MAPPINGS WITHOUT FIXED OR ANTIPODAL POINTS.
SOME GEOMETRIC APPLICATIONS

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Abstract

For $T$ a topological space and $X$ a real normed space $S(T, X)$ denotes the set of continuous mappings from $T$ into $S(X) = \{ x \in X : \|x\| = 1 \}$. Given $f$ in $S(T, X)$ we study the existence of functions $e$ in $S(T, X^*)$ such that $f(t) = e(t) = -f(t), \forall t \in T$. When this holds for every $f$, we say that $S(T, X)$ is plentiful. If dim $X$ is an even integer or infinite this last property is automatic for any $T$. We show that it also verifies if $T$ is a contractible compact space and $X$ is an arbitrary normed space with dim $X \geq 2$. From this we deduce that if $T$ is completely regular and dim $T < \text{dim} X - 1$, then $S(T, X)$ is plentiful, where dim $T$ stands for the covering dimension of $T$. If $C(T, X)$ denotes the space of continuous and bounded functions from $T$ into $X$ endowed with the sup norm, we study the geometry of the unit ball of $C(T, X)$ for $X$ strictly convex and $S(T, X)$ plentiful. For $T$ completely regular and dim $X < \infty$, we prove the following:

The necessary and sufficient condition for every $f$ in the unit ball of $C(T, X)$ to be the mean of 3 extreme points is that dim $T < \text{dim} X$.

Moreover, if $X$ is infinite-dimensional, then the previously mentioned representation remains true without any restriction about $T$.

1. Introduction

Let $X$ be a real normed space. The closed unit ball and the unit sphere of $X$ will be denoted, respectively, by $B(X)$ and $S(X)$. Moreover, $E(X)$ will stand for the set of extreme points of $B(X)$ and $\text{co}(E(X))$ for the convex hull of $E(X)$.

If $T$ is a topological space we will denote by $C(T, X)$ the space of continuous and bounded mappings from $T$ into $X$ with its usual uniform norm. To simplify the notation we will frequently write $Y$ instead of $C(T, X)$. Furthermore $S(T, X)$ will be the set of continuous functions from $T$ into $S(X)$. Let us observe that if $X$ is strictly convex, then $S(T, X) = E(Y)$.

Most of the known results about the extremal structure of the unit ball of $C(T, X)$ depend on the existence of continuous functions $v : S(X) \to S(X)$ verifying

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The existence of such functions was proved in [3, Proposition 12] for $X$ an infinite-dimensional Banach space. On the other hand, if $X$ has finite dimension, such a $v$ exists if, and only if, the dimension of $X$ is even.

In Section 2 we consider a more general situation. Namely we study, among other things, when every continuous function $f$ from $T$ into $S(X)$ admits another continuous mapping $e : T \to S(X)$ such that

$$e(t) \neq f(t), \quad e(t) \neq -f(t), \quad \forall t \in T.$$ 

When this occurs, we say that the set $S(T, X)$ is plentiful.

This last property is automatic if there exists a continuous mapping $v$ from $S(X)$ into itself without fixed or antipodal points. We will show that there exists a wide class of pairs $(T, X)$ such that $S(T, X)$ is plentiful but $X$ has odd dimension. We will also prove that, when $X$ is a normed space with infinite dimension, $S(T, X)$ is plentiful for every topological space $T$. As an immediate consequence the existence of continuous mappings $v$ from $S(X)$ into $S(X)$ satisfying $x \neq v(x) \neq -x, \forall x \in S(X)$ is obtained, but now without assuming the hypothesis of completeness.

Section 3 is devoted to the study of the geometry of the unit ball of $C(T, X)$ for $X$ strictly convex and $S(T, X)$ plentiful. First we show that, when $X$ is strictly convex, a topological property ($S(T, X)$ is plentiful) is equivalent to a geometric property (every element in the unit ball of $C(T, X)$, omitting the origin, is a convenient convex combination of two extreme points). This fact makes possible to extend a technique introduced in [4] for $C^*$-algebras to the $C(T, X)$ spaces and so, we can prove that every convex combination of extreme points of $B(Y)$ is a mean of the same number of extreme points.

For each $f$ in $Y$ we define $\alpha(f) = \text{dist}(f, Y^{-1})$ where $Y^{-1}$ denotes the set of the functions in $Y$ which omit the origin.

Theorem 14 shows that every $f$ in $B(Y)$ with $\alpha(f) < 1$ can be expressed as a convex combination of extreme points. In fact, for any $\lambda_1, \ldots, \lambda_n \in [0, 1]$ such that $\lambda_1 + \cdots + \lambda_n = 1$ and $\lambda_k < \frac{1}{2}(1 - \alpha(f))$ for all $k$, there are extreme points $e_1, \ldots, e_n$ in $B(Y)$ such that

$$f = \lambda_1 e_1 + \cdots + \lambda_n e_n.$$ 

The remainder of the section explores the consequences of this theorem. In Corollary 15 we show that each element of the open unit ball of $C(T, X)$ is a mean of $n$ extreme points for some $n \geq 2$. Corollary 17 determines the set of points in $B(Y)$ which are expressible as a convex combination of elements of $E(Y)$. Namely,
Theorem 18 provides various equivalent assertions to the possibility of expressing each point of \( B(Y) \) as a convex combination of extreme points.

As we have already said most of the known results on the extremal structure of the unit ball of \( C(T, X) \) with \( X \) strictly convex (see [12], [3] and [11]) only consider the cases \( \dim X \) even or infinite. In these papers they get to express every point in the unit ball of \( C(T, X) \) as an average of three ([11]) or four ([12], [3]) extreme points by assuming that \( T \) is (at least) a completely regular space and \( \dim T < \dim X \) (where \( \dim T \) denotes the covering dimension of \( T \), see [7] for definitions). Nevertheless in [5] and [10] the general case (\( \dim X \geq 2 \) arbitrary) is studied, but now every element in \( B(Y) \) is expressed as a mean of eight extreme points (with the same condition on the dimensions of \( T \) and \( X \)). Cantwell conjectured that this number can be improved.

