HARMONIC MEASURE AND HYPERBOLIC DISTANCE IN JOHN DISKS

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1. Introduction.

Suppose that \( D \) is a domain in the complex plane \( \mathbb{C} \). Let \( D^* = \mathbb{C} \setminus \bar{D} \) be the exterior of \( D \) in \( \mathbb{C} \) and let \( B(z, r) = \{ \zeta : |\zeta - z| < r \} \) for \( z \in \mathbb{C} \) and \( r > 0 \).

In this paper, we find several characterizations of John disks which have analogues in the class of quasidisks. John disks can be thought of as “one-sided quasidisks”. For example, a Jordan domain \( D \subset \mathbb{C} \) is a quasidisk if and only if \( D \) and \( D^* \) are John disks. Also, every quasidisk is a John disk [GM3]. The results presented here are likewise one-sided versions of characterizations of quasidisks. These characterizations involve the conformal invariants harmonic measure and hyperbolic distance.

A simply-connected bounded domain \( D \subset \mathbb{C} \) is said to be a \( c \)-John disk if there exist a point \( z_0 \in D \) and a constant \( c \geq 1 \) such that each point \( z_1 \in D \) can be joined to \( z_0 \) by an arc \( \gamma \) in \( D \) satisfying

\[
\ell(\gamma(z_1, z)) \leq c \cdot (z, \partial D)
\]

for each \( z \in \gamma \), where \( \ell(\gamma(z_1, z)) \) is the euclidean length of the subarc of \( \gamma \) with endpoints \( z_1, z \). We call \( z_0 \) a John center, \( c \) a John constant and \( \gamma \) a \( c \)-John arc. We say that \( D \) is John if it is \( c \)-John disk for some \( c \).

A bounded domain \( D \subset \mathbb{C} \) is John if and only if each pair of points \( z_1, z_2 \in D \) can be joined by an arc \( \gamma \) which satisfies

\[
\min_{j=1,2} \ell(\gamma(z_j, z)) \leq c \cdot (z, \partial D)
\]

for all \( z \in \gamma \). We call \( \gamma \) a double \( c \)-cone arc. This definition can be used to define the unbounded John disks \( D \subset \mathbb{C} \) as well [NV, 2.26].

A domain \( D \subset \mathbb{C} \) is said to be \( c \)-uniform if there is a constant \( c \geq 1 \) such

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that each pair of points $z_1, z_2 \in D$ can be joined by an arc $\gamma \subset D$ which satisfies (1.1) and
\[
\ell(\gamma) \leq c |z_1 - z_2|.
\]
We say that $D$ is \textit{uniform} if it is $c$-uniform for some $c \geq 1$.

We say that a domain $D \subset \mathbb{C}$ is a $K$-\textit{quasidisk}, $1 \leq K < \infty$, if it is the image of the unit disk $B$ under a $K$-quasiconformal self mapping of $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$. A Jordan domain $D \subset \mathbb{C}$ is uniform if and only if it is a quasidisk [MS].

In section 2, we show that a bounded Jordan domain $D \subset \mathbb{C}$ satisfies a harmonic doubling condition if and only if $D$ is a John disk. This is a one-sided analogue of a characterization for quasidisks due to Jerison and Kenig [JK]. It is also a one-sided version of a characterization for quasidisks due to Krzyż who compares the harmonic measures of adjacent arcs on the boundary when considered from inside and outside the domain [Kr].

In section 3, we characterize John disks $D$ in terms of various properties of the hyperbolic geodesics in $D$; in particular, the position of the euclidean midpoint of the geodesic or the quasextremal distance property in $D$ with respect to the geodesic. The first of these leads to a third characterization in terms of the Hölder continuity of analytic functions in $D$ similar to a well-known theorem of Hardy and Littlewood [HL]. Finally, we characterize unbounded Jordan John disks in terms of the hyperbolic geodesics in their exteriors.

In section 4, we characterize John disks in terms of a euclidean estimate for the hyperbolic distance between points of $D$. This is again a one-sided analogue of a theorem due to Gehring and Osgood [GO], who showed that a domain $D$ is uniform if and only if it satisfies
\[
k_D(z_1, z_2) \leq c j_D(z_1, z_2) + d
\]
for all $z_1, z_2 \in D$ and some constants $c$ and $d$, where $k_D$ is the quasihyperbolic metric in $D$ and
\[
j_D(z_1, z_2) = \frac{1}{2} \log \left( \frac{|z_1 - z_2|}{\operatorname{dist}(z_1, \partial D)} + 1 \right) \left( \frac{|z_1 - z_2|}{\operatorname{dist}(z_2, \partial D)} + 1 \right).
\]
Our result replaces the euclidean distance $|z_1 - z_2|$ with the inner distance between these points and yields an analogous estimate for $h_D(z_1, z_2)$, the hyperbolic distance between $z_1$ and $z_2$.

We will repeatedly use a result of Gehring and Hayman.

\textbf{Lemma 1.3} [GH, Theorem 2], [Ja]. \textit{Suppose that $D$ is a simply connected domain in $\mathbb{C}$. If $\gamma$ is a hyperbolic geodesic in $D$ and if $\alpha$ is any curve which joins the endpoints of $\gamma$ in $D$, then}
where $k$ is an absolute constant, $4.5 \leq k \leq 17.5$.


A bounded Jordan domain $D \subset \mathbb{C}$ is said to satisfy a *harmonic doubling condition* if for some $z_0 \in D$ and some constant $c_0 > 0$,

\begin{equation}
\omega(z_0, \alpha; D) \leq c_0 \omega(z_0, \beta; D)
\end{equation}

for each pair of consecutive arcs $\alpha, \beta$ on $\partial D$ with $\text{dia}(\alpha) \leq 2 \text{dia}(\beta)$, where $\omega(z_0, \gamma; D)$ is the harmonic measure of $\gamma$ at the point $z_0$ with respect to $D$.

