\frac{1}{2}}\xi_1, s^{\frac{1}{2}}\xi_2, s^{\frac{1}{2}}\xi_3)\) and for \((t_1, t_2) \in \mathbb{R}^2\), \(s \cdot (t_1, t_2) = (s^{\frac{1}{2}}t_1, st_2)\). For \(g : \mathbb{R}^2 \rightarrow C\), \(s > 0\), we set \((s \cdot g)(t_1, t_2) = g(s \cdot (t_1, t_2))\), so we have \(M_{j,r} = 2^{-r} \cdot M_{j,0}\) and \(m_{j,r} = 2^{-r} \cdot m_{j,0}\).

Let \(Q_{j,r}\) be the operator with multiplier \(M_{j,r}\), let \(C_0\) be a large constant and define \(\tilde{Q}_{j,r} = \sum_{|l| = 0} Q_{j,l}\). So \(\tilde{Q}_{j,r}\) is the operator with multiplier \(\tilde{M}_{j,r} = \sum_{|l| \leq C_0} M_{j,l}\). Let \(\tilde{m}_{j,r} = \sum_{|l| \leq C_0} m_{j,l}\), so \(\tilde{m}_{j,r} = 2^{-r} \cdot \tilde{m}_{j,0}\). We choose \(C_0\) in such a way that \(\tilde{m}_{j,r} \equiv 1\) on \(\text{supp } m_{j,r}\).

For \(\epsilon_{r_k} = \pm 1\), \(\left\{ Q_{k,r_k} \right\}_{r_k \in N}\) satisfies \(\left\| \sum_{r_k \in N} \epsilon_{r_k} \tilde{Q}_{k,r_k} \right\|_{p,p} \leq c\), with \(c\) independent of \(\{\epsilon_{r_k}\}\). Indeed, this follows from the Marcinkiewicz multiplier theorem (see [S], p. 109). As in [S], p. 105, we get the Littlewood Paley inequality

\[
\left\| \left( \sum_{r_k \in N} \left| \tilde{Q}_{k,r_k} f \right|^2 \right)^{1/2} \right\|_p \leq c \|f\|_p.
\]

By replacing \(C_0\) by a larger constant we may define operators \(Q'_{k,r_k}\) with the same properties of the operators \(\tilde{Q}_{k,r_k}\), and such that \(\tilde{Q}_{k,r_k} \circ \tilde{Q}_{k,r_k} = \tilde{Q}_{k,r_k}\).

Let \(h \in C_0^\infty (\mathbb{R}^2)\) be identically one in a neighborhood of the origin, let \(H_{j,r}(\xi_1, \xi_2, \xi_3) = h(2^{-\frac{r}{\gamma}}\xi_1, 2^{-r}\xi_3)\) and let \(P_{j,r}\) be the Fourier multiplier operator with symbol \(H_{j,r}\). As in [F-G-U], Lemmas 2.4 and 2.5, there exists \(c > 0\) such that, for \(R \in N, k = 1, 2\)

\[
(4.1) \quad \left\| \sum_{1 \leq r_k \leq R} T_{v_{1,2}} P_{k,r_k} \right\|_{p,q} \leq c \left\| \sum_{1 \leq r_k \leq R} T_{v_{1,2}} \right\|_{p,q} 1 < p, q < \infty;
\]

and

\[
(4.2) \quad \left\| \sum_{1 \leq r_k \leq R} T_{v_{1,2}} (I - P_{k,r_k}) \left(I - \tilde{Q}_{k,r_k} \right) \right\|_{p,q} \leq c \left\| \sum_{1 \leq r_k \leq R} T_{v_{1,2}} \right\|_{p,q} 1 < p, q < \infty.
\]
Let \( \mathcal{J}_{j,r} \) be defined as in the proof of Lemma 2.3. Taking account of proposition 1 in ([St], p. 331), we note that, if \( C_0 \) is large enough, then

\[
\mathcal{J}_{j,0}(1 - \bar{h})(1 - \bar{m}_{j,0}) \in S(\mathbb{R}^2).
\]