Theorem 18 gives an optimal representation of the points in \( B(Y) \) as convex combination (and mean) of three extreme points when \( S(T, X) \) is plentiful. This hypothesis includes the cases \( \dim X \) even or infinite. Moreover, we give examples of pairs \( (T, X) \) with such property, but with \( \dim X \) odd. In fact, we have obtained results on a wide class of \( C(T, X) \) spaces with \( \dim X \) odd (Corollaries 22 and 23). On the other hand, when \( X \) is infinite-dimensional, our results do not require the completeness of \( X \) (in [3], [10] and [11] \( X \) is complete) or the compactness of \( T \) (in [12] \( T \) is compact).

So, it is clear that our new point of view permits to generalize all the known results on the geometry of the unit ball in \( C(T, X) \) spaces with \( X \) strictly convex. However, the aforementioned problem of minimal decompositions remains open when \( S(T, X) \) is nonplentiful.

2. Sufficient conditions for \( S(T, X) \) to be plentiful

Let \( T \) be a topological space and \( X \) a normed space. For every \( f \in S(T, X) \), let us denote

\[
E_f = \{ e \in S(T, X) : f(t) \neq e(t) \neq -f(t), \forall t \in T \}.
\]

Observe that if \( S(T, X) \) is plentiful, then \( E_f \neq \emptyset, \forall f \in S(T, X) \).

It is obvious that \( f \notin E_f \). However, if \( E_f \neq \emptyset \) we have the following result.

**Lemma 1.** Let \( T \) be a topological space, \( X \) a normed space and \( f \in S(T, X) \) such that \( E_f \neq \emptyset \). Then \( f \in \overline{E_f} \).

**Proof.** Given \( \epsilon > 0 \), let us consider \( \lambda \in ]\frac{1}{2}, 1[ \) such that \( \frac{2(1-\lambda)}{2\lambda-1} < \epsilon \) and let \( u \) be in \( E_f \). Define \( v \) on \( T \) by

\[
v(t) = \begin{cases} 
0 & \text{if } f(t) \in [0, u(t)] \\
u(t) & \text{otherwise}
\end{cases}
\]
Clearly $v$ is a continuous function from $T$ into $S(X)$. Now, taking into account that $\|f(t)\| = \|u(t)\| = 1$ for each $t$ in $T$, we have

$$2\lambda - 1 \leq \|\lambda f(t) + (1 - \lambda)u(t)\| \leq 1, \quad \forall t \in T$$

and therefore

$$|\lambda - \|\lambda f(t) + (1 - \lambda)u(t)\|| \leq 1 - \lambda, \quad \forall t \in T.$$ 

Consequently, if $t$ is in $T$, then

$$\|v(t) - f(t)\| = \left\| \frac{\lambda f(t) + (1 - \lambda)u(t)}{\|\lambda f(t) + (1 - \lambda)u(t)\|} - f(t) \right\| =$$

$$\leq \frac{|\lambda - \|\lambda f(t) + (1 - \lambda)u(t)\||f(t) + (1 - \lambda)u(t)|}{\|\lambda f(t) + (1 - \lambda)u(t)\|} \leq$$

$$\leq \frac{2(1 - \lambda)}{\|\lambda f(t) + (1 - \lambda)u(t)\|} \leq \frac{2(1 - \lambda)}{2\lambda - 1}.$$ 

Hence $\|v - f\| \leq \frac{2(1 - \lambda)}{2\lambda - 1} < \epsilon$. Finally, to see that $v \in E_f$, let us assume, to obtain a contradiction, that there is a $t \in T$ such that $v(t) = f(t)$. Then

$$\|\lambda f(t) + (1 - \lambda)u(t)\|f(t) = \lambda f(t) + (1 - \lambda)u(t),$$

that is,

$$(-\lambda + \|\lambda f(t) + (1 - \lambda)u(t)\||f(t) = (1 - \lambda)u(t) \quad \text{(*)}.$$ 

Taking norms it follows that

$$|\lambda f(t) + (1 - \lambda)u(t)\| = 1 - \lambda,$$

which implies that

$$-\lambda + \|\lambda f(t) + (1 - \lambda)u(t)\| = 1 - \lambda$$

or

$$-\lambda + \|\lambda f(t) + (1 - \lambda)u(t)\| = -(1 - \lambda).$$

From (*) we get $f(t) = u(t)$ or $-f(t) = u(t)$ which is impossible since $u \in E_f$. So, $v(t) \neq f(t)$ for every $t$ in $T$.

In the same way it is proved that $v(t) \neq -f(t)$ for each $t$ in $T$. This completes the proof.
Lemma 2. Let \( T \) be a topological space, \( X \) a normed space and \( f \in S(T, X) \) such that \( E_f \neq \emptyset \). Then \( E_g \neq \emptyset \) for every \( g \in S(T, X) \) with \( \|g - f\| < 1 \).

Proof. Let \( u \) be in \( E_f \) and \( g \in S(T, X) \) with \( \|g - f\| < 1 \). By Lemma 1, there is no loss of generality in assuming that

\[
\|u - f\| < 1 - \|g - f\|.
\]

Let \( e : T \to S(X) \) be the function defined by

\[
e(t) = \frac{g(t) + u(t) - f(t)}{\|g(t) + u(t) - f(t)\|}, \quad \forall t \in T.
\]

Note that if \( g(t) + u(t) - f(t) = 0 \) for some \( t \) in \( T \), then \( g(t) - f(t) = u(t) \) and so \( \|g(t) - f(t)\| = 1 \) but this cannot be. Clearly \( e \) is continuous and the proof will be completed if we prove that \( g(t) \neq e(t) \neq -g(t) \) for every \( t \in T \).