**Remark 2.2.** If $D$ satisfies (2.1) for some $z_0 \in D$, then it satisfies (2.1) for every $z_1 \in D$ with a constant $c_1$ which depends on $c_0$, $z_0$ and $z_1$.

**Proof.** Fix $z_1 \in D$ and fix consecutive arcs $\alpha, \beta \subset \partial D$ with $\text{dia}(\alpha) \leq 2\text{dia}(\beta)$. Since $\omega$ is nonnegative and harmonic,

\[
\frac{\omega(z_1, \alpha; D)}{\omega(z_0, \alpha; D)} \leq k \quad \text{and} \quad \frac{\omega(z_0, \beta; D)}{\omega(z_1, \beta; D)} \leq k,
\]

where $k = e^{b(z_0, z_1)}$. (See, for example, [H, Theorem 6].) Thus by hypothesis we have

\[
\frac{\omega(z_1, \alpha; D)}{\omega(z_0, \beta; D)} \leq \frac{\omega(z_0, \alpha; D)}{\omega(z_0, \beta; D)} k^2 \leq c_0 k^2 = c_1
\]

and hence (2.1) holds for every $z_1 \in D$ with $c_1 = c_0 (e^{b(z_0, z_1)})^2$.

**Theorem 2.3.** A bounded Jordan domain $D \subset \mathbb{C}$ is a $c$-John disk if and only if it satisfies a harmonic doubling condition.

To prove Theorem 2.3 we need a lemma.

**Lemma 2.4.** Suppose that $D$ is a bounded Jordan domain in $\mathbb{C}$ and let $z_0 \in D$. Then the following conditions are equivalent, where the constants in each condition need not be the same but depend on each other:

1. $D$ is a $c$-John disk.
2. There exist constants $c$ and $\delta > 0$ such that

\begin{equation}
\frac{\text{dia}(\alpha_1)}{\text{dia}(\alpha)} \leq c \left( \frac{\omega(z_0, \alpha_1; D)}{\omega(z_0, \alpha; D)} \right)^\delta
\end{equation}

for all arcs $\alpha_1 \subset \alpha \subset \partial D$.
3. There exists a constant $c > 1$ such that

\begin{equation}
\omega(z_0, \alpha; D) \leq c \omega(z_0, \alpha_1; D)
\end{equation}
for all arcs $\alpha_1 < \alpha < \partial D$ with $\text{dia}(\alpha) \leq 2 \text{dia}(\alpha_1)$.

**Proof.** The equivalence of (1) and (2) is proved in [P, Theorem 1]. To prove the equivalence of (2) and (3), we first assume that (2) holds and let $\alpha_1 < \alpha < \partial D$ with $\text{dia}(\alpha) \leq 2 \text{dia}(\alpha_1)$. Then

$$\frac{\omega(z_0, \alpha_1; D)}{\omega(z_0, \alpha; D)} \geq c^{\frac{1}{2}} \left( \frac{\text{dia}(\alpha_1)}{\text{dia}(\alpha)} \right)^{\frac{1}{2}} \geq (2c)\frac{1}{2}$$

and hence we have (2.6) with a constant $(2c)^{\frac{1}{2}}$. Next suppose that (3) holds. Then by induction it is not difficult to show that

$$\omega(z_0, \alpha; D) \leq c^n \omega(z_0, \alpha_1; D)$$

for all arcs $\alpha_1 < \alpha < \partial D$ with $\text{dia}(\alpha) \leq 2^n \text{dia}(\alpha_1)$ and for each integer $n > 0$.

Now given any arcs $\alpha_1 < \alpha < \partial D$, there exists an integer $n > 0$ such that

$$2^{n-1} \text{dia}(\alpha_1) \leq \text{dia}(\alpha) \leq 2^n \text{dia}(\alpha_1).$$

Then by (2.7) we have

$$\omega(z_0, \alpha; D) \leq c^n \omega(z_0, \alpha_1; D).$$

Let $\delta = \frac{\log 2}{\log c}$. Then by (2.8) and (2.9) we obtain

$$\frac{\omega(z_0, \alpha; D)}{\omega(z_0, \alpha_1; D)} \leq c^{2^n} = c^{2^{n-1}} \leq c \left( \frac{\text{dia}(\alpha)}{\text{dia}(\alpha_1)} \right)^{\frac{1}{2}}.$$

Hence we get (2.5) with a constant $c^\delta$.

**Proof of Theorem 2.3.** For the necessity suppose that a harmonic doubling condition does not hold for $D$. Then for $j = 1, 2, \ldots$ there are consecutive arcs $\alpha_j, \beta_j$ on $\partial D$ such that

$$\text{dia}(\alpha_j) \leq 2 \text{dia}(\beta_j) \quad \text{and} \quad \omega(z_0, \alpha_j; D) \geq 3^j \omega(z_0, \beta_j; D).$$

Thus $\text{dia}(\alpha_j \cup \beta_j) \leq 3 \text{dia}(\beta_j)$ and hence by Lemma 2.4 (2) and by (2.10)

$$\frac{1}{2} \leq \frac{\text{dia}(\beta_j)}{\text{dia}(\alpha_j \cup \beta_j)} \leq c \left( \frac{\omega(z_0, \beta_j; D)}{\omega(z_0, \alpha_j \cup \beta_j; D)} \right)^{\delta} \leq c \left( \frac{\omega(z_0, \beta_j; D)}{\omega(z_0, \alpha_j; D)} \right)^{\delta} \leq c (3^j)^{\delta}$$

which yields a contradiction as $j \to \infty$.

For the sufficiency, by Lemma 2.4 it suffices to show that $D$ satisfies (2.6). Let $\alpha_1 \subset \alpha$ be arcs of $\partial D$ with $\text{dia}(\alpha) \leq 2 \text{dia}(\alpha_1)$.

