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OUTER CONJUGACY CLASSES OF TRACE SCALING AUTOMORPHISMS OF STABLE UHF ALGEBRAS

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Introduction.

Connes [C] classified automorphisms of the II_∞ approximately finite-dimensional (AFD) factor $R_{0,1}$ up to outer conjugacy. An automorphism θ of $R_{0,1}$ determines a modulus or scaling factor λ on a trace $\tau, \tau\theta = \lambda\tau$. It follows from this classification and [CT] that automorphisms which do not leave a trace invariant are classified up to conjugacy by their modulus; for each $\lambda \in (0,1)$ there is up to conjugacy a unique automorphism θ_λ of $R_{0,1}$ which scales the trace by λ. Consequently, for such λ, there is a unique factor of type III_λ, the Powers factor R_λ, whose associated type II_∞ factor is $R_{0,1}$; necessarily, $R_\lambda = R_{0,1} \rtimes_{\theta_\lambda} \mathbf{Z}$.

A key idea for model building in Connes's work on the classification of automorphisms of AFD factors [C] and in subsequent work of Jones [J], Ocneanu [O], and Kawahigashi, Sutherland and Takesaki [KST] on amenable group actions is a non-commutative Rohlin lemma for an aperiodic automorphism of a finite von Neumann algebra. Recall that an automorphism θ of a von Neumann algebra M is said to be properly outer if the restriction to M_e is outer for each non-zero invariant projection e, and aperiodic if every non-zero power is properly outer. Connes showed that an automorphism θ of M is properly outer if and only if for each non-zero projection e in M, and $\varepsilon > 0$, there exists a non-zero projection f in M such that $\|f\theta(f)\| < \varepsilon$ and $f \le e$. With an abundance of such projections, whose translates are almost orthogonal, Connes could deduce a non-commutative Rohlin lemma. If θ is an aperiodic automorphism of a finite (not necessarily separable) von Neumann algebra leaving invariant a normalised faithful normal trace τ, then for each positive integer n and each $\varepsilon > 0$, there exist projections $F_1, \cdots, F_n$ in M such that $\sum_{j=1}^n F_j = 1$ and $\|\theta(F_j) - F_{j+1}\|_2 \le \varepsilon$, $j = 1, \ldots, n$, where $F_{n+1} = F_1$ and $\|x\|_2 = \tau(x^*x)^{\frac{1}{2}}$.

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The Rohlin lemma can be applied to study an automorphism θ of a von Neumann algebra M which is not necessarily finite through the algebra of central sequences which may be of type II_1. If ω is an ultrafilter on N, let M_ω denote the asymptotic centraliser (see [C]), the von Neumann algebra of centralising sequences (bounded sequences (x_n) such that $\lim_{n\to\omega}\|[x_n,\psi]\| = 0$ for all $\psi \in M_*$), modulo ω-null sequences (bounded sequences converging$*$-strongly to zero).Then $M \cong M \otimes R_0$ if and only if M_ω is of type II_1 (in this case M is said to be strongly stable), where R_0 is the hyperfinite II_1 factor, and the Rohlin lemma may be applied to the induced automorphism θ_ω of M_ω. From the Rohlin lemma, one can deduce a stability or 1-cohomology property. In the algebra of sequences, this takes the form that if θ is an automorphism of a strongly stable factor and $\theta_\omega^n \neq 1$ for all $n \neq 0$, then θ_ω is stable in the sense that any unitary of M_ω is of the form $v\theta_\omega(v^*)$ for some unitary v in M_ω.

These ideas concerning the Rohlin lemma and stability were taken over to a C^*-algebraic setting in [HJ1], [HJ2], and [HO] to classify product cyclic actions on UHF C^*-algebras A up to outer conjugacy and conjugacy. In this case the appropriate sequence algebra A^∞ is $l^\infty(\mathsf{N}, A)$ modulo sequences which converge to zero in norm, and the algebra of central sequences is $A_\infty = A^\infty \cap A'$. These ideas concerning the Rohlin property and stability were taken further in [R1], [R2], in the programme to classify separable amenable C^*-algebras by K-theoretic invariants.

The Rohlin property on which these and later articles were based was that established in [BKRS] for the shift on the Pauli algebra$M_{2^\infty} = \bigotimes_{\mathsf{Z}} M_2$. This was obtained via quasi-free techniques, applied to the quasi-free shift on the (even) Fermion algebra. The Rohlin property for the shift on M_{2^∞} immediately implies the Rohlin property for the shift on the UHF algebra $M_{(2n)^\infty} = M_{2^\infty} \otimes M_{n^\infty}$. A weak Rohlin property was established in [BEK1] for the shift on an odd UHF algebra M_{m^∞} (where m is odd) via the embedding [CE] of the gauge-invariant Fermion algebra in the UHF algebra M_{m^∞}, which is compatible with the shift. In [K1], it was shown that this weaker form of the Rohlin property actually guaranteed the Rohlin property itself for the shift on M_{m^∞} for an arbitrary m.

