A NOTE ON THE LOCAL INVERTIBILITY OF SOBOLEV FUNCTIONS

ROBERTO VAN DER PUTTEN

Abstract.
We give some topological and analytical conditions in order that a continuous Sobolev function be a local homeomorphism. The results are obtained in the setting of the spaces $W^{1,n} (\Omega; \mathbb{R}^n)$ and $W^{2,p} (\Omega; \mathbb{R}^n)$.

1. Introduction.

In this paper we deal with the local invertibility of continuous mappings and, more precisely, with the properties of the branch set of such mappings; we recall that, if $\Omega$ is an open subset of $\mathbb{R}^n$ and $f : \Omega \rightarrow \mathbb{R}^n$ a continuous mapping, the branch set of $f$, denoted by $B_f$, is the set of all points $x \in \Omega$ where $f$ does not define a local homeomorphism. It is well known that if $f \in C^1$, then $B_f \subset Z_f$ where $Z_f = \{x \in \Omega : Df(x) \text{ exists and } \det Df(x) = 0\}$, but the study of $B_f$ becomes more difficult beyond the class of smooth mappings. Some results have been obtained under topological assumptions: if $f$ is light and sense-preserving (see below for definitions) then the topological dimension of $B_f$ and $f(B_f)$ is not greater than $n - 2$ and

$$B_f \subset Z_f \cup S_f$$

where $S_f = \{x \in \Omega : f$ is not weakly differentiable at $x\}$ ([11] and [3]).

However, it is not known under what analytical conditions a mapping is light and sense-preserving; some results can be found in [7] (mappings with finite dilatation) and in the monograph of Rickman ([11]) on quasiregular mappings.

Invertibility has been studied also in the setting of nonlinear elasticity: in fact this requirement guarantees that interpenetration of matter does not occur. In this case Ball and Šverak ([2], [13]) have found analytical conditions which implies the global invertibility of Sobolev functions.

---

Received March 11, 1996.
In this paper we present three results in the setting of Sobolev spaces: the first two concern mappings belonging to \( W^{1,n}(\Omega; \mathbb{R}^n) \) and they are slight improvements of the recalled result in [11] (Chap. I, Lemma 4.11); we prove that (1.1) holds if \( \det Df \geq 0 \) almost everywhere in \( \Omega \) and \( f \) is either open or light. The topological degree is widely used in the proofs. The third theorem concerns mappings belonging to \( W^{2,p}(\Omega; \mathbb{R}^n) \). First we prove that \( S_f \) is a set of zero capacity if \( p > \frac{n(n-1)}{2n-1} \); then we use this result to show that (1.1) holds if \( p > n-1 \), \( Z_f \) is a set of zero capacity and \( \det Df > 0 \) almost everywhere in \( \Omega \).

2. Notations and preliminaries.

Throughout this paper \( \Omega \) is a nonempty, bounded and open set in \( \mathbb{R}^n \), with \( n \geq 2 \).

We write \( \mathcal{L}^n \) for the Lebesgue measure in \( \mathbb{R}^n \) and \( \| \| \) for the norm in the same space. Given \( x \in \mathbb{R}^n \) and \( r > 0 \), \( B(x,r) \) is the open ball of center \( x \) and radius \( r \); \( Q(x,r) \) is the set \( \{ y \in \mathbb{R}^n : |x_i - y_i| < r, \ i \in \{1,\ldots,n\} \} \), where \( x = (x_1,\ldots,x_n) \) and \( y = (y_1,\ldots,y_n) \). If \( A \subset \mathbb{R}^n \), \( D(A) \) will be the set of accumulation points of \( A \).

For \( 1 \leq p \leq +\infty \) and \( m \geq 1 \), let \( L^p(\Omega; \mathbb{R}^m) \) be the collection of all \( m \)-tuples \((f_1,\ldots,f_m)\) of real functions in \( L^p(\Omega) \). For \( k \geq 1 \), we say that \( f \in W^{k,p}(\Omega; \mathbb{R}^m) \) if \( f \in L^p(\Omega; \mathbb{R}^m) \) together with its derivatives (in the sense of distribution) up to \( k \)th order; \( Df \) will be the distributional Jacobian matrix of \( f \).