Suppose first that $\alpha_1, \alpha$ have a common endpoint. Then $\text{dia}(\alpha \setminus \alpha_1) \leq 2 \text{dia}(\alpha_1)$ and hence by (2.1), $\omega(z_0, \alpha \setminus \alpha_1; D) \leq c_0 \omega(z_0, \alpha_1; D)$ for some $z_0 \in D$. Thus
Next suppose that $\alpha \setminus \alpha_1$ consists of two disjoint subarcs $\alpha_2, \alpha_3$. Then for $j = 2, 3$, $\text{diam}(\alpha_1 \cup \alpha_j) \leq 2 \text{diam}(\alpha_1)$ and hence $\omega(z_0, \alpha_1 \cup \alpha_j; D) \leq (c_0 + 1) \omega(z_0, \alpha_1; D)$ by what was proved above. Thus

$$\omega(z_0, \alpha; D) \leq 2(c_0 + 1) \omega(z_0, \alpha_1; D).$$

Therefore by (2.11) and (2.12) $D$ satisfies (2.6) with $c = 2(c_0 + 1)$.

3. Hyperbolic geodesics in John disks.

We say that a domain $D \subset \mathbb{C}$ is a $M$-quasiextremal distance or $M$-QED domain with respect to $E \subset D, 1 \leq M < \infty$, if for each pair of disjoint continua $F_1, F_2 \subset E$

$$\text{mod}(\Gamma) \leq M \text{mod}(\Gamma_D),$$

where $\Gamma$ and $\Gamma_D$ are the families of curves joining $F_1$ and $F_2$ in $\mathbb{C}$ and in $D$, respectively.

**Theorem 3.2.** Suppose that $D$ is a bounded simply connected domain in $\mathbb{C}$. Then the followings are equivalent:

1. $D$ is a $c$-John disk.
2. There exists a constant $c > 0$ such that for each hyperbolic geodesic $\gamma \subset D$

$$\ell(\gamma) \leq c(z_0, \partial D),$$

where $z_0$ is the euclidean midpoint of $\gamma$.

3. There exists a constant $c > 0$ such that if $f$ is analytic with

$$|f'(z)| \leq 1$$

in $D$, then for all $z_1, z_2 \in D$

$$|f(z_1) - f(z_2)| \leq c \text{dist}(z_0, \partial D),$$

where $z_0$ is the euclidean midpoint of the hyperbolic geodesic $\gamma \subset D$ joining $z_1$ to $z_2$.

4. $D$ is a $M$-QED domain with respect to all hyperbolic geodesics in $D$ with a given point $z_0 \in D$ as an endpoint.

Here the constants in each condition need not be the same but depend on each other. In particular, from (4) we obtain a John constant $c$ in (1), which depends on $M$ and a given point $z_0$.

**Proof of equivalence of (1) and (2).** Let $D \subset \mathbb{C}$ be a bounded $c$-John disk and let $z_0$ be the euclidean midpoint of a hyperbolic geodesic $\gamma$ with...
endpoints $z_1$ and $z_2$ in $D$. By [GHM, 2.16 Lemma], there exists a crosscut $\alpha$ of $D$ containing $z_0$ which separates the components of $\gamma \setminus \{z_0\}$ in $D$ and
\[
(3.6) \quad \ell(\alpha) \leq c_1 \dist(z_0, \partial D),
\]
where $c_1$ is an absolute constant. Next since $D$ is a $c$-John disk, there exists a John center $x_0$, a $c$-John arc $\beta_1$ from $z_1$ to $x_0$, and a $c$-John arc $\beta_2$ from $z_2$ to $x_0$.

If $x_0$ is in the component of $D \setminus \alpha$ which contains $z_2$, then by (3.6) there exists a point $w$ in $\alpha \cap \beta_1$ such that
\[
\ell(\beta_1(z_1, w)) \leq c \dist(w, \partial D) \leq c \ell(\alpha) \leq c_1 \dist(z_0, \partial D).
\]
Since $\beta_1(z_1, w) \cup \alpha(z_0, w)$ is a curve which joins $z_1$ to $z_0$ in $D$ and since $\gamma(z_1, z_0)$ is a hyperbolic geodesic in $D$ with $z_1$ and $z_0$ as its end points, Lemma 1.3 and (3.6) imply that
\[
\ell(\gamma) = 2\ell(\gamma(z_1, z_0)) \leq 2k\ell(\beta_1(z_1, w) + \ell(\alpha(z_0, w))) = 2kc_1(c + 1) \dist(z_0, \partial D)
\]
where $k$ is an absolute constant. If $x_0$ is in the component of $D \setminus \alpha$ which contains $z_1$, then the above argument applied to the arc $\beta_2$ yields the desired inequality. Finally if $x_0 \in \alpha$, then by Lemma 1.3 and (3.6),
\[
\ell(\gamma) = 2\ell(\gamma(z_1, z_0)) \leq 2k(\ell(\beta_1) + \ell(\alpha(x_0, z_0))) + \dist(x_0, \partial D) + c \dist(z_0, \partial D).
\]
Since $\alpha$ joins $x_0$ to $\partial D$,
\[
\ell(\gamma) \leq 2k(c \ell(\alpha) + c_1 \dist(z_0, \partial D)) \leq 2kc_1(c + 1) \dist(z_0, \partial D).
\]