Furthermore, it was shown in [K2] that an automorphism of an arbitrary UHF algebra has the Rohlin property if and only if every non-zero power is uniformly outer (i.e., the weak extension is outer in the tracial representation), or, equivalently, the crossed product $A \rtimes \mathsf{Z}$ has a unique tracial state, or the crossed product is of real rank zero. It was also shown that all such automorphisms are outer conjugate (e.g. to an infinite tensor product automorphism$\left(\bigotimes_{n=1}^\infty M_{m_n}, \bigotimes_{n=1}^\infty \mathrm{Ad}\, U_n\right)$ with the eigenvalues of $\bigotimes_{n=l}^\infty U_n$ being uniformly distributed for any l). Hence the crossed products are all isomorphic and $A\mathsf{T}$ – inductive limits of direct sums of matrix algebras over the continuous functions on the circle.

Here we shall show that trace scaling automorphisms of a stable UHF algebra, with the same non-trivial scaling factor, are outer conjugate. An automorphism of a stable UHF algebra $B \otimes \mathcal{K}$, with B a unital UHF algebra, which scales the trace by $p/q \neq 1$ with p and q relatively prime, induces a partial endomorphism γ, an isomorphism of $B \cap C_1'$ onto $B \cap C_2'$ where C_1 and C_2 are subalgebras of B isomorphic to M_p and M_q. This partial endomorphism γ extends to a partial endomorphism of R, the hyperfinite factor generated by B in its tracial representation, taking $R \cap C_1'$ to $R \cap C_2'$. This determines an automorphism γ_ω of R_ω. Lemma 1 shows that γ is outer in the sense that for each full matrix algebra N of R containing C_1, every non-zero projection f in $R \cap N'$, and any unitary U in R the following holds:

$$\inf\left\{\|eU\gamma(e)\|; 0 \neq e \leq f\,, e = e^2 = e^* \in R \cap N'\right\} = 0\,.$$

It then follows (Lemma 2) that any non-zero power of the automorphism γ_ω is properly outer, that the partial endomorphism γ of the von Neumann algebra R has the Rohlin property (Lemma 3), and, hence, that the partial endomorphism γ on the C^*-algebra B has the Rohlin property (Lemma 4). From this, stability of the partial endomorphism γ on B can be deduced (Lemma 5), from which γ is seen to be outer conjugate with $\gamma \otimes \sigma$ for any automorphism σ of the UHF algebra $M_{(pq)^\infty} = B_0$ (note $B \cong B \otimes B_0$). Arguments parallel to those in the von Neumann algebra setting then yield that all trace scaling automorphisms of the stable UHF algebra A, with the same non-trivial scaling factor $p/q \neq 1$, are outer conjugate. The classification of simple purely infinite amenable C^*-algebras which satisfy the UCT ([R2], [Kir], [Phi]) shows that if $A = M_{(pq)^\infty} \otimes \mathcal{K}$ the crossed product $A \rtimes \mathsf{Z}$ is isomorphic to the stable Cuntz algebra $O_{|p-q|+1} \otimes \mathcal{K}$, if p and q are relatively prime.

The arguments here in fact establish the stronger assertion (Corollary 8) that unital endomorphisms of a UHF algebra with image the commutant of a non-trivial matrix algebra are classified up to outer conjugacy by the order of that algebra, and similarly for unital endomorphisms of the hyperfinite factor of type II_1.

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The Main Results.

Let A be a stable UHF algebra; that is, let A be a C^*-algebra isomorphic to $B \otimes \mathcal{K}$ where B is a unital UHF algebra and $\mathcal{K}$ is the algebra of compact operators on an infinite-dimensional separable Hilbert space. There is a densely defined lower semi-continuous trace τ on A; τ is unique up to constant multiple. Let α be an automorphism of A. We define $s(\alpha) \in \mathsf{R}$ by $\tau\alpha = s(\alpha)\tau$. It follows that $s(\alpha)$ is a positive rational number and the map α_* on $K_0(A)$ is multiplication by $s(\alpha)$, identifying $K_0(A)$ with a dense subgroup of R containing Z. We shall say that automorphisms α and β of A are outer conjugate if there is an automorphism σ of A and a unitary U in the multiplier algebra $M(A)$ of A such that $\alpha = (\mathrm{Ad}\, U)\sigma\beta\sigma^{-1}$. If α and β are outer conjugate then $s(\alpha) = s(\beta)$.