For it, let \( t \) be in \( T \) such that \( e(t) = \pm g(t) \).

Taking \( \alpha = \|g(t) + u(t) - f(t)\| \), we have \( \pm \alpha g(t) = g(t) + u(t) - f(t) \). From here, \( (\pm \alpha - 1)g(t) = u(t) - f(t) \) and hence \( |\pm \alpha - 1| \in [0, 1] \). Now, for \( \lambda = \frac{1}{\pm \alpha - 1} \), we obtain

\[
\|f(t) - g(t)\| = \|f(t) + \lambda (u(t) - f(t))\| \geq \|1 - \lambda\| - |\lambda| = 1
\]

and this contradicts our assumption.

It is now clear that the set \( \Omega = \{f \in S(T, X) : E_f \neq \emptyset\} \) is open and closed in \( S(T, X) \). Therefore it is interesting to clarify when \( S(T, X) \) is connected. First an elementary result is given without proof.

Lemma 3. Let \( T \) be a topological space and \( X \) a normed space such that \( \dim X \geq 2 \). The following statements are equivalent:

1. Any two functions \( f, g \) in \( S(T, X) \) are uniformly homotopic, that is, there is a continuous function \( \Phi : [0, 1] \times T \to S(X) \) satisfying
   
   (a) \( \Phi(0, t) = f(t) \), \quad \Phi(1, t) = g(t), \quad \forall t \in T. \)

   (b) For every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that
   
   \[
s, s' \in [0, 1], |s - s'| < \delta \quad \Rightarrow \quad \|\Phi(s, t) - \Phi(s', t)\| < \epsilon, \quad \forall t \in T.
   \]

2. Every function \( f \) in \( S(T, X) \) is uniformly nullhomotopic.

3. \( S(T, X) \) is path-connected.

The following known concept is useful for our aforementioned purpose.

Definition 4. Let \( E \) be a metric space and \( \epsilon > 0 \). \( E \) is said to be \( \epsilon \)-enchained if for any \( f, g \in E \) there is a finite sequence \( f_0, ..., f_n \) in \( E \) with \( f_0 = f \) and \( f_n = g \) such that \( d(f_k, f_{k+1}) < \epsilon \) for all \( k \in \{0, ..., n - 1\} \). We will say that \( E \) is enchained if \( E \) is \( \epsilon \)-enchained for every \( \epsilon > 0 \).
Theorem 5. Let $T$ and $X$ be as in 3. The following six properties are equivalent:

1. Any two functions $f, g \in S(T, X)$ are uniformly homotopic.
2. Every function $f$ in $S(T, X)$ is uniformly nullhomotopic.
3. $S(T, X)$ is path-connected.
4. $S(T, X)$ is connected.
5. $S(T, X)$ is enchained.
6. $S(T, X)$ is $2$-enchained.

Moreover, any of the above assertions implies that $S(T, X)$ is plentiful.

Proof. 1 $\iff$ 2 $\iff$ 3 is the above lemma and 3 $\implies$ 4 $\implies$ 5 $\implies$ 6 hold in every metric space. To prove 6 $\implies$ 3 let $f, g$ be in $S(T, X)$. By hypothesis, there exists a finite sequence $f_0, ..., f_n$ in $S(T, X)$ with $f_0 = f$ and $f_n = g$ such that $\|f_k - f_{k+1}\| < 2$, $\forall k \in \{0, ..., n-1\}$. Then

$$f_k(t) \neq -f_{k+1}(t), \quad \forall t \in T, \quad \forall k \in \{0, ..., n-1\}.$$ 

Let us define $\gamma : [0, 1] \to S(T, X)$ by

$$\gamma(s)(t) = \frac{(ns - k)f_{k+1}(t) + (1 + k - ns)f_k(t)}{\|(ns - k)f_{k+1}(t) + (1 + k - ns)f_k(t)\|}, \quad \forall t \in T,$$

$$\forall s \in \left[\frac{k}{n}, \frac{k+1}{n}\right], \quad \forall k \in \{0, ..., n-1\}.$$ 

$\gamma$ is a path in $S(T, X)$ running from $f$ to $g$ and so we have 3.

Finally, since $\dim X \geq 2$, $\Omega := \{f \in S(T, X) : E_f \neq \emptyset\}$ is nonempty (it contains the constant mappings), and by Lemma 2, $\Omega$ is open and closed in $S(T, X)$. If one of the above conditions holds, then $S(T, X)$ is connected. Therefore, $\Omega = S(T, X)$ and so $S(T, X)$ is plentiful.

$S(T, X)$ may be plentiful and not path-connected. For example, if we take $T = S(\mathbb{R}^{2n})$ and $X = \mathbb{R}^{2n}$, then $S(T, X)$ is plentiful (there exists a continuous mapping $v$ from $S(X)$ into itself without fixed or antipodal points) and, however, it is not path-connected by [6, Chap. XVII, Corollary 2.2].

The next results show that there is an extensive range of pairs $(T, X)$ such that $S(T, X)$ is plentiful.

Proposition 6. Let $T$ be a compact topological space and $X$ a normed space with $\dim X \geq 2$. Assume that one of the following properties holds:

1. $T$ is contractible.
2. Every $f \in S(T, X)$ is nonsurjective.