Conversely, suppose that (2) holds and let $L = \sup \ell(\gamma)$, where the supremum is taken over all possible hyperbolic geodesics $\gamma$ with endpoints in $D$. Then there exist two points $z_1, z_2 \in D$ such that $\ell(\gamma) = \frac{L}{2}$, where $\gamma$ is the hyperbolic geodesic joining $z_1$ to $z_2$ in $D$. Let $z_0$ be the euclidean midpoint of $\gamma$. Then by (3.3),
\[
(3.7) \quad \text{dia}(D) \geq \dist(z_0, \partial D) \geq \frac{1}{c} \ell(\gamma) = \frac{1}{2c} L.
\]
Now fix a point $z \in D$ and let $w_0$ be the euclidean midpoint of the hyperbolic geodesic $\alpha$ joining $z$ to $z_0$ in $D$. If $x \in \alpha(w_0, z)$, then we can find a point $x_1 \in \alpha(x, z_0)$ with $\ell(\alpha(z, x)) = \ell(\alpha(x, x_1))$ and by (3.3) applied to $\alpha(z, x_1)$,
\[
(3.8) \quad \ell(\alpha(z, x)) = \frac{1}{2} \ell(\alpha(z, x_1)) \leq \frac{c}{2} \dist(x, \partial D).
\]
If $x \in \alpha(z_0, w_0)$, then we can find a point $x_2 \in \alpha(x, z)$ with $\ell(\alpha(x, z_0)) = \ell(\alpha(x_2, x))$. Then again by (3.3) applied to $\alpha(z_0, x_2)$ and by (3.7),
(3.9) \( \ell(\alpha(z, x)) \leq L \leq 2c \text{dist}(z_0, \partial D) \leq 2c \left( \ell(\alpha(x_2, z_0)) + \text{dist}(x, \partial D) \right) \)
\[ \leq 2c(c + 1) \text{dist}(x, \partial D). \]
Hence by (3.8) and (3.9) \( D \) is a \( c_1 \)-John disk with \( c_1 = 2c(c + 1) \).

**Proof of equivalence of (2) and (3).** First suppose that \( D \) satisfies (2). Then \( D \) is a \( b \)-John disk, where \( b \) depends only on \( c \). Let \( f \) be analytic and satisfy

(3.10) \[ |f'(z)| \leq \text{dist}(z, \partial D)^{\alpha - 1} \]
for some \( 0 < \alpha \leq 1 \) in \( D \). Fix \( z_1, z_2 \in D \), and let \( \gamma \) be the hyperbolic geodesic joining \( z_1 \) to \( z_2 \) in \( D \). Next let \( s \) denote arclength measured along \( \gamma \) from \( z_1 \), let \( z(s) \) denote the corresponding representation for \( \gamma \), and set \( g(s) = f(z(s)) \). Then

\[ |g'(s)| = |f'(z(s))| \]
while
\[ \min(s, l - s) \leq b_1 \text{dist}(z(s), \partial D), \quad l = \ell(\gamma), \]
where \( b_1 \geq 1 \) is a constant depending only on \( b \), by [GHM, Theorem 4.1]. Thus

\[ |g'(s)| \leq \text{dist}(z(s), \partial D)^{\alpha - 1} \leq \left( \frac{\min(s, l - s)}{b_1} \right)^{\alpha - 1} \]
for \( 0 < s < l \), and hence

(3.11) \[ |f(z_1) - f(z_2)| = |g(l) - g(0)| \]
\[ \leq \int_0^l |g'(s)| ds \leq 2b_1^{1 - \alpha} \int_0^l s^{\alpha - 1} ds \]
\[ = \frac{2b_1^{1 - \alpha}}{\alpha} \left( \frac{l}{2} \right)^{-\frac{1}{\alpha}} \leq \frac{c_1}{\alpha} \text{dist}(z_0, \partial D)^{\alpha}, \]
where \( c_1 = b_1 c \). If \( f \) satisfies (3.4) in \( D \), then \( f \) satisfies (3.10) with \( \alpha = 1 \). Hence, \( f \) satisfies (3.11) with \( \alpha = 1 \), i.e. \( f \) satisfies (3.5).

Now suppose that (3.5) holds for any analytic function \( f \) on \( D \) which satisfies (3.4). By [KW, Theorem 1] with \( k = 1 \), for \( z_1, z_2 \in D \)
\[ \inf_{\beta} \int_{\beta} |d\zeta| \leq c_1 \sup_f |f(z_1) - f(z_2)|, \]
where the infimum is taken over all Jordan arcs \( \beta \) in \( D \) joining \( z_1 \) to \( z_2 \), \( c_1 \) is
an absolute constant, and the supremum is taken over all analytic functions \( f \) on \( D \) with \( |f'(z)| \leq 1 \). Thus by Lemma 1.3,
\[
\ell(\gamma) \leq k \inf_\beta \ell(\beta) \leq k c \sup_f |f(z_1) - f(z_2)| \leq k c_1 \text{dist}(z_0, \partial D)
\]
for an absolute constant \( k \), where \( \gamma \) is the hyperbolic geodesic joining \( z_1 \) to \( z_2 \) in \( D \) and \( z_0 \) is the euclidean midpoint of \( \gamma \).

**Remark 3.12.** Note that this proof shows that if (3.4) implies (3.5), then \( D \) satisfies (2) and hence
\[
|f'(z)| \leq \text{dist}(z, \partial D)^{\alpha - 1}
\]
in \( D \) implies
\[
|f(z_1) - f(z_2)| \leq \frac{c}{\alpha} \text{dist}(z_0, \partial D)^\alpha
\]
for any \( 0 < \alpha \leq 1 \).

In order to prove the equivalence of (1) and (4), we need a lemma which shows that each hyperbolic line in \( D \) which joins two points on \( \partial D \) lies in the middle of \( D \). See [R, Lemma 4.13] and [PR, Theorem 3.3].

**Lemma 3.13.** Suppose that \( D \) is a simply connected proper subdomain in \( \mathbb{C} \) and that \( \gamma \subset D \) is a hyperbolic line joining \( w_1, w_2 \in \partial D \) and dividing \( D \) into disjoint subdomains \( D_1 \) and \( D_2 \). Then
\[
\frac{1}{b} \leq \frac{\text{dist}(z, \alpha_1)}{\text{dist}(z, \alpha_2)} \leq b, \quad b = 3 + 2\sqrt{2}
\]
for all \( z \in \gamma \), where \( \alpha_j = \partial D_j \setminus \gamma, j = 1, 2 \).