Now we introduce the Bessel capacity. Let \( g \) be the Bessel kernel, that is the function whose Fourier transform is

\[
(g)(x) = (2\pi)^{-n}(1 + \|x\|^2)^{-\frac{n}{2}};
\]

for \( p > 1 \), we define the Bessel capacity for any set \( A \subset \mathbb{R}^n \) as

\[
B_{1,p}(A) = \inf \left\{ \int_{\mathbb{R}^n} |f(x)|^p \, dx : f \in L^p(\mathbb{R}^n), g \ast f \geq 1 \text{ on } A, f \geq 0 \right\},
\]

where \( g \ast f \) is the convolution of \( g \) and \( f \) (the elementary properties of Bessel capacity can be found in [15]).

Let \( A \subset \mathbb{R}^n \). The Hausdorff dimension of \( A \) is defined by

\[
\dim_H(A) = \sup \{ \alpha \geq 0 : H^\alpha(A) > 0 \}
\]

with the convention \( \dim_H(\emptyset) = 0 \), where \( H^\alpha \) is the \( \alpha \)-dimensional Hausdorff measure (see [4]).

Now let \( f : \Omega \rightarrow \mathbb{R}^n \) be a continuous mapping. We say that \( f \) satisfies the condition \((N)\) on \( \Omega \) if \( \mathcal{L}^n(f(A)) = 0 \) whenever \( A \subset \Omega \) is such that \( \mathcal{L}^n(A) = 0 \). If \( A \subset \Omega \) and \( y \in \mathbb{R}^n \), we denote by \( N(f,A,y) \) the number (possibly infinite) of elements of the set \( A \cap f^{-1}(y) \). The map \( f \) is said to be light if \( f^{-1}(y) \) is totally disconnected for every \( y \in \mathbb{R}^n \).
In the following \( A \) will be a domain such that \( A \subset \Omega \) and \( y \in \mathbb{R}^n \).

Now we introduce the topological degree. Suppose that \( y \notin f(\partial A) \). Then there exists \( r \in \mathbb{R}, 0 < r < 1, r \) small enough, such that \( f \) induces a homomorphism of cohomology groups

\[
f^*: H_{n+1}(B(y,r^{-1});B(y,r)\setminus B(y,r)) \to H_{n+1}(\overline{A};\partial A).
\]

If \( g_1, g_2 \) are suitable generators of the cohomology groups, there exists an integer, we denote it \( \mu(y,f,A) \), such that \( f^*(g_1) = \mu(y,f,A)g_2 \); \( \mu(y,f,A) \) is called the topological degree of \( y \) with respect to the pair \((f,A)\).

We say that \( f \) is sense-preserving (weakly sense-preserving) if \( \mu(y,f,A) > 0(\mu(y,f,A) \geq 0) \) for every domain \( A \subset \subset \Omega \) and \( y \notin f(\partial A) \).

Now, resorting to the topological degree, we may define some multiplicity functions. Given a domain \( B \) such that \( \overline{B} \subset A \), we say that \( B \) is a positive (negative) indicator domain for \((y,f,A)\) if \( y \notin f(\partial B) \) and \( \mu(y,f,B) > 0(<0) \). A finite (possibly empty) collection of pair-wise disjoint positive (negative) indicator domains for \((y,f,A)\) is called a positive (negative) indicator system for \((y,f,A)\) and is denoted by \( \sigma^+(y,f,A)(\sigma^-(y,f,A)) \). Finally we define the multiplicity functions

\[
K^+(y,f,A) = \sup \left\{ \sum_{B \in \sigma^+} \mu(y,f,B) : \sigma^+ = \sigma^+(y,f,A) \right\},
\]

\[
K^-(y,f,A) = \sup \left\{ -\sum_{B \in \sigma^-} \mu(y,f,B) : \sigma^- = \sigma^-(y,f,A) \right\},
\]

\[
K(y,f,A) = K^+(y,f,A) + K^-(y,f,A)
\]

with the convention that \( K^+(y,f,A) = 0(K^-(y,f,A) = 0) \) if there is no positive (negative) nonempty indicator system. The multiplicity function \( K \) is related with the concept of essential maximal model continua (e.m.m.c.). We say that \( C \subset \mathbb{R}^n \) is an e.m.m.c. for \((y,f,A)\) if \( C \) is a component of \( A \cap f^{-1}(y) \) which is a continuum and if for every open set \( D \) such that \( C \subset D \subset A \) there exists a positive or negative indicator domain \( B \) for \((y,f,A)\) such that \( C \subset B \subset \overline{B} \subset D \). If either \( K(y,f,A) \leq 1 \) or \( K(y,f,A) = +\infty \) then \( K(y,f,A) \) agrees with the number of e.m.m.c. for \((y,f,A)\) ([10], II.3.4., Thm. 3).