Let α be an automorphism of A such that $s(\alpha) \neq 1$. Let $s(\alpha) = p/q$ where p and q are relatively prime positive integers. Since $K_0(A) = (p/q)K_0(A)$, it follows that

$$K_0(A) = K_0(A) \otimes \mathsf{Z}[1/pq]\,.$$

That is, A is isomorphic to $A \otimes M_{(pq)^\infty}$ where M_{n^∞} is the (unital) UHF algebra of type n^∞.

Let e be a projection in A whose equivalence class is 1 in $K_0(A)$, and set $B = eAe$. Then $A \cong B \otimes \mathcal{K}$ and $B \cong M_{(pq)^\infty} \otimes D$, where D is a UHF algebra (or a matrix algebra) such that $K_0(D)$ is not divisible by p or q. Let us identify B with $M_{(pq)^\infty} \otimes D$ and choose a subalgebra C_1 of $M_{(pq)^\infty} \otimes 1$ isomorphic to M_p and a subalgebra C_2 of $M_{(pq)^\infty} \otimes 1$ isomorphic to M_q. Set $B_i = B \cap C_i'$. Then B_1 and B_2 are isomorphic to B. Let us identify B with $B_1 \otimes C_1$ and with$B_2 \otimes C_2$.

Since the minimal projections in $C_1 \otimes \mathcal{K}$ $(\subset B \otimes \mathcal{K})$ correspond to $1/p$ in $K_0(A)$, the minimal projections in $\alpha(C_1 \otimes \mathcal{K})$ $(\subset B \otimes \mathcal{K})$ correspond to $1/q$. Hence there is a unitary U in $M(B \otimes \mathcal{K})$ such that

$$(\mathrm{Ad}\, U)\alpha(C_1 \otimes \mathcal{K}) = C_2 \otimes \mathcal{K}\,.$$

Let us denote by $\overline{\alpha}$ the extension of α to an automorphism of $M(A)$. Then it follows that $(\mathrm{Ad}\, U)\,\overline{\alpha}$ maps B_1 $(\cong B_1 \otimes 1 \subset M(B \otimes \mathcal{K}))$ onto B_2. Set

$$\theta = (\mathrm{Ad}\, U)\alpha | C_1 \otimes \mathcal{K}\,,$$
$$\gamma = (\mathrm{Ad}\, U)\overline{\alpha} | B_1\,.$$

Then θ is an isomorphism of $C_1 \otimes \mathcal{K}$ onto $C_2 \otimes \mathcal{K}$ – note that such an isomorphism is unique up to inner automorphisms (arising from unitaries in $M(C_2 \otimes \mathcal{K})$) – and γ is an isomorphism of B_1 onto B_2. Note that $(\mathrm{Ad}\, U)\alpha$ is obtained as follows :

$$B \otimes \mathcal{K} = B_1 \otimes (C_1 \otimes \mathcal{K}) \xrightarrow{\gamma \otimes \theta} B_2 \otimes (C_2 \otimes \mathcal{K}) = B \otimes \mathcal{K}.$$

Lemma 1. *Let R denote the AFD factor of type II_1 and let C_1 and C_2 be unital finite-dimensional subalgebras of R such that $C_1 \cong M_p$ and $C_2 \cong M_q$ with $p \neq q$. Let γ be an isomorphism of $R \cap C_1'$ onto $R \cap C_2'$. Then for any full matrix subalgebra N of R with $N \supset C_1$, for any non-zero projection f in $R \cap N'$ and for any unitary U in R it follows that*

$$\inf\{\|eU\gamma(e)\|; e \leq f, 0 \neq e = e^2 = e^* \in R \cap N'\} = 0.$$

Proof. Let θ be an isomorphism of $C_1 \otimes I_\infty$ onto $C_2 \otimes I_\infty$ where I_∞ is a factor of type I_∞. Then one can define an automorphism α of the von Neumann algebra tensor product $R \otimes I_\infty$ by

$$R \otimes I_\infty = R \cap C_1' \otimes C_1 \otimes I_\infty \xrightarrow{\gamma \otimes \theta} R \cap C_2' \otimes C_2 \otimes I_\infty = R \otimes I_\infty.$$

Since $\tau\alpha = (p/q)\tau$ where τ is a trace on $R \otimes I_\infty$, α is (properly) outer.