Then $S(T, X)$ is plentiful.
Proof. Let \( f \) be in \( STX \). If \( T \) is contractible there exist \( t_0 \) in \( T \) and a continuous mapping \( \varphi : [0, 1] \times T \to T \) such that
\[
\varphi(0, t) = t, \quad \varphi(1, t) = t_0, \quad \forall t \in T.
\]
In this case we can consider \( x_0 = f(t_0) \) and \( \Phi = f \circ \varphi \). On the other hand, if \( 2 \) holds then there is \( x_0 \in S(X) \setminus f(T) \) and now \( \Phi \) is defined by
\[
\Phi(s, t) = \frac{(1 - s)f(t) + sx_0}{\| (1 - s)f(t) + sx_0 \|}, \quad \forall (s, t) \in [0, 1] \times T \quad (x_0 = -z_0).
\]
In both cases \( \Phi : [0, 1] \times T \to S(X) \) is continuous and satisfies that
\[
\Phi(0, t) = f(t), \quad \Phi(1, t) = x_0, \quad \forall t \in T.
\]
By the compactness of \([0, 1] \times T\), \( \Phi \) is an uniform homotopy and so \( f \) is uniformly nullhomotopic. By the previous theorem, \( STX \) is plentiful.

If \( X \) is an infinite-dimensional normed space, Y. Benyamini and Y. Sternfeld proved in [2] that the unit sphere of \( X \) is Lipschitz contractible. This permits us to obtain the following result.

**Proposition 7.** Let \( X \) be a normed space with infinite dimension. Then \( STX \) is plentiful for any topological space \( T \).

**Proof.** By [2], there are \( x_0 \) in \( S(X) \) and a Lipschitz function \( \Gamma \) from \([0, 1] \times S(X) \) into \( S(X) \) satisfying \( \Gamma(0, x) = x, \Gamma(1, x) = x_0, \forall x \in S(X) \).

Hence, given \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that
\[
(s, x), (s', x') \in [0, 1] \times S(X), \quad |s - s'| + \|x - x'\| < \delta \Rightarrow \\
\Rightarrow \| \Gamma(s, x) - \Gamma(s', x') \| < \varepsilon.
\]

Let \( T \) be an arbitrary topological space. Given \( f : T \to S(X) \) continuous, consider \( \Phi : [0, 1] \times T \to S(X) \) defined by
\[
\Phi(s, t) = \Gamma(s, f(t)), \quad \forall (s, t) \in [0, 1] \times T.
\]
Evidently \( \Phi \) is continuous and satisfies:
1. \( \Phi(0, t) = \Gamma(0, f(t)) = f(t), \quad \Phi(1, t) = \Gamma(1, f(t)) = x_0, \quad \forall t \in T. \)
2. Given \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that
\[
s, s' \in [0, 1], \quad |s - s'| < \delta \Rightarrow \| \Phi(s, t) - \Phi(s', t) \| < \varepsilon, \quad \forall t \in T.
\]
So, every function \( f \in STX \) is uniformly nullhomotopic and \( STX \) is plentiful by Theorem 5.

Our above result permits to prove the following fact which is known for infinite-dimensional Banach spaces [3, Proposition 12] (now the completeness of \( X \) is not required).
Proposition 8. Let $X$ be an infinite-dimensional normed space. Then there is a continuous mapping $v : S(X) \rightarrow S(X)$ such that

$$v(x) \neq x, \quad v(x) \neq -x, \quad \forall x \in S(X).$$

Proof. It is sufficient to take $T = S(X)$, $f$ the identity function onto $S(X)$ and to apply the preceding proposition.

Let $T$ be a topological space and $X$ a normed space with $\dim X \geq 2$. If $X$ is infinite-dimensional $S(A, X)$ is plentiful for any subset $A$ of $T$ by Proposition 7. If $X$ has finite dimension we have the following easy lemma.

Lemma 9. Let $T$ be a compact Hausdorff topological space, let $X$ be a finite-dimensional normed space with $\dim X \geq 2$ and assume that $S(T, X)$ is plentiful and $\dim T < \dim X$. Then $S(A, X)$ is plentiful for any closed subset $A$ of $T$.

We now need the following topological concept.

For any topological space $T$, the cone $CT$ over $T$ is the quotient space $T \times I/R$, where $I = [0, 1]$ and $R$ is the equivalence relation defined on $T \times I$ by

$$(t, s)R(t', s') \iff (t, s) = (t', s') \quad \text{or} \quad s = s' = 1.$$ 

Intuitively, $CT$ is obtained from $T \times I$ by pinching $T \times 1$ to a single point. The elements of $CT$ are denoted by $(t, s)$. It is trivial to verify that the map $t \mapsto (t, 0)$ is a homeomorphism, so we can identify $T$ with the subspace $\{(t, 0) : t \in T\}$ in $CT$. Also it is easy to check that if $T$ is compact Hausdorff, then $CT$ is it too. Moreover, $CT$ is always contractible and it is known that if the covering dimension of $T$ is finite, then $\dim CT = \dim T + 1$.

Proposition 10. Let $T$ be a completely regular topological space and $X$ a finite-dimensional normed space with $\dim X \geq 2$. Each assertion implies the following one:

1. $\dim T < \dim X - 1$.

2. $S(\beta(T), X)$ is plentiful where $\beta(T)$ is the Stone-Cech compactification of $T$.

3. $S(T, X)$ is plentiful.

Proof. 1 $\Rightarrow$ 2 : Let $T$ and $X$ satisfy 1. By the above remark, $C\beta(T)$ is compact (Hausdorff) and contractible. By Proposition 6, $S(C\beta(T), X)$ is plentiful. Since $\dim C\beta(T) < \dim X$ (dim $\beta(T) = \dim T$ by [7, Theorem 7.1.17]) and $\beta(T)$ is closed in $C\beta(T)$, $S(\beta(T), X)$ is plentiful by the above lemma.