A sequence \( \{B_k\}_{k \in \mathbb{N}} \) of nonempty domains is called a determining sequence for \((y,f,A)\) if \( B_{k+1} \subset B_k \subset \subset A, y \in f(\overline{B_k}) \setminus f(\partial B_k) \), for every \( k \in \mathbb{N} \) and \( \lim_{k \to \infty} \text{diam}(f(\overline{B_k})) = 0 \).

Now let \( x_0 \in \Omega \); we say that \( f \) has a weak differential at \( x_0 \) if there exists a linear mapping \( L : \mathbb{R}^n \to \mathbb{R}^n \) and a set \( B \subset \mathbb{R} \) such that \( 0 \) is a point of right density of \( B \) satisfying
\[ \lim_{t \to 0^+} \sup_{z \in \mathbb{B}} \left\{ \left\| \frac{f(x_0 + tz) - f(x_0)}{t} - L(z) \right\| : z \in \partial \Omega(0,1) \right\} = 0. \]

The following result by Goffman and Ziemer ([6], Thm. 3.4) states the weak differentiability properties of Sobolev functions.

**Theorem 2.1.** If \( f \in W^{1,p}(\Omega; \mathbb{R}^n) \) with \( p > n - 1 \), then \( f \) is weakly differentiable almost everywhere in \( \Omega \).

3. **Local invertibility of Sobolev functions.**

**Theorem 3.1.** Let \( f \in W^{1,n}(\Omega; \mathbb{R}^n) \) be a continuous, open mapping such that \( \det Df \geq 0 \) almost everywhere in \( \Omega \). Then \( B_f \subset Z_f \cup S_f \).

**Proof.** Let \( x_0 \not\in Z_f \cup S_f \); we shall prove that \( x_0 \not\in B_f \). We use the link between the weak differential and the topological degree ([10], p. 329); if \( x_0 \not\in Z_f \cup S_f \), there exist \( r_1, r_2 > 0 \) such that \( Q(x_0, r_1) \subset \subset \Omega \) and \( \mu(y, f, Q(x_0, r_1)) = 1 \) for every \( y \in B(f(x_0), r_2) \).

We show that \( N(y, f, Q(x_0, r_1)) = 1 \) for every \( y \in B(f(x_0), r_2) \). Since \( f \) is weakly sense-preserving ([11], Ch. VI, Lemma 5.1), we have \( \sigma^-(y, f, Q(x_0, r_1)) = \emptyset \) for every \( y \in \mathbb{R}^n \); therefore \( K^-(y, f, Q(x_0, r_1)) = 0 \) and \( K(y, f, Q(x_0, r_1)) = K^+(y, f, Q(x_0, r_1)) \) for every \( y \in \mathbb{R}^n \). Furthermore

\[ K^+(y, f, Q(x_0, r_1)) = \mu(y, f, Q(x_0, r_1)) = 1 \]

for every \( y \in B(f(x_0), r_2) \) such that \( K(y, f, Q(x_0, r_1)) < +\infty \) ([10], II. 3.4, Thm. 2 and Thm. 4). Now we note that \( f \) satisfies the condition \( \text{(N)} \) ([8], Corollary B) and, by Theorem 2.1, \( f \) is weakly differentiable almost everywhere in \( \Omega \). This implies that \( K(\cdot, f, Q(x_0, r_1)) \in L^1(\mathbb{R}^n) \) ([10], V. 3.3., Thm. 5) and then \( K(y, f, Q(x_0, r_1)) < +\infty \) for almost every \( y \in \mathbb{R}^n \). By (3.1), \( K(y, f, Q(x_0, r_1)) = 1 \) for almost every \( y \in B(f(x_0), r_2) \) and, therefore, also \( N(y, f, Q(x_0, r_1)) = 1 \) for almost every \( y \in B(f(x_0), r_2) \) ([10], V. 3.3. Thm. 2).

