K-THEORY FOR C*-ALGEBRAS ASSOCIATED WITH
SUBSHIFTS

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Abstract.
We present K-theory formula for C*-algebras associated with subshifts. The formula is a generalization of K-theory formula for Cuntz-Krieger algebras, which are associated with topological Markov shifts. The dimension group for a general subshift is introduced to be the dimension group for the associated AF-algebra.

1. Introduction.
In [Ma], the author has introduced and studied a class of C*-algebras associated with subshifts in the theory of symbolic dynamics. The class of C*-algebras is a generalized one of the Cuntz-Krieger algebras which are associated with topological Markov shifts. Each of the C*-algebras associated with subshifts has generators of partial isometries with mutually orthogonal ranges. It also has universal properties subject to some operator relations ([Ma; Theorem 4.9 and 5.2]) so that it becomes purely infinite and simple in many cases including Cuntz-Krieger algebras. It is an analogy to the Cuntz-Krieger algebras that AF-subalgebras are appeared inside of the C*-algebras as the algebras of all fixed points of certain one-parameter group actions, called gauge actions. However, these AF-subalgebras have more complicated structure than the AF-subalgebras appeared inside of the Cuntz-Krieger algebras.

For a subshift (A, σ), we denote by ℂ_A and F_A the C*-algebra associated with the subshift (A, σ) and the corresponding AF-subalgebra inside of it respectively. If a subshift is a topological Markov shift, then the K_0-group of the AF-subalgebra, as an ordered group, becomes the dimension group for the topological Markov shift considered in [Kr1] and [Kr2]. Hence for a general subshift, it seems to be natural to define "the dimension group" for a

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subshift \((A, \sigma)\) as the \(K_0\)-group \(K_0(\mathcal{F}^\infty_A)\) of the AF-algebra \(\mathcal{F}^\infty_A\) as an ordered group.

In this paper, we present K-theory formula of these \(C^*\)-algebras \(\mathcal{O}_A\) and \(\mathcal{F}^\infty_A\) (Theorem 3.11 and Theorem 4.9). We first compute the \(K_0\)-group \(K_0(\mathcal{F}^\infty_A)\) of the AF-algebra \(\mathcal{F}^\infty_A\) inside of it and show that the \(K_0\)-group is realized as an inductive limit of a sequence of the \(K_0\)-groups of the finite dimensional and commutative \(C^*\)-algebras generated by support projections of canonical generators of partial isometries (Theorem 3.11). We will next show that the AF-algebra \(\mathcal{F}^\infty_A\) is stably isomorphic to the crossed product of the \(C^*\)-algebra \(\mathcal{O}_A\) by the gauge action. Hence, \(\mathcal{O}_A\) is stably isomorphic to the crossed product of the tensor product \(C^*\)-algebra of \(\mathcal{F}^\infty_A\) and the \(C^*\)-algebra of all compact operators on a Hilbert space by an action of \(\mathbb{Z}\). Thus it becomes to be possible to compute K-groups for the \(C^*\)-algebra \(\mathcal{O}_A\) by using the Pimsner-Voiculescu six-term exact sequence for K-theory. The resulting K-group formula (Theorem 4.9) includes the K-group formula of the Cuntz-Krieger algebras ([C2]).

We will finally compute the K-group for the \(C^*\)-algebra associated with a certain sofic subshift but not conjugate to a topological Markov shift. Computation of K-groups for \(C^*\)-algebras associated with other concrete subshifts will appear in some papers (cf. [KMW]).

We remark that the \(C^*\)-algebras associated with subshifts are nuclear purely infinite simple and satisfy the Universal Coefficient Theorem in many cases. Hence, by recent results of Kirchberg and Phillips in [Ki] and [Ph], they can be completely classified by their own K-theory (Corollary 4.11).