We also have

\[
\mathcal{J}_{j,r}(1 - \bar{m}_{j,r})(t_1, t_2) = 2^{-\frac{\gamma_j}{2}} \mathcal{J}_{j,0}(1 - \bar{m}_{j,0})(2^{-r} \bullet (t_1, t_2)).
\]

For \( R \in \mathbb{N} \) we decompose

\[
\sum_{1 \leq \nu_1, \nu_2 \leq R} T_{\nu_1, \nu_2} = \sum_{1 \leq \nu_1, \nu_2 \leq R} T_{\nu_1, \nu_2} P_{k, r_k} +
\]

\[
+ \sum_{1 \leq \nu_1, \nu_2 \leq R} T_{\nu_1, \nu_2} (I - P_{k, r_k}) (I - \bar{Q}_{k, r_k}) + \sum_{1 \leq \nu_1, \nu_2 \leq R} T_{\nu_1, \nu_2} (I - P_{k, r_k}) \bar{Q}_{k, r_k}.
\]

**Lemma 4.4.** If \( 0 < \gamma_1, \gamma_2 < 1 \) then the kernel of the convolution operator

\[
\sum_{1 \leq \nu_1, \nu_2 \leq R} T_{\nu_1, \nu_2} (I - P_{2, r_2}) (I - \bar{Q}_{2, r_2})
\]

belongs to weak-\( L^{\frac{\gamma_1 + 1}{\gamma_2}} \) with weak constant less than \( e2^{-\frac{\gamma_1}{\gamma_2}} \). Also

\[
\sum_{1 \leq \nu_1, \nu_2 \leq R} T_{\nu_1, \nu_2} (I - P_{1, r_1}) (I - \bar{Q}_{1, r_1})
\]

belongs to weak-\( L^{\frac{\gamma_1 + 1}{\gamma_2}} \) with weak constant less than \( e2^{-\frac{\gamma_1}{\gamma_2}} \).

**Proof.** We follow the proof of Lemma 2.6 in [F-G-U] to obtain that the kernel \( K_{r_1, r_2} \) of the convolution operator \( T_{\nu_1, \nu_2} (I - P_{2, r_2}) (I - \bar{Q}_{2, r_2}) \) satisfies

\[
K_{r_1, r_2} = 2^{\frac{\gamma_1}{2} (1 - \gamma_2) + r_2} (\eta_1 \otimes_1 \delta) * (2^{\gamma_1} \bullet_2 G_2 \otimes_2 \delta)
\]

where \( \eta_1 \) is the measure defined by \( \eta_1(E) = \int \Phi_1(2^{\gamma_1} s) \chi_E(s, |s|^{1 - \alpha_1}) |s|^{-1} ds \) and \( G_2 = (\mathcal{J}_{2,0}(1 - \bar{h})(1 - \bar{m}_{2,0})) \). We compute this convolution for \( f \in S(\mathbb{R}^3) \). We get

\[
K_{r_1, r_2}(x_1, x_2, x_3) = 2^{\frac{\gamma_1}{2} (1 - \gamma_2) + r_2} (2^{\gamma_1} \bullet_2 G_2)(x_2, x_3 + |x_1|^{\alpha_1}) \Phi_1 \left( \frac{2^{\gamma_1} x_1}{|x_1|^{\alpha_1}} \right) |x_1|^{-1}.
\]

So

\[
\sum_{r_2} |K_{r_1, r_2}(x_1, x_2, x_3)| \leq 12^{\frac{\gamma_1}{2} (1 - \gamma_2)} \chi_{V_{r_1}^2}(x_1, x_2, x_2) \sum_{r_2} 2^{\frac{\gamma_1}{2} (1 - \gamma_2) + r_2} |2^{\gamma_1} \bullet_2 G_2(x_2, x_3 + |x_1|^{\alpha_1})|
\]

where \( V_{r_1}^2 = \left\{ (x_1, x_2) \in Q : 2^{-\frac{\gamma_1 - 1}{\alpha_1}} \leq |x_1| \leq 2^{-\frac{\gamma_1 - 4}{\alpha_1}} \right\} \). So we obtain

\[
\sum_{r_2} |K_{r_1, r_2}(x_1, x_2, x_3)| \leq 2^{\frac{\gamma_1}{2} (1 - \gamma_2)} \chi_{V_{r_1}^2}(x_1, x_2) (|x_2|^{\alpha_2} + |x_3 + |x_1|^{\alpha_1}|)^{\frac{\gamma_1 - 1}{\alpha_1}}.
\]
From this we get the first statement of the lemma. The second one is analogous.