Since \( f \) is open, then \( N(\cdot, f, Q(x_0, r_1)) \) is lower semicontinuous in \( \mathbb{R}^n \) ([5], Chap. 5, Thm. 1.3) and this implies that \( N(\cdot, f, Q(x_0, r_1)) \leq 1 \) everywhere in \( B(f(x_0), r_2) \). On the other hand \( B(f(x_0), r_2) \subset f(Q(x_0, r_1)) \) because \( \mu(y, f, Q(x_0, r_1)) = 1 \) for every \( y \in B(f(x_0), r_2) \) and then \( N(\cdot, f, Q(x_0, r_1)) = 1 \) everywhere in \( B(f(x_0), r_2) \).

Finally, let \( A_{x_0} = f^{-1}(B(f(x_0), r_2)) \cap Q(x_0, r_1) \); since \( f(A_{x_0}) \subset B(f(x_0), r_2) \), the restriction of \( f \) to \( A_{x_0} \) is one-to-one and, as \( f \) is an open mapping, \( f \) is a homeomorphism from \( A_{x_0} \) to \( f(A_{x_0}) \), that is \( x_0 \not\in B_f \).

**Definition 3.2.** Let \( f : \Omega \to \mathbb{R}^n \) be a continuous mapping. For every \( y \in \mathbb{R}^n \) we define \( R_y = \{ x \in f^{-1}(y) \} \); there exists a sequence \( \{ V_m \}_{m \in \mathbb{N}} \) of open
neighbourhood of $x$ such that $\text{diam}(V_m) < \frac{1}{m}$, there exists $\mu(y, f, V_m)$ and it is positive for every $m \in \mathbb{N}$.

**Remark 3.3.** If $f$ is light, weakly sense-preserving and $A \subset \Omega$ is a domain, we have that $R_y \cap A$ is the set of the e.m.m.c. for $(y, f, A)$.

In the following lemma, we recall a property of the set $R_y$ for such mappings (see also [14]).

**Lemma 3.4.** Let $f : \Omega \rightarrow \mathbb{R}^n$ be a continuous, light and weakly sense-preserving mapping. Then $D(R_y) \cap \Omega = \emptyset$ for every $y \in \mathbb{R}^n$.

**Proof.** By contradiction we suppose that there exists $y \in \mathbb{R}^n$ such that $D(R_y) \cap \Omega \neq \emptyset$. Then let $x \in \Omega$ be a limit point of a sequence $\{x_m\}_{m \in \mathbb{N}}$ in $R_y$. Since $f$ is continuous and light, then $x \in f^{-1}(y)$ and there exists an open neighbourhood $B$ of $x$ such that $B \subset \Omega$ and $f^{-1}(y) \cap \partial B = \emptyset$. Let $\mu(y, f, B) = \alpha \geq 0, k \in \mathbb{N}$ satisfying $k > \alpha$ and let $\{x_1, x_2, \ldots, x_k\}$ be $k$ points of the sequence $\{x_m\}_{m \in \mathbb{N}}$ belonging to $B$. Then, for every $i \in \{1, 2, \ldots, k\}$, there exists an open neighbourhood $V_i$ of $x_i$ such that $V_i \subset B$, $V_j \cap V_i = \emptyset$ for every $j, l \in \{1, 2, \ldots, k\}$ and $\mu(y, f, V_i) \geq k$.

Finally let $V = \bigcup_{i=1}^{k} V_i$ and $\Omega = B \setminus V$. Then $f^{-1}(y) \cap \partial V = \emptyset$, $f^{-1}(y) \cap \partial W = \emptyset$ and therefore $f^{-1}(y) \cap B \subset \bigcup_{i=1}^{k} V_i \cup W$. Consequently

$$\alpha = \mu(y, f, B) = \sum_{i=1}^{k} \mu(y, f, V_i) + \mu(y, f, W) \geq k$$

and this is a contradiction.

**Theorem 3.5.** Let $f \in W^{1,n}(\Omega; \mathbb{R}^n)$ be a continuous, light mapping such that $\det Df \geq 0$ almost everywhere in $\Omega$. Then $B_f \subset Z_f \cup S_f$.

**Proof.** As in Theorem 3.1, we consider $x_0 \notin Z_f \cup S_f$ and we prove that $x_0 \notin B_f$. Analogously we can show that there exist $r_1, r_2 > 0$ such that $K(y, f, Q(x_0, r_1)) = 1$ for every $y \in B(f(x_0), r_2)$ satisfying $K(y, f, Q(x_0, r_1)) < +\infty$.