**Proof of equivalence of (1) and (4).** Suppose that \( D \subset \mathbb{C} \) is a bounded \( c \)-John disk with fixed John center \( z_0 \). Fix \( z_1 \in D \) and let \( \gamma \) be the hyperbolic geodesic joining \( z_0 \) to \( z_1 \) in \( D \). Fix two disjoint continua \( F_1, F_2 \) of \( \gamma \). Then by [K, Theorem 2.1] and the construction on [GO, pp. 67-68], there is a \( K \)-quasidisk \( G_1 \) in \( D \) such that \( \gamma \in \overline{G_1} \), where \( K \) depends only on \( c \). Thus by [GM3, Remark 2.23], \( G_1 \) is \( M \)-QED with respect to \( G_1 \) for some constant \( M, 1 \leq M < \infty \), which depends only on \( K \), and hence only on \( c \). Therefore, since \( \Gamma_{G_1} \subset \Gamma_D \),
\[
\text{mod}(\Gamma) \leq M \text{mod}(\Gamma_{G_1}) \leq M \text{mod}(\Gamma_D),
\]
where \( \Gamma, \Gamma_{G_1}, \Gamma_D \) are the families of curves which join \( F_1 \) and \( F_2 \) in \( \mathbb{C} \), \( G_1, D \), respectively.

Suppose next that \( z_0 \) is a point in \( D \) and that \( D \) is \( M \)-QED with respect to
all hyperbolic geodesics in \( D \) which have \( z_0 \) as an endpoint. Fix \( z_1 \in D \), \( z_1 \neq z_0 \) and let \( \gamma \) be the hyperbolic geodesic joining \( z_0 \) to \( z_1 \) in \( D \).

We show first that for some constant \( a > 1 \) and for all \( z \in \gamma \)
\[
\min(|z_0 - z|, |z - z_1|) \leq a \, \text{dist}(z, \partial D). \tag{3.14}
\]
Suppose otherwise. Then for each constant \( a > 1 \), there is a point \( z \in \gamma \) such that \( \min(|z_0 - z|, |z - z_1|) > a \, \text{dist}(z, \partial D) \). Fix a constant \( a > 1 \) and let \( b = 3 + 2\sqrt{2} \). Then for a constant \( ab > 1 \) there is a point \( z \in \gamma \) such that
\[
\min(|z_0 - z|, |z - z_1|) > ab \, \text{dist}(z, \partial D).
\]
Consider the hyperbolic line in \( D \) which contains \( \gamma \) and which has the endpoints \( w_1, w_2 \in \partial D \) and let \( \alpha_1, \alpha_2 \) be as described in Lemma 3.13. Then \( \text{dist}(z, \partial D) = \min_{i=1,2} \text{dist}(z, \alpha_i) \). Thus we may assume that \( \text{dist}(z, \partial D) = \text{dist}(z, \alpha_1) \) and hence by Lemma 3.13
\[
\text{dist}(z, \alpha_2) \leq b \, \text{dist}(z, \partial D). \tag{3.15}
\]
Let \( r = b \, \text{dist}(z, \partial D) \). By means of a preliminary similarity mapping we may assume that \( z = 0 \). Then \( z_0, z_1 \notin B(0, ar) \). Let \( A = B(0, ar) \setminus \overline{B}(0, \sqrt{ar}) \). For \( j = 0, 1 \), let \( F_j \) denote a component of \( A \cap \gamma(0, z_j) \) which joins the two boundary circles of \( A \). Then by [V, Theorem 10.12],
\[
\text{mod}(\Gamma) \geq \text{mod}(\Gamma_A) = \frac{2}{\pi} \log \sqrt{a}, \tag{3.16}
\]
where \( \Gamma, \Gamma_A \) are the families of curves joining \( F_0 \) and \( F_1 \) in \( C \) and in \( A \), respectively. Now let \( B = B(0, \sqrt{ar}) \setminus \overline{B}(0, r), E = \partial B(0, r), \) and \( F = \partial B(0, \sqrt{ar}) \). Then by (3.15), \( \Gamma_D \) is minorized by \( \Gamma_B \) and hence by [V, 7.5]
\[
\text{mod}(\Gamma_D) \leq \text{mod}(\Gamma_B) = 2\pi \left( \frac{\log \sqrt{ar}}{r} \right)^{-1} = \frac{2\pi}{\log \sqrt{a}}, \tag{3.17}
\]
where \( \Gamma_B \) is the family of curves joining \( E \) and \( F \) in \( B \) and \( \Gamma_D \) is the family of curves joining \( F_0 \) and \( F_1 \) in \( D \). Then by the hypothesis, (3.16) and (3.17)
\[
\frac{2}{\pi} \log \sqrt{a} \leq \text{mod}(\Gamma) \leq M \text{mod}(\Gamma_D) \leq \frac{2\pi M}{\log \sqrt{a}}
\]
and hence
\[
M \geq \left( \frac{\log \sqrt{a}}{\pi} \right)^2.
\]
This holds for each constant \( a > 1 \) and it leads a contradiction, which establishes (3.14).
Next to show that \( D \) is a \( c \)-John disk, by [NV, Lemma 2.10] we need to prove that for some constant \( c \geq 1 \) and for all \( z \in \gamma \)
\begin{equation}
|z - z_1| < c \operatorname{dist}(z, \partial D).
\end{equation}
For this let \( L = \max\{|z_0 - z| : z \in \partial D\} \), \( k = \frac{L}{\operatorname{dist}(z_0, \partial D)} \) and \( c_1 = \max(a, k) \). If \( |z - z_1| < |z - z_0| \), then by (3.14)
\begin{equation}
|z - z_1| < a \operatorname{dist}(z, \partial D).
\end{equation}
If \( |z - z_1| > |z - z_0| \), then \( |z_0 - z_1| \leq L \) and (3.14) give
\[
\frac{|z - z_1|}{c_1} < \frac{|z - z_0|}{a} + \frac{|z_0 - z_1|}{k} < \operatorname{dist}(z, \partial D) + \operatorname{dist}(z_0, \partial D)
\leq |z - z_0| + 2 \operatorname{dist}(z, \partial D) < (a + 2) \operatorname{dist}(z, \partial D).
\]
Hence
\begin{equation}
|z - z_1| < c_1(a + 2) \operatorname{dist}(z, \partial D).
\end{equation}
Therefore by (3.19) and (3.20) we obtain (3.18) with \( c = c_1(a + 2) \), which depends on \( M \) and \( z_0 \).