2 $\Rightarrow$ 3 : Let $f$ be in $S(T, X)$. Since $S(X)$ is compact there exists a unique
continuous mapping $F : \beta(T) \to S(X)$ such that $F(t) = f(t)$, $\forall t \in T$. If 2 holds, there is a $\overline{\sigma}$ in $E_F$. Then it is clear that the restriction of $\overline{\sigma}$ to $T$ belongs to $E_f$. Hence $S(T, X)$ is plentiful.

3. The main results

Let $Y$ be a normed space. In [1] Aron and Lohman introduced the $\lambda$-function on elements $f$ of $B(Y)$ to be the supremum, $\lambda(f)$, of numbers $\lambda$ in $[0, 1]$, for which there is a pair $(e, g)$ in $E(Y) \times B(Y)$, such that

$$f = \lambda e + (1 - \lambda)g.$$ 

The space $Y$ is said to have the $\lambda$-property if $\lambda(y) > 0$ for all $y$ in $B(Y)$, and $Y$ has the uniform $\lambda$-property if $Y$ verifies the $\lambda$-property and, in addition, satisfies

$$\inf\{\lambda(y) : y \in B(Y)\} > 0.$$ 

A complete study of the $\lambda$-property in functions spaces $C(T, X)$ with $T$ a topological space and $X$ a strictly convex normed space was carried out in [8]. Among other things, they got a general expression of the $\lambda$-function in these spaces. Namely,

$$\lambda(f) = \frac{1}{2} (1 + m(f) - \alpha(f)), \quad \forall f \in B(Y)$$

where $m(f) = \inf\{\|f(t)\| : t \in T\}$ and $\alpha(f) = \text{dist}(f, Y^{-1})$.

Let $T$ be a topological space and $X$ a normed space. In this section we assume $Y$ denotes the space $C(T, X)$. Moreover, we suppose, unless otherwise stated, that $X$ is strictly convex and $S(T, X)$ is plentiful. First we show that this property on $S(T, X)$ is equivalent to the fact that every function in $Y^{-1} \cap B(Y)$ is a mean of two extreme points of $B(Y)$.

The proof of the ”if” half of our next result is similar to the proof of Theorem 4 in [11].

However, for the sake of completeness, we include it.

**Proposition 11.** Let $T$ be a topological space and $X$ a strictly convex normed space. The following conditions are equivalent:

1. $S(T, X)$ is plentiful.
2. For every continuous function $h$ from $T$ into $B(X)$ which omits the origin and, for any $\lambda$ in $[\frac{1}{2}, \lambda(h)]$, there are extreme points $e_1$ and $e_2$ of $B(Y)$ such that

$$h = \lambda e_1 + (1 - \lambda)e_2.$$ 

**Proof.** 1 $\Rightarrow$ 2 : Let $h$ and $\lambda$ satisfy the hypotheses of 2. Then it is obvious that $m(h) \geq 2\lambda - 1$ and therefore $\|h(t)\| \geq 2\lambda - 1 = |2\lambda - 1|$, $\forall t \in T$. 


If \( \lambda = 1 \), then \( h \in E(Y) \) and we can take \( e_1 = e_2 = h \).

Let us suppose \( \lambda < 1 \). Let \( f \) be in \( S(T, X) \) defined by \( f(t) = \frac{h(t)}{\|h(t)\|} \) for every \( t \) in \( T \). By 1, there is an element \( e \) in \( E_f \).

Let us define \( g : [0, 2] \times T \to X \) by

\[
g(s, t) = \begin{cases} 
(1 - s)f(t) + se(t) & \text{if } 0 \leq s \leq 1 \\
2 - s)e(t) - (s - 1)f(t) & \text{if } 1 \leq s \leq 2
\end{cases}
\]

Then \( g \) is continuous and \( g(s, t) \neq 0 \), \( \forall (s, t) \in [0, 2] \times T \). We define \( \Gamma \) on \([0, 2] \times T\) in the following way

\[
\Gamma(s, t) = \frac{g(s, t)}{\|g(s, t)\|}, \quad \forall (s, t) \in [0, 2] \times T.
\]

Evidently \( \Gamma \) is continuous and if we fix \( t \) in \( T \), it follows that

\[
\left\| \frac{h(t)}{1 - \lambda} - \frac{\lambda}{1 - \lambda} \Gamma(0, t) \right\| = \frac{\|h(t) - \lambda h(t)\| h(t)\|}{1 - \lambda} = \frac{\|h(t)\| - \lambda}{1 - \lambda} \leq 1
\]

and

\[
\left\| \frac{h(t)}{1 - \lambda} - \frac{\lambda}{1 - \lambda} \Gamma(2, t) \right\| = \frac{\|h(t) + \lambda h(t)\| h(t)\|}{1 - \lambda} = \frac{\|h(t)\| + \lambda}{1 - \lambda} \geq 1
\]

so there is some \( s \) in \([0, 2]\) such that

\[
\left\| \frac{h(t)}{1 - \lambda} - \frac{\lambda}{1 - \lambda} \Gamma(s, t) \right\| = 1, \quad (*)
\]

that is,

\[
\left\| \frac{h(t)}{\lambda} - \Gamma(s, t) \right\| = \frac{1 - \lambda}{\lambda}.
\]

Now, by [11, Lemma 1], there is only one \( s \) for which the above equality (*) holds; if we denote it by \( s(t) \), we now claim that the mapping \( t \to s(t) \) from \( T \) into \([0, 2]\) is continuous. If not, there is a point \( t \in T \) and a net \( \{t_n\} \) converging to \( t \) such that \( \{s(t_n)\} \to s \neq s(t) \). Using the continuity of \( \Gamma \) we find that

\[
\left\| \frac{h(t_n) - \lambda \Gamma(s(t_n), t_n)}{1 - \lambda} \right\| \to \left\| \frac{h(t) - \lambda \Gamma(s(t), t)}{1 - \lambda} \right\|.
\]

So \( \left\| \frac{h(t) - \lambda \Gamma(s(t), t)}{1 - \lambda} \right\| = 1 \), this contradicts the uniqueness of \( s(t) \) and the continuity of \( t \to s(t) \) is established.