Let N, f and U be given. Replacing γ by $(\mathrm{Ad}\, U)\gamma$ and C_2 by $(\mathrm{Ad}\, U)C_2$ we may suppose that $U = 1$. Let p_i (resp. p) be a minimal projection in C_i (resp I_∞). We may assume that $\theta(p_1 \otimes p) = p_2 \otimes p$. Let q_1 be a minimal projection in $N \cap C_1'$ and set F equal to $fp_1q_1 \otimes p$, which is a projection in $R \otimes I_\infty$. Let a projection $E \in R \otimes I_\infty$ with $E \leq F$ be given. Necessarily, E is of the form $ep_1q_1 \otimes p$ where e is a projection in $R \cap N'$. If $N \cong M_{rp}$, then there is a family $(V_{1k}; k = 1, \ldots, rp)$ of unitaries in N such that

$$\sum_k V_{1k} p_1 q_1 V_{1k}^* = 1.$$

There is also a family $(V_{2k}; k = 1, \ldots, rq)$ of unitaries in $\gamma(N \cap C_1') \vee C_2$ such that

$$\sum_k V_{2k} p_2 q_2 V_{2k}^* = 1,$$

where $q_2 = \gamma(q_1)$. It follows that

$$\sum_{k,l} V_{1k} E V_{1k}^* V_{2l} \alpha(E) V_{2l}^* = e\gamma(e) \otimes p.$$

Hence,

$$\|e\gamma(e)\| \leq pqr^2 \max\{\|EV_{1k}^* V_{2l}\alpha(E)\|; k = 1, \ldots, rp\,, l = 1, \ldots, rq\}\,.$$

By [C, 1.2.2], the infimum of the right hand side over non-zero projections $E \in R \otimes I_\infty$ with $E \leq F$ is zero – as α is outer. The infimum of the left hand side over non-zero projections $e \in R \cap N$ with $e \leq f$ is therefore also zero, as desired.

REMARK. In the situation of Lemma 1 there is a unitary $U_1 \in R$ such that $\mathrm{Ad}\,U_1(C_2) \subset C_1'$. Then $\gamma(\mathrm{Ad}\,U_1)\gamma$ is defined as an isomorphism of $R \cap C_{11}'$ onto $R \cap C_{21}'$ where $C_{11} = C_1 \vee ((\mathrm{Ad}\,U_1)\gamma)^{-1}(C_1) \cong M_p \otimes M_p$ and $C_{21} = C_2 \vee \gamma(\mathrm{Ad}\,U_1)(C_2) \cong M_q \otimes M_q$. Thus, $\gamma(\mathrm{Ad}\,U_1)\gamma$ also satisfies the conclusion of Lemma 1. Similarly one can find unitaries U_i such that $\gamma(\mathrm{Ad}\,U_{n-1})\gamma(\mathrm{Ad}\,U_{n-2})\cdots(\mathrm{Ad}\,U_1)\gamma$ is well defined as an isomorphism of $R \cap C_{1n}'$ onto $R \cap C_{2n}'$ where $C_{1n} \cong M_{p^n}$ and $C_{2n} \cong M_{q^n}$. As in [C] and in the introduction, define, for a free ultrafilter ω on N, the asymptotic centraliser R_ω, which is a finite von Neumann algebra. Then the partial endomorphism γ of R defines an *automorphism* γ_ω of R_ω, and it follows that

$$(\gamma(\mathrm{Ad}\,U_{n-1})\gamma\cdots(\mathrm{Ad}\,U_1)\gamma)_\omega = \gamma_\omega^n\,,$$

independently of the choice of $U_1, U_2, \ldots, U_{n-1}$.

LEMMA 2. *Let γ be as in Lemma 1 and let γ_ω, R_ω be as above. Then γ_ω^n is properly outer for any $n \neq 0$.*

PROOF. Let N be a full matrix subalgebra of R such that $N \supset C_1$. As in the proof of [C, 1.2.5], for a fixed $\delta > 0$ denote by Q the set of couples (e, V) such that

(a) e is a projection in $R \cap N'$,

(b) V is a unitary in R with $\|V - 1\|_1 \leq \delta\tau(e)$,

(c) $V\gamma(e)V^*$ is a projection in $R \cap N'$ and $V\gamma(e)V^*e = 0$.

We define an order on Q as follows : $(e, V) \leq (e', V')$ if

(1) $e \leq e'$,

(2) $\|V' - V\|_1 \leq \delta\tau(e' - e)$.