In a similar way we obtain

**Lemma 4.5.** If \(0 < \gamma_1, \gamma_2 < 1\), then the kernel of the convolution operator

\[
\sum_{1 \leq r_2 \leq R} T_{\nu_1, r_2} P_{2, r_2}
\]

belongs to weak-\(L^{\frac{\gamma_2 + 1}{\gamma_2}}\) with weak constant less than \(c2^{-\frac{\gamma_2 + 1}{\gamma_2}}\). Also

\[
\sum_{1 \leq r_1 \leq R} T_{\nu_1, r_2} P_{1, r_1}
\]

belongs to weak-\(L^{\frac{\gamma_1 + 1}{\gamma_1}}\) and its weak constant is less than \(c2^{-\frac{\gamma_1 + 1}{\gamma_1}}\).

**Remark 4.6.** To prove the main result we will need Lemma 2.2 in [F-G-U] which we now state.

Let \(\{\sigma_r\}_{r \in \mathbb{N}}\) be a sequence of positive measures on \(\mathbb{R}^{n+1}\), and let \(T sf = \sigma_r * f, f \in S(\mathbb{R}^{n+1})\). Suppose \(1 \leq k \leq n, 1 < p \leq 2\) and \(p \leq q < \infty\). If there exists \(A > 0\) such that \(\sup_{r \in \mathbb{N}} \|T_s\|_{p, q} \leq A\), \(\|\sum_{1 \leq r \leq R} T_s \sigma_{r, k}\|_{p, q} \leq A\) and \(\|\sum_{1 \leq r \leq R} T_s (I - P_{k, r})(I - \tilde{Q}_{k, r})\|_{p, q} \leq A\) for all \(R \in \mathbb{N}\), then there exists \(c > 0, c\) independent of \(A, R\) and \(\{\sigma_r\}_{r \in \mathbb{N}}\), such that \(\|\sum_{1 \leq r \leq R} T_s\|_{p, q} \leq c A\).

**Theorem 4.7.** If \(\frac{1}{2} \leq \gamma_1, \gamma_2 \leq 1\) and \(\frac{\alpha_1 + 2}{\gamma_1} \leq \frac{\alpha_2 + 2}{\gamma_2}\) then \(E^{\mu}_{\nu}\) is the closed convex polygonal region \(\sum_{\gamma_1, \gamma_2} \gamma_1 \gamma_2\).

**Proof.** We will prove the theorem for each one of the cases described in paragraph 2. In the case I the theorem follows from (i), Theorem 3.1. In the case II, taking account of (ii), Theorem 3.1, it is enough to check that \(B_{\alpha_2, \gamma_2} \in E^{\mu}_{\nu}\). In the case IV we must only show that \(B_{\alpha_2, \gamma_2} \in E^{\mu}_{\nu}\). In the case V we must prove that \(\sum_{1 \leq r \leq R} T_{\nu_1, r_2} P_{2, r_2}\) and \(\sum_{1 \leq r \leq R} T_{\nu_1, r_2} (I - P_{2, r_2})(I - \tilde{Q}_{2, r_2})\) are of weak type \((1, \frac{\alpha_2 + 1}{\gamma_2})\), then (i) in Lemma 3.4, (4.1), (4.2), the Marcinkiewicz inter-
Itos theorem (see [B-S], Remark 4.15, (d)) and a brief computation show that there exists \( c > 0 \), such that for \( R \in \mathbb{N} \)

\[
\left\| \sum_{1 \leq r_2 \leq R} T_{\nu_{12}} P_{2,r_2} \right\|_{B_{\alpha_2 \gamma_2}} \leq c^{\frac{n_1}{n_2}} \left( \frac{\gamma_2 (n_1 + 1 - \gamma_1)}{n_2} \right)
\]

and

\[
\left\| \sum_{1 \leq r_2 \leq R} T_{\nu_{12}} (I - P_{2,r_2}) (I - \hat{Q}_{2,r_2}) \right\|_{B_{\alpha_2 \gamma_2}} \leq c^{\frac{n_1}{n_2}} \left( \frac{\gamma_2 (n_1 + 1 - \gamma_1)}{n_2} \right).
\]