It is now clear how \( e_1 \) and \( e_2 \) are to be defined on \( T \)

\[
e_1(t) = \Gamma(s(t), t), \quad e_2(t) = \frac{h(t) - \lambda \Gamma(s(t), t)}{1 - \lambda}, \quad \forall t \in T.
\]

This completes the proof of the implication \( 1 \Rightarrow 2 \).
2 ⇒ 1: Let $f$ be in $S(T, X)$. Clearly the function $h = \frac{1}{2}f$ is an element in $B(Y) \cap Y^{-1}$. Applying 2, for $\lambda = \frac{1}{2}$ there exist $e_1$ and $e_2$ in $E(Y)$ such that $h = \frac{1}{2}(e_1 + e_2)$. Let us take $e = e_1$ or $e = e_2$. An easy verification shows that $e$ is in $E_f$.

Our next result is a generalization of Proposition 5 in [11] and can be proved similarly.

**Proposition 12.** Let be $u \in E(Y)$, $g \in B(Y)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}^+$ such that $
abla \alpha > \beta$, $\alpha + \beta = \gamma + \delta$ and $\gamma, \delta \in [\beta, \alpha]$. Then there exist $e_1, e_2 \in E(Y)$ verifying that

$$\alpha u + \beta g = \gamma e_1 + \delta e_2.$$  

In [9] Kadison and Pedersen proved, by using a very laborious method, that every convex combination of extreme points of the unit ball of a $C^*$-algebra can be expressed as a mean of the same number of extreme points.

The above proposition permits us to obtain this same conclusion in any $C(T, X) \ast$ space such that $X$ is strictly convex and $S(T, X)$ is plentiful.

**Corollary 13.** Each convex combination of extreme points of $B(Y)$ is a mean of the same number of extreme points.

**Proof.** The proof is by induction on $n$. If $n = 2$, let $\alpha e_1 + \beta e_2$ be a convex combination with $e_1, e_2 \in E(Y)$. If $\alpha = \beta = \frac{1}{2}$, then we have the desired conclusion. In other case, we can suppose, without loss of generality, that $\alpha > \beta$. Let be $\gamma = \delta = \frac{1}{2}$. By the above result, there are $u_1, u_2 \in E(Y)$ such that

$$\alpha e_1 + \beta e_2 = \frac{1}{2}(u_1 + u_2).$$

Assume that the property holds for $n$, we will prove it for $n + 1$. Let us consider

$$f = \lambda_1 e_1 + \cdots + \lambda_n e_n \quad (\ast)$$

with $\lambda_1, \ldots, \lambda_n, \lambda_{n+1}$ in $[0, 1]$, $\lambda_1 + \cdots + \lambda_{n+1} = 1$ and $e_1, \ldots, e_{n+1} \in E(Y)$. First let us suppose that some of the $\lambda_i$ is $\frac{1}{n+1}$. For example, $\lambda_{n+1} = \frac{1}{n+1}$. Then

$$f = (1 - \lambda_{n+1})\left(\frac{\lambda_1}{1 - \lambda_{n+1}} e_1 + \cdots + \frac{\lambda_n}{1 - \lambda_{n+1}} e_n \right) + \lambda_{n+1} e_{n+1}.$$  

By the hypotheses of induction, we have

$$\frac{\lambda_1}{1 - \lambda_{n+1}} e_1 + \cdots + \frac{\lambda_n}{1 - \lambda_{n+1}} e_n = \frac{1}{n} \left(u_1 + \cdots + u_n\right)$$

for some $u_1, \ldots, u_n \in E(Y)$.

It follows that
Repeating the argument we find Proposition 12.

Now we are ready to prove our main result in this section.

By the above proposition, there are $e_n, u_{n+1} \in E(Y)$ such that

\[
\lambda_n e_n + \lambda_{n+1} u_{n+1} = \lambda_n' u_n + \lambda_{n+1}' u_{n+1}
\]

By substituting in (*) we have

\[
f = \lambda_1 e_1 + \cdots + \lambda_{n-1} e_{n-1} + \lambda_n' u_n + \lambda_{n+1}' u_{n+1}
\]

and since $\lambda_{n+1}' = \frac{1}{n+1}$ we can apply the previous argument and the proof is complete.

Let us observe that the Corollary 13 provides, in particular, the aforementioned result by Kadison and Pedersen for commutative $C^*$-algebras.

Now we are ready to prove our main result in this section.