One can show that $(Q, \leq)$ is inductive, and, by using Lemma1, that a maximal element (e, V) satisfies $e + V\gamma(e)V^* = 1$. Since

$$\begin{aligned}\tau((e-\gamma(e))^2) &= \tau(e+\gamma(e)-e\gamma(e)-\gamma(e)e)\\ &= 1-2\tau(e\gamma(e)e)\\ &= 1-2\tau(e(1-V)\gamma(e)e)\,,\end{aligned}$$

and

$$\begin{aligned}|\tau(e(1-V)\gamma(e)e)| &\le \tau((1-V)(1-V)^*)^{\frac{1}{2}}\tau(e\gamma(e)e)^{\frac{1}{2}}\\ &\le \sqrt{2}\tau(|1-V|)^{\frac{1}{2}}\tau(e)^{\frac{1}{2}}\\ &= \tau(|1-V|)^{\frac{1}{2}}\\ &\le \left(\frac{\delta}{2}\right)^{\frac{1}{2}},\end{aligned}$$

we obtain

$$\|e-\gamma(e)\|_2 \ge \left(1-\sqrt{2\delta}\right)^{\frac{1}{2}}.$$

Hence if $\delta = 1/8$, $\|e-\gamma(e)\|_2 \ge 1/\sqrt{2}$. For an increasing sequence (N_n) of full matrix subalgebras of R such that $C_1 \subset N_1$ and $\bigcup_{n=1}^{\infty} N_n$ is dense in R, we choose $e_n \in R \cap N_n'$ as above, and set $e = (e_n) \in R_\omega$. Then it follows that $\|e-\gamma_\omega(e)\|_2 \ge 1/\sqrt{2}$. Thus we have that γ_ω is properly outer as in the proof of [C, 2.1.2]. By the Remark above, this argument also applies to any non-zero power of γ_ω.

LEMMA 3. *In the situation of the previous lemma, γ has the Rohlin property, i.e., for any $n \in \mathsf{N}$, $\varepsilon > 0$, and finite subset S of $R \cap C_1'$ there exists a partition $(F_0, \dots, F_{n-1})$ of unity by projections in $R \cap C_1'$ such that*

$$\begin{aligned}&\|\gamma(F_i) - F_{i+1}\|_2 < \varepsilon\,,\\ &\|[x, F_i]\|_2 < \varepsilon\end{aligned}$$

for $i = 0, \dots, n-1$ with $F_n = F_0$ and for $x \in S$, where $\|y\|_2 = \tau(y^*y)^{\frac{1}{2}}$ for $y \in R$.

PROOF. This follows from [C, 1.2.5] in view of [C, 1.1.3].

LEMMA 4. *Let B be a* UHF *algebra and let C_1 and C_2 be unital finite-dimensional subalgebras of B such that $C_1 \cong M_p$ and $C_2 \cong M_q$ with $p \neq q$. Let γ be an isomorphism of $B \cap C_1'$ onto $B \cap C_2'$. Then γ has the Rohlin property, i.e., for any n, $\varepsilon > 0$, and finite subset S of $B \cap C_1'$ there is a partition $(F_i; i = 0, 1, \dots, (pq)^n - 1)$ of unity of projections in $B \cap C_1'$ such that*

$$\|\gamma(F_i) - F_{i+1}\| < \varepsilon\,,$$

$$\|[x, F_i]\| < \varepsilon$$

for $i = 0, 1, \ldots, (pq)^n - 1$ with $F_{(pq)^n} = F_0$ and for $x \in S$.

PROOF. Let N be a full matrix subalgebra of B with $N \supset C_1$. For any n and $\varepsilon > 0$ we have to construct a partition $(F_i; i = 0, 1, \ldots, (pq)^n - 1)$ of unity in $B \cap N'$ such that

$$\|\gamma(F_i) - F_{i+1}\| < \varepsilon$$

for $i = 0, 1, \ldots, (pq)^n - 1$. By having Lemma 3 at hand this can be done exactly as in [K2].

LEMMA 5. *In the situation of the previous lemma, let* (U_n) *be a central sequence of unitaries in* B. *Then there exists a central sequence* (V_n) *of unitaries in* $B \cap C_1'$ *such that*

$$\left\|\gamma(V_n^*)V_n - U_n\right\| \to 0\,.$$

PROOF. Set $A^\infty = l^\infty(\mathsf{N}, A)/c_0(\mathsf{N}, A)$ and imbed A into A^∞ as the constant sequences. Set $A_\infty = A^\infty \cap A'$. Then γ defines an automorphism γ_∞ of A_∞ and by Lemma 4, γ_∞ has the Rohlin property. Hence by [HO] we have a unitary $v \in A_\infty$ for $u = (U_n) \in A_\infty$ such that

$$\gamma_\infty(v^*)v = u\,.$$

We may represent v as a sequence (V_n) of unitaries in $B \cap C_1'$. This completes the proof.