Remark 4.6 implies

\[
(4.8) \quad \left\| \sum_{1 \leq r_2 \leq R} T_{\nu_{12}} \right\|_{B_{\alpha_2 \gamma_2}} \leq c^{\frac{n_1}{n_2}} \left( \frac{\gamma_2 (n_1 + 1 - \gamma_1)}{n_2} \right).
\]

Since we are in case II, we can perform the sum on \( r_1 \), to obtain the theorem, in this case.

Case V: As in case II we obtain that \( B_{\alpha_2 \gamma_2} \in E_{\mu} \), (i) in Lemma 3.4, (4.8) and the Riesz Thorin theorem give

\[
(4.9) \quad \left\| \sum_{1 \leq r_2 \leq R} T_{\nu_{12}} \right\|_{B_{\alpha_2 \gamma_2}} \leq c,
\]

with \( c \) independent of \( r_1 \) and \( R \).

Now, Lemmas 4.4, 4.5 and the weak Young’s inequality imply that the operators \( \sum_{1 \leq r_1 \leq R} T_{\nu_{12}} P_{1,r_1} \) and \( \sum_{1 \leq r_1 \leq R} T_{\nu_{12}} (I - P_{1,r_1}) (I - \hat{Q}_{1,r_1}) \) are of weak type \((1, \frac{\alpha_1 + 1 - \gamma_1}{\alpha_1 + 1 - \gamma_1})\), with weak constant \( 2^{-\frac{\gamma_2}{\gamma_2}} \). Also (4.1), (4.2) and (ii) Lemma 3.4 imply that they have \( ||A_{\alpha_1 \gamma_1} || \) less than \( c^{\frac{n_1}{n_2}} \frac{\gamma_2 (n_1 + 1 - \gamma_1)}{\alpha_1 + 2} \). We set \( t \in (0, 1] \), such that

\[
t (\alpha_2 - 2) (\alpha_1 + 1 - \gamma_1) + (1 - \gamma_2) (3 \alpha_1 + 2) - (1 - t) \gamma_2 = 0
\]

and we define \( B = t A_{\alpha_1 \gamma_1} + (1 - t) \left( \frac{\alpha_1 + 1 - \gamma_1}{\alpha_1 + 1} \right) \). So the operators

\[
\sum_{1 \leq r_2 \leq R} \sum_{1 \leq r_1 \leq R} T_{\nu_{12}} P_{1,r_1}
\]

and
\[ \sum_{1 \leq r_2 \leq R} \sum_{1 \leq r_1 \leq R} T_{\nu_1, \nu_2}(I - P_{1,r_1})(I - \tilde{Q}_{1,r_1}) \]

are bounded on the open polygon with vertices \((1, \frac{\alpha_1+1-\gamma_1}{\alpha_1+1}), B, A_0, (1/2, 1/2)\) and \((1, 1)\) with bounds independent of \(R\). It is easy to check that \(B^0_{\alpha_2, \gamma_2}\) belongs to this polygon, so (4.9) and Remark 4.6 imply that \(B^0_{\alpha_2, \gamma_2} \in E_\nu\).

Case IV: (4.8) says, in this case, that

\[ \left\| \sum_{1 \leq r_2 \leq R} \sum_{1 \leq r_1 \leq R} T_{\nu_1, \nu_2} P_{1,r_1} \right\|_{B^0_{\alpha_2, \gamma_2}} \leq c \]

and

\[ \left\| \sum_{1 \leq r_2 \leq R} \sum_{1 \leq r_1 \leq R} T_{\nu_1, \nu_2}(I - P_{1,r_1})(I - \tilde{Q}_{1,r_1}) \right\|_{B^0_{\alpha_2, \gamma_2}} \leq c. \]

Since \(B^0_{\alpha_2, \gamma_2} = B_{\alpha_2, \gamma_2}\), the theorem follows by Remark 4.6.