Now we prove that $\{y \in B(f(x_0), r_2) : K(y, f, Q(x_0, r_1)) = +\infty\} = \emptyset$. By contradiction we suppose that there exists $y \in B(f(x_0), r_2)$ such that $K(y, f, Q(x_0, r_1)) = +\emptyset$. Then let $\{C_i\}_{i \in \mathbb{N}}$ be a sequence of e.m.m.c. for $(y, f, Q(x_0, r_1))$; since $f$ is light, we have $C_i = \{x_i\}$ for every $i \in \mathbb{N}$, and $x_i \in R_y$ for every $i \in \mathbb{N}$. Therefore $R_y \cap Q(x_0, r_1)$ contains a bounded, infinite subset and this implies that $D(R_y) \cap \Omega \neq \emptyset$, which contradicts Lemma 3.4. Hence we have $K(y, f, Q(x_0, r_1)) = 1$ for every $y \in B(f(x_0), r_2)$ and, by Remark 3.3, this implies that $R_y \cap Q(x_0, r_1) = \{x_i\}$ for every $y \in B(f(x_0), r_2)$.

Now let $g(y) = x_i$ for every $y \in B(f(x_0), r_2)$. We prove that $g$ is a homeomorphism and $g = f^{-1}$ in a neighbourhood of $f(x_0)$.
First we show that $g$ is continuous on $B(f(x_0), r_2)$. Let $y \in B(f(x_0), r_2)$ be a limit point of a sequence $\{y_m\}_{m \in \mathbb{N}}$ in $B(f(x_0), r_2)$ and suppose for contradiction that there exists a subsequence $\{x_{j_m}\}_{k \in \mathbb{N}}$ which converges to $x \in \overline{Q(x_0, r_1)}$ and $x \neq x_j$. We have that $x \in f^{-1}(y)$ and therefore $x \in Q(x_0, r_1)$. Hence $x \not\in \mathcal{N}$, and since $f$ is light, this implies that there exists an open neighbourhood $W$ of $x$ such that $\mu(y, f, W) = 0$; consequently, there exists $h \in \mathbb{N}$ such that $\mu(y_{j_m}, f, W) = 0$ for every $k \geq h$. On the other hand there exists an open neighbourhood $V_m$ of $x_{j_m}$ such that $\operatorname{diam}(V_m) < \frac{1}{m}$ and $\mu(y, f, V_m) \geq 1$; then, if we take $h^* \in \mathbb{N}$ such that $h^* > h$ and $V_{m_k} \subset W$ for every $k \geq h^*$ we have

$$\mu(y_{j_m}, f, W) = \mu(y_{j_m}, f, V_{m_k}) + \mu(y_{j_m}, f, W \setminus V_{m_k}) = 0$$

for every $k \geq h^*$. Hence $\mu(y_{j_m}, f, V_{m_k}) = 0$ and this is a contradiction. Therefore $g$ is continuous on $B(f(x_0), r_2)$.

Now we observe that, by the definition of $R_y$, $f(g(y)) = y$ for every $y \in B(f(x_0), r_2)$ and then $g$ is one-to-one. Therefore, by the invariance domain theorem, $g$ is open and then it is a homeomorphism from $B(f(x_0), r_2)$ to $g(B(f(x_0), r_2))$; consequently $f = g^{-1}$ is a homeomorphism from $g(B(f(x_0), r_2))$ to $B(f(x_0), r_2)$. Finally we observe that $g(f(x_0)) = x_0$ ([10], pag 329) and $g(B(f(x_0), r_2))$ is an open neighbourhood of $x_0$. Therefore $x_0 \not\in B_f$.

**Remark 3.6.** Theorems 3.1 and 3.5 are not true without the topological assumptions about $f$, i.e. if $f$ is neither open nor light, as it is shown by the following example: let $k \in \mathbb{N}, I = (-2, 2), I_k = (k^{-1}, k^{-1} + k^{-2}), F = \bigcup_{k \in \mathbb{N}} I_k$ and $E = F \cup (-F)$; it is easily verified that $x_0 = 0$ is a point of density 1 for $E$ (see [4] for the definition). We set

$$g(x) = \int_0^x \chi_E(t) dt \text{ and } f(x, y) = (g(x), y) \text{ if } (x, y) \in I \times I$$

where $\chi_E$ is the characteristic function of $E$. Then $f \in W^{1, \infty}(I \times I; \mathbb{R}^2)$ and $\det Df(x, y) = \chi_E(x) \geq 0$ for almost every $(x, y) \in I \times I$.