LEMMA 6. *In the situation of Lemma 4, consider in addition the UHF algebra* $B_0 = M_{(pq)^\infty}$ *and let* σ *be an automorphism of* B_0. *Then* γ *and* $\gamma \otimes \sigma$ *are outer conjugate; that is, there is an isomorphism* φ *of* $B \otimes B_0$ *onto* B *such that* $\varphi(C_1 \otimes 1) = C_1$ *and*

$$(\mathrm{Ad}\, U)\gamma = \varphi(\gamma \otimes \sigma)\varphi^{-1} \ \textit{on}\ B \cap C_1'$$

for some unitary $U \in B$.

PROOF. First note that $B \cong B \otimes B_0$. Therefore, we may choose a central sequence $(e_{ij}^n; i, j = 1, \ldots, pq)$ of $pq \times pq$ matrix units in $B \cap C_1'$. Choose an element λ of T of infinite order. Since γ takes $B \cap C_1'$ onto $B \cap C_2'$, there exists a central sequence (v_n) of unitaries in B such that

$$\gamma(e_{ij}^n) = \lambda^{i-j} v_n e_{ij}^n v_n^*\,.$$

By Lemma 5, there exists a central sequence (w_n) of unitaries in $B \cap C_1'$ such that $\|\gamma(w_n^*)w_n - v_n\| \to 0$. Set $f_{ij}^n = w_n e_{ij}^n w_n^*$. Then $(f_{ij}^n : i,j = 1,\ldots,pq)$ is a central sequence of systems of $pq \times pq$ matrix units in $B \cap C_1'$, and

$$\gamma(f_{ij}^n) - \lambda^{i-j} f_{ij}^n \to 0\,.$$

We may choose a subsequence of (f_{ij}^n) and modify it slightly so that the resulting sequence (f_{ij}^n) consists of mutually commuting systems of matrix units and there is a unitary $U \in B$ with

$$(\mathrm{Ad}\, U)\gamma(f_{ij}^n) = \lambda^{i-j} f_{ij}^n\,.$$

This implies that B has a factorization $B_1 \otimes B_0$ with respect to which $(\mathrm{Ad}\, U)\gamma$ factorizes as $\gamma_1 \otimes \sigma_0$, where γ_1 is a partially defined endomorphism of B_1 of the same type as γ and σ_0 is the automorphism of B_0 consisting of the infinite tensor product of copies of

$$\mathrm{Ad}\begin{pmatrix} 1 & & & 0 \\ 0 & \lambda & & 0 \\ & & \ddots & \\ 0 & & & \lambda^{pq-1} \end{pmatrix}.$$

Since σ_0 has the Rohlin property ([BEK2], [K1]), for any automorphism σ of B_0 there is (by [K2]) a unitary V in $B_0 \otimes B_0$ such that

$$\sigma_0 \cong \mathrm{Ad}\, V(\sigma_0 \otimes \sigma)\,,$$

where $\cong$ denotes conjugacy. This implies that

$$(\mathrm{Ad}\, U)\gamma \cong \mathrm{Ad}(1 \otimes V)(\gamma_1 \otimes \sigma_0 \otimes \sigma) \cong \mathrm{Ad}(\widetilde{V}(U \otimes 1))\gamma \otimes \sigma$$

where $\widetilde{V}$ denotes the image of $1 \otimes V$ under the second isomorphism. This completes the proof.

THEOREM 7. *Let A be a stable* UHF *algebra and let α, β be automorphisms of A. If $s(\alpha) = s(\beta) \neq 1$, then α and β are outer conjugate.*

PROOF. Let $A \cong B \otimes \mathcal{K}$ with B a unital UHF algebra and write $s(\alpha) = p/q$ with p and q relatively prime positive integers. As shown above we may assume that α has the decomposition

$$A = B \otimes \mathcal{K} = B \cap C_1' \otimes (C_1 \otimes \mathcal{K}) \xrightarrow{\gamma \otimes \theta} B \cap C_2' \otimes (C_2 \otimes \mathcal{K}) = A\,,$$

where γ is an isomorphism of $B \cap C_1'$ onto $B \cap C_2'$, $C_1 \cong M_p$, $C_2 \cong M_q$, and θ is an isomorphism of $C_1 \otimes \mathcal{K}$ onto $C_2 \otimes \mathcal{K}$. We may identify $\mathcal{K}$ with $\mathcal{K} \otimes \mathcal{K}$. Since θ is outer conjugate to $\theta \otimes \mathrm{id}_{\mathcal{K}}$, this implies that α is outer conjugate to $\alpha \otimes \mathrm{id}_{\mathcal{K}}$.