**Theorem 14.** For every $f \in B(Y)$ with $\alpha(f) < 1$ and any $\lambda_1, \ldots, \lambda_n < 0, 1$ such that $\lambda_1 + \cdots + \lambda_n = 1$ and $\lambda_k < \frac{1}{2} (1 - \alpha(f))$ for all $k$, there are extreme points $e_1, \ldots, e_n$ in $B(Y)$ such that

\[
f = \lambda_1 e_1 + \cdots + \lambda_n e_n.
\]

**Proof.** Of course we can suppose that $\lambda_1 \geq \lambda_k$ for every $k$ in $\{1, \ldots, n\}$. Let be $\lambda_1' = \lambda_1 + \epsilon$ and $\lambda_2' = \lambda_2 - \epsilon$ with $\epsilon > 0$ sufficiently small, so that $0 < \lambda_2'$ and $\lambda_1' < \frac{1}{2} (1 - \alpha(f))$. Evidently $\lambda_1' < \lambda(f)$ ([8]). By [1, Proposition 1.2.c)], there are $e \in E(Y)$ and $g \in B(Y)$ such that

\[
f = \lambda_1' e + (1 - \lambda_1') g = \lambda_1' e + (\lambda_2' + \lambda_3 + \cdots + \lambda_n) g.
\]

Since $\lambda_1' > \lambda_2'$, we have $\lambda_1' e + \lambda_2' g = \lambda_1' u_2 + \lambda_2' e_2'$ for some $u_2, e_2' \in E(Y)$ by Proposition 12.

Repeating the argument we find $u_3, e_3$ in $E(Y)$ such that

\[
\lambda_1' u_2 + \lambda_3 g = \lambda_1' u_3 + \lambda_3 e_3
\]

and after $n - 1$ steps we have found extreme points $u_n, e_n'$, $e_n$, in $B(Y)$ such that
Now use Proposition 12 on the element $\lambda'_1 u_n + \lambda'_2 e_2'$ to obtain extreme points $e_1, e_2$ in $E(Y)$ such that

$$
\lambda'_1 u_n + \lambda'_2 e_2' = (\lambda_1 + \epsilon) u_n + (\lambda_2 - \epsilon) e_2' = \lambda_1 e_1 + \lambda_2 e_2.
$$

Inserting this in the above decomposition we have the desired expression.

The above theorem was proved in [4, Theorem 3.3] in case $Y$ is a $C^*$-algebra.

**Corollary 15.** If $f$ is an element of $Y$ such that $\|f\| < 1$, then there are $n$ extreme points $e_1, \ldots, e_n$ in $B(Y)$ such that $f = \frac{1}{n}(e_1 + \cdots + e_n)$ for some integer $n$ greater than \( \frac{2}{1 - \|f\|} \). So, $B(Y)$ is the closed convex hull of $E(Y)$ and $Y$ is the linear expansion of $E(Y)$.

Let us observe that if one considers $X = C$ in the above corollary, then we obtain the result by R.R. Phelps [13, Th. 1].

Let $f$ be in $B(Y)$. Let $u(f)$ denote the least integer $n$ such that $f$ is a convex combination of $n$ extreme points in $B(Y)$, $u(f)$ will be called by extremal rank of $f$. Set $u(f) = \infty$ if $f$ is not expressible as such a convex combination.

In the next result, we relate the extremal rank, $u(f)$, of a element $f$ in the unit ball of $Y$ to the distance, $\|f\|$, from $f$ to the set $Y^{-1}$.

**Corollary 16.** For each $f$ in $B(Y)$ and $n \geq 2$, $u(f) \leq n$ implies $\alpha(f) \leq 1 - \frac{2}{n}$ and $\alpha(f) < 1 - \frac{2}{n}$ implies $u(f) \leq n$.

**Proof.** Suppose $u(f) \leq n$ with $n \geq 2$. There exist $\lambda_1, \ldots, \lambda_n$ in $[0,1]$ such that $\lambda_1 + \cdots + \lambda_n = 1$ and $e_1, \ldots, e_n$ in $E(Y)$ such that $f = \lambda_1 e_1 + \cdots + \lambda_n e_n$. If $\alpha(f) = 0$ evidently $\alpha(f) \leq 1 - \frac{2}{n}$.

If $\alpha(f) > 0$, then $f \notin Y^{-1}$ and, by applying [8], we have that

$$
\alpha(f) = \frac{2}{1 - \alpha(f)}
$$

and therefore $\alpha(f) \leq 1 - \frac{2}{n}$.

Conversely if $\alpha(f) < 1 - \frac{2}{n}$, then $\frac{1}{n} < \frac{1}{2} (1 - \alpha(f))$. By Theorem 14, taking $\lambda_k = \frac{1}{n}$ for $k = 1, \ldots, n$, we see that $f$ is a mean of $n$ elements of $E(Y)$. Thus $u(f) \leq n$.

In Corollary 15 we proved that every point in the open unit ball of $Y$ belongs to $\text{co}(E(Y))$. Now, we see which points in $S(Y)$ are not expressible as a convex combination in $E(Y)$. 

\[ f = \lambda'_1 u_n + \lambda'_2 e_2' + \lambda_3 e_3 + \cdots + \lambda_n e_n. \]
Corollary 17. \( B(Y) \backslash \text{co}(E(Y)) = \{ f \in B(Y) : \alpha(f) = 1 \} \)

Proof. If \( f \in B(Y) \backslash \text{co}(E(Y)), \) we have that \( \alpha(f) = 1 \) by Theorem 14. Conversely, if \( f \in \text{co}(E(Y)) \) then \( u(f) \leq n \) for some \( n, \) and, by Corollary 16, it follows that \( \alpha(f) < 1. \)

In the following theorem we collect all the information about the extremal structure of the unit ball of \( C(T, X) \) in case \( X \) is strictly convex and \( S(T, X) \) is plentiful.