Now we observe that $(0, 0) \in B_{y_J} \setminus ((Z_f \cup S_f)$. Indeed, if $k \in \mathbb{N}$, let $J_k = ([k + 1]^{-1} + [k + 1]^{-2}, k^{-1})$; we note that $g'(x) = 0$ for every $x \in \bigcup_{k \in \mathbb{N}} J_k$ and therefore $g$ is constant on each $J_k$. Then, for every $\delta > 0$ we may choose $m \in \mathbb{N}$ such that $J_m \subset (-\delta; \delta)$ and $f(J_m \times \{0\})$ is a point. This implies that $(0, 0) \in B_{y_J}$. On the other hand, since $x_0 = 0$ is a point of density 1 for $E$, we have that $f$ is weakly differentiable in $(0, 0)$ and $\det Df(0, 0) = 1$, that is $(0, 0) \not\in (Z_f \cup S_f)$. Finally, we note that $f$ is neither open nor light; indeed $f(J_k \times I) = \{g(k^{-1})\} \times I$ and $f^{-1}(g(k^{-1}), 0) = J_k \times \{0\}$ for every $k \in \mathbb{N}$. 


4. Local invertibility of Sobolev functions of higher order.

The following lemma concerns a local property of the fibers $f^{-1}(y)$; the proof is similar to the one of Theorem 1, (iv) in [2].

**Lemma 4.1.** Let $f \in W^{1,n}(\Omega;\mathbb{R}^n)$ be a continuous mapping such that $\det Df > 0$ almost everywhere in $\Omega$ and let $x_0 \notin Z_f \cup S_f$.

Then there exist $r_1, r_2 > 0$ such that $f^{-1}(y) \cap Q(x_0, r_1)$ is a continuum for every $y \in B(f(x_0), r_2)$.

**Proof.** Since $x_0 \notin Z_f \cup S_f$, there exist $r_1, r_2 > 0$ such that $Q(x_0, r_1) \subset \subset \Omega$ and $\mu(y, f, Q(x_0, r_1)) = 1$ for every $y \in B(f(x_0), r_2)$ (see the proof of Theorem 3.1).

First we observe that $f^{-1}(y) \cap Q(x_0, r_1)$ is closed. In fact for every $y \in B(f(x_0), r_2)$, $f^{-1}(y) \cap \partial Q(x_0, r_1)$ is empty and since $Q(x_0, r_1) \subset \subset \Omega$ and $f^{-1}(y)$ is closed in $\Omega$, we have $f^{-1}(y) \cap Q(x_0, r_1) \subset Q(x_0, r_1)$ and this implies that $f^{-1}(y) \cap Q(x_0, r_1)$ is closed.

Now let’s suppose by contradiction that $f^{-1}(y) \cap Q(x_0, r_1)$ is not connected and consider a partition of compact sets $\{C_i\}_{i=1,2}$ of $f^{-1}(y) \cap Q(x_0, r_1)$; besides let $A_1, A_2$ be disjoint open sets such that $C_i \subset A_i \subset Q(x_0, r_1) \ (i = 1, 2)$. Since $y \notin f(\partial A_1) \cup f(\partial A_2)$ we have

$$1 = \mu(y, f, Q(x_0, r_1)) = \mu(y, f, A_1) + \mu(y, f, A_2).$$

Without loss of generality we can suppose $\mu(y, f, A_1) \leq 0$. Then there exists $\alpha > 0$ such that $B(y, \alpha) \cap [f(\partial A_1) \cup f(\partial A_2)]$ is empty and $\mu(z, f, A_1) \leq 0$ for every $z \in B(y, \alpha)$; now, $f$ satisfies condition $(N)$ on $\Omega$ ([9], Corollary 3.13) and therefore, as in Theorem 3.1, one can prove that $K(y, f, A_1) < +\infty$ for almost every $y \in \mathbb{R}^n$. Hence the following transformation formula holds ([10], II.3.4., Thm 2 and V.3.4., Thm 1)

$$\int_{A_1} \chi_{B(y, \alpha)}(f(x)) \det Df(x) \, dx = \int_{B(y, \alpha)} \mu(z, f, A_1) \, dz$$

and consequently $\mathcal{L}^n(A_1 \cap f^{-1}(B(y, \alpha))) = 0$. Since $A_1 \cap f^{-1}(B(y, \alpha))$ is open, it follows that $A_1 \cap f^{-1}(B(y, \alpha)) = \emptyset$ and therefore $C_1 \cap f^{-1}(y) = \emptyset$, which is a contradiction.