By Lemma 6 applied to γ, α is outer conjugate to $\alpha \otimes \sigma$ for any automorphism σ of $B_0 = M_{(pq)^\infty}$.

Let ν be an automorphism of $B_0 \otimes \mathcal{K}$ such that $s(\nu) = s(\alpha)$. Then $s(\alpha \otimes \nu^{-1}) = 1$, and so $\alpha \otimes \nu^{-1}$ is outer conjugate to $\sigma \otimes \mathrm{id}_{\mathcal{K}}$ on $B \otimes \mathcal{K}$ for some automorphism σ of B. By Lemma 4 applied to γ, it follows that σ has the Rohlin property and so $\alpha \otimes \nu^{-1}$ is outer conjugate to $\sigma_0 \otimes \mathrm{id}_B \otimes \mathrm{id}_{\mathcal{K}}$ on $B_0 \otimes B \otimes \mathcal{K}$ with σ_0 an automorphism of B_0 with the Rohlin property. Similarly, $\nu \otimes \nu^{-1}$ is outer conjugate to $\sigma_0 \otimes \mathrm{id}_{\mathcal{K}}$. Thus we have that

$$\begin{aligned}\alpha &\sim \alpha \otimes \mathrm{id}_{\mathcal{K}} \sim \alpha \otimes \sigma_0 \otimes \mathrm{id}_{\mathcal{K}} \sim \alpha \otimes \nu \otimes \nu^{-1} \otimes \mathrm{id}_{\mathcal{K}} \\ &\sim \nu \otimes \sigma_0 \otimes \mathrm{id}_B \otimes \mathrm{id}_{\mathcal{K}} \sim \nu \otimes \sigma_0 \otimes \mathrm{id}_B \sim \nu \otimes \mathrm{id}_B\,,\end{aligned}$$

where outer conjugacy is denoted by $\sim$. Since the right hand side does not depend on α, we have the assertion.

Remark. Let A be a stable UHF algebra and let α be an automorphism of A with $s(\alpha) \neq 1$. Then by Theorem 7 the isomorphism class of the crossed product $A \rtimes_\alpha \mathsf{Z}$ depends only on $s(\alpha)$. Since $A \rtimes_\alpha \mathsf{Z}$ is purely infinite [R2], this fact also follows from [R2], [Kir] and [Phi], by using the Pimsner-Voiculescu exact sequence. When $A = M_{(pq)^\infty} \otimes \mathcal{K}$ and $s(\alpha) = p/q$, where p, q are relatively prime, calculation yields that

$$K_0(A \rtimes_\alpha \mathsf{Z}) = \mathsf{Z}/(p-q)\mathsf{Z}\,, \qquad K_1(A \rtimes_\alpha \mathsf{Z}) = 0.$$

Hence by [Kir] and [Phi], $A \rtimes_\alpha \mathsf{Z}$ is isomorphic to the stable Cuntz algebra $O_{|p-q|+1} \otimes \mathcal{K}$.

Corollary 8. *Let α and β be unital endomorphisms of a* UHF *algebra B such that* $\operatorname{Im}\alpha = B \cap B'_\alpha$ *and* $\operatorname{Im}\beta = B \cap B'_\beta$ *where B_α and B_β are unital subalgebras of B with $B_\alpha \cong M_p \cong B_\beta$ with $p \geq 2$. Then α and β are outer conjugate.*

Proof. Denote by σ the unilateral shift on M_{p^∞} and by B_σ the first copy of M_p in M_{p^∞}. Then $\operatorname{Im}\sigma = M_{p^\infty} \cap B'_\sigma$ and σ is an endomorphism of M_{p^∞} of the type considered in the corollary.

We define an automorphism (α^{-1}, σ) of $B \otimes M_{p^\infty}$ by

$$B \otimes M_{p^\infty} = \alpha(B) \otimes B_\alpha \otimes M_{p^\infty} \xrightarrow{\alpha^{-1} \otimes \theta_{\alpha\sigma} \otimes \sigma} B \otimes B_\sigma \otimes \sigma(M_{p^\infty}) = B \otimes M_{p^\infty}$$

where $\theta_{\alpha\sigma}$ is an isomorphism of B_α onto B_σ. Thus, (α^{-1}, σ) is defined uniquely up to inner automorphisms. Similarly we define an automorphism (α, σ^{-1}) of $B \otimes M_{p^\infty}$, which is the inverse of (α^{-1}, σ) up to inner automorphisms, and also an automorphism (σ^{-1}, σ) of $M_{p^\infty} \otimes M_{p^\infty}$.