Note that in the proof of equivalence of (1) and (4) in Theorem 3.2, what we get from (4) is the John condition on all hyperbolic geodesics with a given point \( z_0 \) as an end point and a fixed constant \( c = c(z_0, M) \). If \( c \) were independent of \( z_0 \), we are in the uniform domain case as follows.

**Corollary 3.21.** Suppose that \( D \) is a bounded finitely connected domain in \( \mathbb{C} \). Then \( D \) is \( c \)-uniform if and only if \( D \) is a \( M \)-QED domain with respect to all hyperbolic geodesics in \( D \). Here \( c \) and \( M \) depend only on each other.

**Proof.** Suppose that \( D \) is \( c \)-uniform. Then by [GM3, Theorem 2.22], \( D \) is a \( M \)-QED domain with respect to \( D \), \( M = M(c) \), and hence with respect to all hyperbolic geodesics in \( D \). For the sufficiency, let \( z_1, z_2 \) be two disjoint points in \( D \) and let \( \gamma \) be the hyperbolic geodesic in \( D \) with endpoints \( z_1, z_2 \). Then by an argument similar to that for the proof of (3.14)
\begin{equation}
\min(|z_1 - z|, |z - z_2|) \leq a \operatorname{dist}(z, \partial D)
\end{equation}
for all \( z \in \gamma \) and for some constant \( a > 1 \). Also by the same argument as the proof of [GM3, Lemma 2.7]
\begin{equation}
\ell(\gamma) \leq k |z_1 - z_2|,
\end{equation}
where \( k, 1 < k < \infty \), is a constant depending only on \( M \). Therefore [NV, Theorem 2.16], (3.22) and (3.23) imply that \( D \) is \( c \)-uniform with \( c = \max(a, k) \), which depends only on \( M \).
Next we characterize unbounded Jordan John disks with \( \infty \in \partial D \) in terms of the hyperbolic geodesics in their exteriors.

**Lemma 3.24** [GHM], [NV], [R]. A Jordan domain \( D \subset \mathbb{C} \) is a \( c \)-John disk if and only if each pair of points \( z_1, z_2 \in D^* \) can be joined by a continuum \( E \subset D^* \) with

\[
\text{diam}(E) \leq c_1 |z_1 - z_2|.
\]

Here the constants \( c \) and \( c_1 \) depend only on each other.

**Theorem 3.25.** A Jordan domain \( D \subset \mathbb{C} \) with \( \infty \in \partial D \) is a \( c \)-John disk if and only there is a constant \( c_0 \geq 1 \) such that for each hyperbolic geodesic \( \gamma \) in \( D^* \)

\[
\text{diam}(\gamma) \leq c_0 |z_1 - z_2|,
\]

where \( z_1, z_2 \) are the endpoints of \( \gamma \). Here \( c \) and \( c_0 \) depend on each other.

To prove this we need a lemma which gives the diameter version of the Gehring-Hayman inequality in Lemma 1.3. See [R, Lemma 3.22] and [PR, Theorem 3.2].

**Lemma 3.27.** Suppose that \( \gamma \) is a hyperbolic geodesic in a simply connected proper subdomain \( D \subset \mathbb{C} \) and that \( \alpha \) is an arc which joins the endpoints of \( \gamma \) in \( D \cap B(z_0, r) \) for \( z_0 \in \mathbb{C} \). Then

\[
\gamma \subset \overline{B(z_0, br)}, \quad b = 3 + 2\sqrt{2}.
\]

**Proof of Theorem 3.25.** Suppose first that a Jordan domain \( D \subset \mathbb{C} \) is a \( c \)-John disk with \( \infty \in \partial D \). Then by Lemma 3.24 for each pair of points \( z_1, z_2 \in D^* \) there exists a continuum \( E \subset D^* \) such that \( \text{diam}(E) \leq c_1 |z_1 - z_2| \). Thus by [NV, Lemma 4.3], \( E \) can be replaced by an arc \( \alpha \subset D^* \) with \( \text{diam}(\alpha) \leq c_2 |z_1 - z_2| \) for any \( c_2 > c_1 \). Next let \( \gamma \) be the hyperbolic geodesic joining \( z_1 \) and \( z_2 \) in \( D^* \). Then \( \alpha \) is an arc which joins the endpoints of \( \gamma \). Now choose a point \( z_0 \in \alpha \) such that \( |z_1 - z_0| = |z_2 - z_0| \) and let \( r = \text{diam}(\alpha) \). Thus \( \alpha \subset D^* \cap \overline{B(z_0, r)} \), while \( \gamma \subset \overline{B(z_0, br)} \) with \( b = 3 + 2\sqrt{2} \) by Lemma 3.27. Hence

\[
\text{diam}(\gamma) \leq 2br = 2b \text{diam}(\alpha) \leq 2bc_2 |z_1 - z_2|
\]

and this shows (3.26) with \( c_0 = 2bc_2 \).

Suppose next that (3.26) holds. Then by Lemma 3.24, \( D \) is a \( c \)-John disk.