In the following we suppose that $\Omega$ has a Lipschitz boundary (see [1]).

**Lemma 4.2.** If $f \in W^{2,p}(\Omega;\mathbb{R}^n)$ with $p > \frac{n(n-1)}{2(n-1)}$, then $B_{1,p}(S_f) = 0$.

**Proof.** From the Sobolev inequalities it follows that $f \in W^{1,p}(\Omega;\mathbb{R}^n)$ with $p > n - 1$ and then, according to the proof of Theorem 3.4 in [6], we have
\[
\Omega \setminus S_f \supset \left\{ x \in \Omega : Df(x) \text{ exists and } \lim_{t \to 0^+} \left[ t^{-n} \int_{Q(x; 2t)} \|Df(y) - Df(x)\|^p \, dy + t^{-n-p} \int_{Q(x; 2t)} \|f(y) - f(x) - Df(x)(y - x)\|^p \, dy \right] = 0 \right\}.
\]

Since \( Df \in W^{1,p}(\Omega; \mathbb{R}^n) \), there exists a set \( E \subset \Omega \) such that \( B_{1,p}(E) = 0 \) and

\[
\lim_{t \to 0^+} t^{-n} \int_{Q(x; 2t)} \|Df(y) - Df(x)\|^p \, dy = 0
\]

for every \( x \in \Omega \setminus E \) ([15], Theorem 3.3.3.). Besides, there exists a set \( F \subset \Omega \) such that \( B_{1,p}(F) = 0 \) and

\[
\lim_{t \to 0^+} t^{-n-p} \int_{Q(x; 2t)} \|f(y) - f(x) - Df(x)(y - x)\|^p \, dy = 0
\]

for every \( x \in \Omega \setminus F \) ([15], Theorem 3.4.2.). Finally, if we define \( G = E \cup F \), we obtain \( S_f \subset G \) and \( B_{1,p}(G) = 0 \).

In the next theorem we deal with mappings \( f \) belonging to \( W^{2,p}(\Omega; \mathbb{R}^n) \) with \( p > n - 1 \). We recall that, by the Sobolev inequalities, such mappings have a Hölder continuous representative in their equivalence class; we shall always assume that this representative of \( f \) has been chosen.

**Theorem 4.3.** Let \( p > n - 1 \) and \( f \in W^{2,p}(\Omega; \mathbb{R}^n) \) such that

1. \( \det Df > 0 \) almost everywhere in \( \Omega \).
2. \( B_{1,p}(Z_f) = 0 \).

Then \( B_f \subset Z_f \cup S_f \), \( \dim_H(B_f) \leq n - p \) and \( \dim_H(f(B_f)) \leq \frac{p(n-p)}{2p-n} \).

**Proof.** Let \( x_0 \notin Z_f \cup S_f \); we shall prove that \( x_0 \notin B_f \). First we observe that, by the Sobolev inequalities, \( f \in W^{1,n}(\Omega; \mathbb{R}^n) \) and satisfies the condition (N) ([9], Corollary 3.13); as in the proof of Theorem 3.1, we can show that there exist \( r_1, r_2 > 0 \) and a set \( N \subset B(f(x_0), r_2) \) such that \( \mathcal{L}^n(N) = 0 \) and \( N(y, f, Q(x_0, r_1)) = 1 \) for every \( y \in B(f(x_0), r_2) \setminus N \).

Let \( A_{x_0} = f^{-1}(B(f(x_0), r_2)) \cap Q(x_0, r_1) \) and let’s prove that \( N(y, f, Q(x_0, r_1)) = 1 \) for every \( y \in f(A_{x_0}) \). For this purpose we show that \( f^{-1}(y) \cap Q(x_0, r_1) \subset (Z_f \cup S_f) \cap A_{x_0} \) for every \( y \in N \cap f(A_{x_0}) \).