Since, by Lemma 4, σ has the Rohlin property, (σ^{-1}, σ) also has the Rohlin property and so by [K2] is outer conjugate to any other automorphism of M_{p^∞} with the Rohlin property. Hence by Lemma 6,

$$\alpha \sim \alpha \otimes (\sigma^{-1}, \sigma) \sim (\alpha, \sigma^{-1}) \otimes \sigma \sim \sigma \otimes \mathrm{id}_B\,,$$

where the second outer conjugacy consists of exchanging B_σ in the last factor and B_α in the first factor of the factorisation $B \otimes M_{p^\infty} \otimes M_{p^\infty}$, which is an inner automorphism. Since the right hand side does not depend on α, this completes the proof.

Corollary 9. *If α and β are unital endomorphisms of the hyperfinite type* II_1 *factor R such that $\operatorname{Im}\alpha = R \cap R'_\alpha$ and $\operatorname{Im}\beta = R \cap R'_\beta$ for some unital finite-dimensional subfactors R_α and R_β of R, then α and β are outer conjugate if and only if $R_\alpha \cong R_\beta$.*

Proof. This is proved in exactly the same way as the previous corollary.

Remark. If M is a type I_∞ factor and α is a unital normal endomorphism of M, then $M_\alpha = M \cap \alpha(M)'$ is of type I_n for some n and $\alpha(M) = M \cap M'_\alpha$. The outer conjugacy class of α is uniquely determined by n.

Remark. Let M be a properly infinite factor and let α be a unital normal endomorphism of M such that $\alpha(M) = M \cap M'_\alpha$ for some type I subfactor M_α of M. Suppose that M_α is of type I_{n_α} (with n_α countable). Then there exist n_α isometries (s_i) in M such that $(s_i s_j^*)$ is a system of matrix units for M_α, and α is given by

$$\alpha(x) = \sum s_i \gamma(x) s_i^*$$

for some automorphism γ of M. For a different choice of (s_i), γ differs only by an inner automorphism determined by a unitary in M_α. Denote the class containing γ in $\operatorname{Aut} M/\operatorname{Int} M$ by $\hat{\alpha}$. Then we can assert: Two such endomorphisms α, β of M are outer conjugate if and only if $n_\alpha = n_\beta$ and $\hat{\alpha}$ is conjugate to $\hat{\beta}$. (If M is of type III or type II_∞, it is known [C] that there is an abundance of outer conjugacy classes in Aut M; so the situation is different from the one in Corollary 9.) For example if α and γ are as above and σ is another automorphism of M, then

$$\sum s_i \sigma\gamma\sigma^{-1}(x) s_i^* = (\operatorname{Ad} V)\sigma\alpha\sigma^{-1}(x)$$

where V is the unitary defined by

$$V = \sum s_i \sigma(s_i)^* .$$

The other computations are as easy as this.

Remark. With σ the unilateral shift endomorphism of the UHF algebra M_{2^∞}, define an endomorphism α of M_{2^∞} by

$$\alpha(x) = e_1\sigma(x) + e_2\sigma^2(x) ,$$

where e_1, e_2 is a partition of unity by projections in the first copy of M_2 in M_{2^∞}. Note that the relative commutant of $\operatorname{Im}\alpha$ is isomorphic to $\mathsf{C} \oplus M_2$, and that α does not have the Rohlin property. For if it did, then there would be a unitary $u \in M_{2^\infty}$ such that u commutes with M_2 and $\alpha(u) \approx -u$, which implies that $\sigma(u) \approx -u$ and $\sigma^2(u) \approx -u$, a contradiction. When γ is an automorphism of M_{2^∞} such that $\sigma\gamma = \gamma\sigma$ and the action of $\mathsf{N} \times \mathsf{Z}$ defined by $(m, n) \to \sigma^m\gamma^n$ satisfies the Rohlin property (cf. Remark 4 of [KK]), define an endomorphism β of M_{2^∞} by

$$\beta(x) = e_1\sigma(x) + e_2\gamma\sigma^2(x).$$

Then it follows that $M_{2^\infty} \cap (\operatorname{Im}\beta)' = M_{2^\infty} \cap (\operatorname{Im}\alpha)'$ and that β has the Rohlin property. Compare with Lemma 4 and Corollary 8.

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