We define a one-sided analogue of the function $j_D$ in (1.2) as follows:

$$j_D'(z_1, z_2) = \frac{1}{2} \log \left( \frac{\lambda_D(z_1, z_2)}{\text{dist}(z_1, \partial D)} + 1 \right) \left( \frac{\lambda_D(z_1, z_2)}{\text{dist}(z_2, \partial D)} + 1 \right),$$

where $\lambda_D$ is the inner distance on $D$,

$$\lambda_D(z_1, z_2) = \inf_{\gamma} \ell(\gamma),$$

and the infimum is taken over all paths $\gamma \subset D$ with $z_1$ and $z_2$ as endpoints. The main result of this section relates $h_D$ and $j_D'$ in John disks. As mentioned in the introduction, this is a one-sided analogue of a characterization of quasidisks due to Gehring and Osgood [GO]. Their two-sided version characterizes uniform domains, regardless of connectivity, when the hyperbolic metric is replaced by the quasihyperbolic metric.

**Theorem 4.1.** A simply connected proper subdomain $D \subset \mathbb{C}$ is a $c$-John disk if and only if there exists a constant $b \geq 1$ such that

$$h_D(z_1, z_2) \leq bj'_D(z_1, z_2)$$

for all $z_1, z_2 \in D$. Here the constants $c$ and $b$ depend only on each other.

We will use the following inequality, which is easily derived.

**Lemma 4.3.** For any $c \geq 1$ and $x \geq 0$,

$$\log(cx + 1) \leq c \log(x + 1).$$

**Proof of necessity.** Suppose that $D$ is a $c$-John disk. Then by [GHM, Theorem 4.1] each $z_1, z_2 \in D$ can be joined by a hyperbolic geodesic $\gamma$ in $D$ such that for all $z \in \gamma$

$$\min_{j=1, 2} \ell(\gamma(z_j, z)) \leq c_1 \text{dist}(z, \partial D)$$

for some constant $c_1$ depending only on $c$. Choose $z_0 \in \gamma$ so that $\ell(\gamma(z_0, z_1)) = \ell(\gamma(z_0, z_2))$. Then by the triangle inequality it is sufficient to show that

$$h_D(z_j, z_0) \leq b \log \left( \frac{\lambda_D(z_1, z_2)}{\text{dist}(z_j, \partial D)} + 1 \right)$$

for $j = 1, 2$, where $b = (3c_1 + 2)k$ and $k$ is an absolute constant. By symmetry we may assume that $j = 1$.

Suppose first that
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(4.6) \[ \ell(\gamma(z_1, z_0)) \leq \frac{c_1}{c_1 + 1} \text{dist}(z_1, \partial D). \]

Then \( z_0 \in B(z_1, \frac{c_1}{c_1 + 1} \text{dist}(z_1, \partial D)) \). If \( z \in [z_1, z_0] \), then

\[ \text{dist}(z, \partial D) \geq \text{dist}(z_1, \partial D) - |z_1 - z| \geq \frac{1}{c_1 + 1} \text{dist}(z_1, \partial D) \]

and hence

\[ |z_1 - z| + \text{dist}(z_1, \partial D) \leq c_1 \text{dist}(z, \partial D) + (c_1 + 1) \text{dist}(z, \partial D) \]

\[ \leq (2c_1 + 1) \text{dist}(z, \partial D). \]

If \( \rho_D(z) \) is the hyperbolic density in \( D \), then the Schwarz lemma and the Koebe distortion theorem give the inequalities

\[ \frac{1}{4} \text{dist}(z, \partial D) \leq \rho_D(z) \leq \frac{1}{\text{dist}(z, \partial D)}. \]

Thus Lemma 1.3 and Lemma 4.3 yield

\[ h_D(z_1, z_0) \leq \int_{[z_1, z_0]} \frac{ds}{\text{dist}(z, \partial D)} \]

\[ \leq \int_0^{\text{dist}(z_1, \partial D)} (2c_1 + 1) \frac{ds}{s + \text{dist}(z_1, \partial D)} \]

\[ \leq (2c_1 + 1) \log \left( \frac{\ell(\gamma)}{\text{dist}(z_1, \partial D)} + 1 \right) \]

\[ \leq (2c_1 + 1) k \log \left( \frac{\lambda_D(z_1, z_2)}{\text{dist}(z_1, \partial D)} + 1 \right). \]

where \( k \) is an absolute constant. This implies (4.5).

Next suppose that (4.6) does not hold and choose \( y_1 \in \gamma(z_1, z_0) \) so that

\[ \ell(\gamma(z_1, y_1)) = \frac{c_1}{c_1 + 1} \text{dist}(z_1, \partial D). \]

If \( z \in \gamma(y_1, z_0) \), then

\[ \text{dist}(z, \partial D) \geq \frac{1}{c_1} \ell(\gamma(z_1, z)) \]

by (4.4) and hence again
We also have

\[ h_D(z_1, y_1) \leq (2c_1 + 1)k \log \left( \frac{\lambda_D(z_1, y_1)}{\text{dist}(z_1, \partial D)} + 1 \right) \]

by what was proved above. Then (4.5) follows from the triangle inequality.

**Proof of sufficiency.** Suppose that (4.2) holds. Fix \( z_1, z_2 \in D \) and let \( \gamma \) be the hyperbolic geodesic joining \( z_1 \) to \( z_2 \) in \( D \). We may assume that \( \text{dist}(z_1, \partial D) \geq \text{dist}(z_2, \partial D) \).

Suppose first that

\[ 2\lambda_D(z_1, z_2) \leq \text{dist}(z_1, \partial D) \]

Then \( |z_1 - z_2| \leq \text{dist}(z_1, \partial D)/2 \) and hence

\[ z_2 \in B \left( z_1, \frac{\text{dist}(z_1, \partial D)}{2} \right) \subset D. \]

Thus \( \lambda_D(z_1, z_2) = |z_1 - z_2| \) and since euclidean disks in \( D \) are convex with respect to the hyperbolic geometry in \( D \) [10],

\[ \gamma \subset B \left( \frac{z_1 + z_2}{2}, \frac{|z_1 - z_2|}{2} \right) \subset B \left( z_1, \frac{\text{dist}(z_1, \partial D)}{2} \right). \]

Then by Lemma 1.3

\[ \min_{j=1,2} \ell(\gamma(z_j, z)) \leq \ell(\gamma) \leq k |z_1 - z_2| \leq k \text{dist}(z, \partial D) \]

for all \( z \in \gamma \) and \( k \) is an absolute constant.