**Theorem 4.10.** The interior of \(E_\mu\) agrees with the interior of \(\Sigma_{\alpha_1, \alpha_2, \gamma_1, \gamma_2}\).

**Proof.** It is enough to check that the vertices of \(\Sigma_{\alpha_1, \alpha_2, \gamma_1, \gamma_2}\) belong to the boundary of \(E_\mu\), in the cases III, IV and V. We will consider analytic families of operators of the form \(T_z f = \mu_z * f\) where \(\mu_z\) are complex measures defined for \(\text{Re}(\gamma_1(z)), \text{Re}(\gamma_2(z)) > 0\), by

\[ f(x_1, x_2, \varphi(x_1, x_2))|x_1|^\gamma_1(z)-1|x_2|^\gamma_2(z)-1 dx_1 dx_2 \]

with \(\gamma_j(z) = k_j - (k_j - \gamma_j)(1 - z)\) for a suitable choice of \(k_j\), in each case.

To prove that \(B_{\alpha_2, \gamma_2} \in \partial E_\mu\), we take, in the above construction, \(k_1 = \frac{\alpha_1 + 2}{4}\), \(k_2 = \frac{\alpha_2 + 2}{4}\) and we consider the strip \(-\frac{4\gamma_2}{\alpha_2 + 2 - 4\gamma_2} + \epsilon \leq \text{Re}(z) \leq 1\). For \(\text{Re}(z) = 1, T_z\) is bounded by \(T_\nu\), where \(\nu\) is the measure associated with \(\alpha_1, \alpha_2, k_1\) and \(k_2\). Theorem 3.1, (i) implies that \(\|T_z\|_{4,4} \leq c\). We take \(\epsilon > 0\) small enough. For \(\text{Re}(z) = -\frac{4\gamma_2}{\alpha_2 + 2 - 4\gamma_2} + \epsilon\), it is easy to check that \(\text{Re}(\gamma_1(z)), \text{Re}(\gamma_2(z)) > 0\) and so \(\|T_z\|_{1,1} \leq c\). The complex interpolation theorem implies that the interpolated point \(B_{\alpha_2, \gamma_2}^0\) corresponding to \(z = 0\), belongs to \(E_\mu\).

Since \(B_{\alpha_2, \gamma_2}^0\) tends to \(B_{\alpha_2, \gamma_2}\) as \(\epsilon\) tends to zero, it follows that \(B_{\alpha_2, \gamma_2} \in \partial E_\mu\).

Now we prove that if case V occurs, then \(B_{\alpha_2, \gamma_2}^0\) belongs to \(\partial E_\mu\). Indeed, we take, in the definition of \(T_z\),
and we apply the complex interpolation theorem on the strip

\[
\frac{3\alpha_2 \gamma_1 - 2\gamma_2 + 2\gamma_1 + \alpha_1 \gamma_2}{\alpha_1 \alpha_2 + \alpha_1 - 3\alpha_2 \gamma_1 + 2\gamma_2 - 2\gamma_1 - \alpha_1 \gamma_2} + \epsilon \leq \text{Re}(z) \leq 1
\]

for \(\epsilon > 0\) small enough, to obtain as above that \(B^{0}_{\alpha_2, \gamma_2} \in \partial E_{\mu}\).

Finally, we check that, if the case III occurs then \(B^{0}_{\alpha_2, \gamma_2} \in \partial E_{\mu}\). We take, in the definition of \(T_z\), \(k_1 = \frac{1}{3} (\alpha_1 + 1)\), \(k_2 = \frac{1}{3}\) and we apply the complex interpolation theorem on the strip \(-3\gamma_2 \leq \text{Re}(z) \leq 1\), for \(\epsilon > 0\) small enough.

REFERENCES