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After submitting the first draft of this paper, the author was informed of preprints [KPRR] and [PR] by Kumjian-Pask-Raeburn-Renault and Pask-Raeburn. They study generalization of Cuntz-Krieger algebras from graph theoretic view point, but our generalization of Cuntz-Krieger algebras are different from theirs.

2. Review of the \(C^*\)-algebras associated with subshifts.

We will review the construction of the \(C^*\)-algebras associated with subshifts along [Ma].

In the throughout this paper, a finite set \(\Sigma = \{1, 2, ..., n\}\) is fixed.

Let \(\Sigma^\mathbb{Z}, \Sigma^\mathbb{N}\) be the infinite product spaces \(\prod_{i=-\infty}^{\infty} \Sigma_i, \prod_{i=1}^{\infty} \Sigma_i\) where \(\Sigma_i = \Sigma\), endowed with the product topology respectively. The transformation \(\sigma\) on \(\Sigma^\mathbb{Z}, \Sigma^\mathbb{N}\) given by \((\sigma(x))_i = x_{i+1}, i \in \mathbb{Z}, \mathbb{N}\) is called the (full) shift. Let \(A\) be a
shift invariant closed subset of $\Sigma^Z$ i.e. $\sigma(A) = A$. The topological dynamical system $(A, \sigma|_A)$ is called a subshift. We denote $\sigma|_A$ by $\sigma$ for simplicity. This class of the subshifts includes the class of the topological Markov shifts (cf. [DGS]).

A finite sequence $\mu = (\mu_1, ..., \mu_k)$ of elements $\mu_j \in \Sigma$ is called a block or a word. We denote by $|\mu|$ the length $k$ of $\mu$. A block $\mu = (\mu_1, ..., \mu_k)$ is said to occur in $x = (x_i) \in \Sigma^Z$ if $x_{m} = \mu_1, ..., x_{m+k-1} = \mu_k$ for some $m \in \mathbb{Z}$.

For a subshift $(A, \sigma)$, set for $k \in \mathbb{N}$

$$A^k = \{ \mu : \text{a block with length } k \text{ in } \Sigma^Z \text{ occurring in some } x \in A \}$$

and $A_i = \cup_{k=0}^i A^k$, $A_* = \cup_{k=0}^\infty A^k$ where $A^0$ denotes the empty word $\emptyset$.

Let $\{e_1, ..., e_n\}$ be an orthonormal basis of $n$-dimensional Hilbert space $C^n$.

We put

$$F_0^n = C e_0 \quad (e_0: \text{vacuum vector})$$
$$F_A^k = \text{the Hilbert space spanned by the vectors } e_\mu = e_{\mu_1} \otimes ... \otimes e_{\mu_k}, \mu = (\mu_1, ..., \mu_k) \in A^k,$$
$$F_A = \bigoplus_{k=0}^\infty F_A^k \quad (\text{Hilbert space direct sum})$$

We denote by $T_\nu, (\nu \in A^*)$ the creation operator on $F_A$ of $e_\nu, \nu \in A^* (\nu \neq \emptyset)$ defined by

$$T_\nu e_0 = e_\nu \quad \text{and} \quad T_\nu e_\mu = \begin{cases} e_\nu \otimes e_\mu, & (\nu \mu \in A^*) \\ 0 & \text{else} \end{cases}$$

which is a partial isometry. We put $T_\nu = 1$ for $\nu = \emptyset$. We denote by $P_0$ the rank one projection onto the vacuum vector $e_0$. It immediately follows that

$$\sum_{i=1}^n T_i T_i^* + P_0 = 1.$$ 

We then easily see that for $\mu, \nu \in A^*$, the operator $T_\mu P_0 T_\nu^*$ is the rank one partial isometry from the vector $e_\nu$ to $e_\mu$. Hence, the $C^*$-algebra generated by elements of the form $T_\mu P_0 T_\nu^*, \mu, \nu \in A^*$ is nothing but the $C^*$-algebra $K(F_A)$ of all compact operators on $F_A$. Let $\mathcal{T}_A$ be the $C^*$-algebra on $F_A$ generated by the elements $T_\nu, \nu \in A^*$.