Let \( y \in N \cap f(A_{x_0}) \) and \( x \in f^{-1}(y) \cap Q(x_0, r_1) \).

First we prove that \( x \in (\Omega \setminus A_{x_0}) \cup (Z_f \cup S_f) \). Let’s suppose by contra-
A Note on the Local Invertibility of Sobolev Functions

diiction that \( x \in A_{x_0} \setminus (Z_f \cup S_f) \). By the Hölder continuity of \( f \), there exists a sequence of positive real numbers \( \{\alpha_n\}_{n \in \mathbb{N}} \) such that \( \{Q(x; \alpha_n)\}_{n \in \mathbb{N}} \) is a determining sequence for \((y, f, Q(x_0, r_1))\) ([10], p.329). This implies that \( \{x\} \) is a component of \( f^{-1}(y) \cap Q(x_0, r_1) \) ([10], II.3.1., Lemma 6); besides \( y \in B(f(x_0), r_2) \) and therefore \( f^{-1}(y) \cap Q(x_0, r_1) \) is a continuum by Lemma 4.1; consequently \( \{x\} = f^{-1}(y) \cap Q(x_0, r_1) \) and \( y \notin N \), which is a contradiction. Then \( x \in (\Omega \setminus A_{x_0}) \cup (Z_f \cup S_f) \).

Finally we show that \( x \in A_{x_0} \). Again by contradiction, let’s suppose \( x \notin A_{x_0} \). Since \( x \in Q(x_0, r_1) \), we have \( x \notin f^{-1}(B(f(x_0), r_2)) \) and this implies that \( x \notin f^{-1}(y) \), otherwise \( f(x) = y \in f(A_{x_0}) \subset B(f(x_0), r_2) \), and this is a contradiction. Then \( x \in A_{x_0} \cap [(\Omega \setminus A_{x_0}) \cup (Z_f \cup S_f)] = A_{x_0} \cap (Z_f \cup S_f) \) and consequently \( f^{-1}(y) \cap Q(x_0, r_1) \subset (Z_f \cup S_f) \cap A_{x_0} \) for every \( y \in N \cap f(A_{x_0}) \).

Now, by Lemma 4.2 and (b), we obtain

\[
B_{1,p}(f^{-1}(y) \cap Q(x_0, r_1)) \leq B_{1,p}(Z_f \cup S_f) = 0
\]

for every \( y \in N \cap f(A_{x_0}) \). Since \( p > n - 1 \), we have

\[
\dim_H(f^{-1}(y) \cap Q(x_0, r_1)) < 1
\]

([15], Th. 2.6.16), and, since \( f^{-1}(y) \cap Q(x_0, r_1) \) is a continuum by Lemma 4.1, then \( \dim(f^{-1}(y) \cap Q(x_0, r_1)) = 0 \), that is \( N(y, f, Q(x_0, r_1)) = 1 \). Therefore we have proved that \( N(y, f, Q(x_0, r_1)) = 1 \) for every \( y \in N \cap f(A_{x_0}) \) and then for every \( y \in f(A_{x_0}) \). This implies that the restriction of \( f \) to \( A_{x_0} \) is one-to-one; by the invariance domain theorem it follows that such restriction is open and then \( f \) is a homeomorphism from \( A_{x_0} \) to \( f(A_{x_0}) \), that is \( x_0 \notin B_f \).

In order to show that \( \dim_H(B_f) \leq n - p \), it is enough to recall that, by (b) and Lemma 4.2, \( B_{1,p}(B_f) \leq B_{1,p}(Z_f) + B_{1,p}(S_f) = 0 \). Hence \( H^{n-p+\epsilon}(B_f) = 0 \) for every \( \epsilon > 0 \) ([15], Theorem 2.6.16) and \( \dim_H(B_f) \leq n - p \).

Finally we observe that, by the Sobolev inequalities, \( f \in C^{(2n-p)/p}(\Omega; R^n) \) and therefore \( H^{(n-p)/2}(f(B_f)) \leq H^{n-p}(B_f) = 0 \) for every \( \epsilon > 0 \) ([12], Theorem 29); consequently \( \dim_H(f(B_f)) \leq \frac{p(n-p)}{2p-n} \).

REFERENCES