Next suppose that (4.7) does not hold. By compactness there exists a point \( z_0 \in \gamma \) with

\[ \text{dist}(z_0, \partial D) = \sup_{z \in \gamma} \text{dist}(z, \partial D). \]

Let \( m \) denote the largest integer for which

\[ 2^m \text{dist}(z_1, \partial D) \leq \text{dist}(z_0, \partial D) \]

and let \( y_0 \) be the first point of \( \gamma(z_1, z_0) \) with
\[ \text{dist}(y_0, \partial D) = 2^m \text{dist}(z_1, \partial D) \]

as we traverse \( \gamma \) from \( z_1 \) towards \( z_0 \). Clearly

(4.9) \[ \text{dist}(y_0, \partial D) \leq \text{dist}(z_0, \partial D) < 2 \text{dist}(y_0, \partial D). \]

Let \( y_1 = z_1 \) and choose points \( y_2, \ldots, y_{m+1} \in \gamma(z_1, z_0) \) so that \( y_i \) is the first point of \( \gamma(z_1, z_0) \) for which

(4.10) \[ \text{dist}(y_i, \partial D) = 2^{i-1} \text{dist}(y_1, \partial D) \]

as we traverse \( \gamma \) from \( z_1 \) towards \( z_0 \). Then \( y_{m+1} = y_0 \) and let \( y_{m+2} = z_0 \).

We show first that for \( i = 1, \ldots, m + 1 \)

(4.11) \[ \begin{cases} h_D(y_i, y_{i+1}) \leq 2^i \cdot b^2 \\ \ell(\gamma(y_i, y_{i+1})) \leq 2^i \cdot b^2 \text{dist}(y_i, \partial D) \end{cases} \]

Fix \( i \in \{1, \ldots, m + 1\} \) and set

\[ t = \frac{\ell(\gamma(y_i, y_{i+1}))}{\text{dist}(y_i, \partial D)}. \]

If \( z \in \gamma(y_i, y_{i+1}) \), then by (4.9), (4.10)

\[ \text{dist}(z, \partial D) \leq \text{dist}(y_{i+1}, \partial D) \leq 2 \text{dist}(y_i, \partial D) \]

and hence

\[ t = \int_{\gamma(y_i, y_{i+1})} \frac{|dz|}{\text{dist}(y_i, \partial D)} \leq 8 h_D(y_i, y_{i+1}). \]

Since

\[ j_D(y_i, y_{i+1}) \leq \log \left( \frac{\lambda_D(y_i, y_{i+1})}{\text{dist}(y_i, \partial D)} + 1 \right) \leq \log(t + 1), \]

(4.2) implies that

\[ h_D(y_i, y_{i+1}) \leq b \log(t + 1) \leq b (t + 1)^{1/2}. \]

If \( t \geq 1 \), then

\[ t \leq 8 h_D(y_i, y_{i+1}) \leq 8b (t + 1)^{1/2} \leq 8b (2t)^{1/2} \]

which implies

\[ t \leq 2^7 \cdot b^2 \]

and hence

\[ h_D(y_i, y_{i+1}) \leq b (2 \cdot 2^7 \cdot b^2)^{1/2} = 2^4 \cdot b^2. \]
Thus we obtain (4.11). If $t < 1$, then $t \leq 2^7 \cdot b^2$ and again we obtain (4.11). This completes the proof of (4.11).

Now [GP, Lemma 2.1] and (4.11) imply that for $z \in \gamma (y_i, y_{i+1})$, $i = 1, \ldots, m + 1$

$$\log \frac{\text{dist}(y_{i+1}, \partial D)}{\text{dist}(z, \partial D)} \leq 4h_D(z, y_{i+1}) \leq 4h_D(y_i, y_{i+1}) < 2^6 \cdot b^2 = c_0$$

and thus

$$\text{(4.12)} \quad \text{dist}(y_{i+1}, \partial D) \leq e^{c_0} \text{dist}(z, \partial D).$$

If $z \in \gamma (z_1, z_0)$, then $z \in \gamma [y_{i_0}, y_{i_0+1}]$ for some $i_0 \in \{1, \ldots, m + 1\}$ and hence by (4.10), (4.11) and (4.12)

$$\text{(4.13)} \quad \min_{j=1,2} \ell(\gamma(z_j, z)) \leq \ell(\gamma(z_1, z)) \leq \sum_{i=1}^{i_0} \ell(\gamma[y_i, y_{i+1}])$$

$$\leq 2c_0 \sum_{i=1}^{i_0} \text{dist}(y_i, \partial D) = 2c_0 (2^{i_0} - 1) \text{dist}(y_1, \partial D)$$

$$< 2c_0 \text{dist}(y_{i_0+1}, \partial D) \leq 2c_0 e^{c_0} \text{dist}(z, \partial D).$$

Likewise, if $z \in \gamma (z_2, z_0)$, then we also have (4.13). Therefore by (4.8) and (4.13) $D$ is a $c$-John disk with $c = 2c_0 e^{c_0}$.

**Remark 4.14.** Theorem 4.1 is easily translated into a result for the quasihyperbolic distance in $D$, $k_D$. If we assume that quasihyperbolic geodesics are double $c$-cone arcs in $D$, the result for $k_D$ can be generalized to finitely connected domains in the plane, and to domains in $\mathbb{R}^n$ which are quasi-conformal images of uniform domains. In the quasihyperbolic case, the proof of sufficiency of Theorem 4.1 shows that in a domain $D \subset \mathbb{C}$ satisfying

$$k_D(z_1, z_2) \leq b f_D(z_1, z_2),$$

quasihyperbolic geodesics are double $c$-cone arcs, where $c$ depends only on $b$.

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