Theorem 18. The following conditions are equivalent:

1. For every \( f \) in \( B(Y) \) and for any \( \lambda_1, ..., \lambda_n \in [0, \frac{1}{2}] \) with \( \lambda_1 + \cdots + \lambda_n = 1, \) there are extreme points \( e_1, ..., e_n \) in \( B(Y) \) such that
   \[ f = \lambda_1 e_1 + \cdots + \lambda_n e_n. \]
2. \( B(Y) = E(Y) + \frac{n}{n} E(Y) \) for every \( n \geq 3. \)
3. \( B(Y) = \text{co}(E(Y)). \)
4. \( \lambda(f) = \frac{1}{2}(1 + m(f)) \) for every \( f \) in \( B(Y). \)
5. \( Y \) has the uniform \( \lambda \)-property.
6. \( Y \) has the \( \lambda \)-property.
7. \( \alpha(f) < 1 \) for every \( f \) in \( B(Y). \)
8. \( Y^{-1} \) is dense in \( Y. \)
9. \( (T, X) \) has the extension property.

Moreover if we suppose that \( T \) is completely regular and \( X \) is finite-dimensional with \( \text{dim} X \geq 2, \) then the conditions 1 – 9 are equivalent to
10. \( \text{dim} T < \text{dim} X. \)

The equivalence between the conditions 4 to 10 was established in [8]. On the other hand \( 1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 6 \) is obvious and \( 7 \Rightarrow 1 \) follows from Theorem 14.

The equivalence between 3, 5, 6, 9 and 10 was proved in [10, Corollaries 7 and 9] without assuming that \( S(T, X) \) is plentiful, but in this more general case, 2 was only obtained for \( n = 8. \) On the other hand, in [11] the equivalence between the conditions 1, 2, 3 and 10 was proved when \( X \) is a Banach space and \( \text{dim} X \) is an even integer or infinite.

Let us suppose that \( Y^{-1} \neq Y. \) Then, \( B(Y) \neq \text{co}(E(Y)) \) so that \( u(f) = \infty \) for some \( f \) in \( B(Y). \) Moreover, for \( g \) in \( B(Y) \) with \( \alpha(g) = 1 \) and \( n \geq 3, \) set
\[ f = \beta g \text{ where } 1 - \frac{2}{n-1} < \beta < 1 - \frac{3}{n}. \] Then \( f \) is in \( \text{co}(E(Y)) \) and \( \alpha(f) = \beta, \) so that \( u(f) = n \) by Corollary 16. Clearly \( u(0) = 2 \) and \( u(e) = 1 \) for every \( e \) in \( E(Y); \) so this establishes that
\[ \{ u(f) : f \in B(Y) \} = \mathbb{N} \cup \{ \infty \}. \]

Conversely, if \( Y^{-1} = Y \) then \( u(f) \leq 3 \) for every \( f \) in \( B(Y) \) by Theorem 18.
Since \( u(f) = 1 \) only for extreme points \( f \) in \( B(Y) \), \( \max\{u(f) : f \in B(Y)\} \) is 2 or 3. In [14], for \( C(T, C) \) it was proved that \( u(f) \leq 2, \forall f \in B(Y) \) if, and only if, \( T \) is an \( F \)-space and \( \dim T \leq 1 \). So we have

**Corollary 19.** \( \max\{u(f) : f \in B(Y)\} (= \max\{u(f) : f \in \co(E(Y))\}) \), is 2, 3 or \( \infty \).

Taking into account that, when \( X \) is an infinite-dimensional normed space, \( (T, X) \) has the extension property and \( S(T, X) \) is plentiful for every topological space \( T \), we obtain

**Corollary 20.** Let \( T \) be a topological space and \( X \) an infinite-dimensional strictly convex normed space. Then

\[
B(Y) = \lambda_1 E(Y) + \cdots + \lambda_n E(Y)
\]

for every natural \( n \geq 3 \) and \( \lambda_1, \ldots, \lambda_n \in ]0, \frac{1}{2} [ \) with \( \lambda_1 + \cdots + \lambda_n = 1 \).

Our corollary allows us to get the following interesting result.

**Corollary 21.** Let \( X \) be as in Corollary 20. Then, for each \( x \) in \( B(X) \) and every \( n \geq 3 \), there exist \( e_1, \ldots, e_n \) retractions of the unit ball of \( X \) onto the unit sphere of \( X \) such that

\[
x = \frac{1}{n} (e_1(x) + \cdots + e_n(x)).
\]

Corollaries 20 and 21 were obtained in [11] in case \( X \) is complete. Moreover Proposition 9 in [11] states that it is not possible to improve on the number three in the last corollary.

If \( T \) is contractible and compact, \( S(T, X) \) is plentiful for every normed space \( X \) with \( \dim X \geq 2 \) by Proposition 6. So, when \( X \) is finite-dimensional, we have the following result.

**Corollary 22.** Let \( T \) be a contractible and compact topological space and \( X \) a finite-dimensional strictly convex normed space with \( \dim X \geq 2 \). Suppose that \( \dim T < \dim X \). Then

\[
B(Y) = \lambda_1 E(Y) + \cdots + \lambda_n E(Y)
\]

for every natural \( n \geq 3 \) and \( \lambda_1, \ldots, \lambda_n \in ]0, \frac{1}{2} [ \) with \( \lambda_1 + \cdots + \lambda_n = 1 \).

In [11] the same conclusion is obtained by assuming \( T \) completely regular and \( X \) strictly convex with even dimension. On the other hand, when \( X \) has odd dimension it is known (see [10, Corollary 7]) that every element in \( B(Y) \) can be expressed as an average of eight extreme points if, and only if, \( \dim T < \dim X \).

By using Proposition 10 we improve this result in a particular case.
**Corollary 23.** Let $T$ be a completely regular space and $X$ a strictly convex normed space with $\dim X \geq 2$ (odd or even). If $\dim T < \dim X - 1$, then

$$B(Y) = \lambda_1E(Y) + \cdots + \lambda_nE(Y)$$

for every natural $n \geq 3$ and $\lambda_1, \ldots, \lambda_n \in ]0, \frac{1}{2}]$ with $\lambda_1 + \cdots + \lambda_n = 1$.

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