**Definition** ([Ma]). The $C^*$-algebra $\mathcal{O}_A$ associated with subshift $(A, \sigma)$ is defined as the quotient $C^*$-algebra $\mathcal{T}_A/K(F_A)$ of $\mathcal{T}_A$ by $K(F_A)$.

We denote by $S_i, S_\mu$ the quotient image of the operator $T_i, i \in \Sigma$, $T_\mu, \mu \in A^*$. Hence $\mathcal{O}_A$ is generated by $n$ partial isometries $S_1, \ldots, S_n$ with relation

$$\sum_{i=1}^n S_i S_i^* = 1.$$ 

If $(A, \sigma)$ is a topological Markov shift, the $C^*$-algebra $\mathcal{O}_A$ is nothing but the Cuntz-Krieger algebra associated with the topological Markov shift (cf. [CK],[EFW],[Ev]).
We henceforth fix an arbitrary subshift \((A, \sigma)\) in \(\Sigma^\mathbb{Z}\). We denote by \((X_A, \sigma)\) the associated right one-sided subshift for \((A, \sigma)\).

We will present notation and basic facts for studying the \(C^*\)-algebra \(\mathcal{O}_A\).

Put \(a_\mu = S^{\mu}_* S^{\mu}, \mu \in \Lambda^*\). Since \(T^{\nu}_* T^{\mu}_\nu\) commutes with \(T^{\mu}_* T^{\mu}_\nu, \mu, \nu \in \Lambda^*\), the following identities hold

\[
(*) \quad a_\mu S^{\nu} = S^{\nu} a_{\mu \nu}, \quad \mu, \nu \in \Lambda^*.
\]

We notice that for \(\mu, \nu \in \Lambda^*\) with \(|\mu| = |\nu|\),

\[
S^{\nu}_* S^{\nu} \neq 0 \quad \text{if and only if} \quad \mu = \nu.
\]

We will use the following notation. Let \(k, l\) be natural numbers with \(k \leq l\).

\(A_l\) = The \(C^*\)-subalgebra of \(\mathcal{O}_A\) generated by \(a_\mu, \mu \in A_l\).

\(A_\Lambda\) = The \(C^*\)-subalgebra of \(\mathcal{O}_A\) generated by \(a_\mu, \mu \in \Lambda^*\).

\(\mathcal{F}^l_k\) = The \(C^*\)-subalgebra of \(\mathcal{O}_A\) generated by \(S^{\mu}_* a S^{\mu}_\nu, \mu, \nu \in \Lambda^k, a \in A_1\).

\(\mathcal{F}^\infty_k\) = The \(C^*\)-subalgebra of \(\mathcal{O}_A\) generated by \(S^{\mu}_* S^{\nu}_*, \mu, \nu \in \Lambda^k, a \in A_\Lambda\).

\(\mathcal{F}^\infty_\Lambda\) = The \(C^*\)-subalgebra of \(\mathcal{O}_A\) generated by \(S^{\mu}_* S^{\nu}_*, \mu \in \Lambda^*, |\mu| = |\nu|, a \in A_\Lambda\).

The projections \(\{T^{\mu}_* T^{\mu}_\nu; \mu \in \Lambda^*\}\) are mutually commutative so that the \(C^*\)-algebras \(A_l, l \in \mathbb{N}\) are commutative. Thus we easily see the following lemma (cf. [Ma; Section 3]).

**Lemma 2.1.**

(i) \(A_l\) is finite dimensional and commutative.

(ii) \(A_l\) is naturally embedded into \(A_{l+1}\) so that \(A_\Lambda = \lim_{\longrightarrow} A_l\) is a commutative \(AF\)-algebra.

(iii) Each element of \(\mathcal{F}^l_k\) is a finite linear combination of elements of the form \(S^{\mu}_* a S^{\mu}_\nu, \mu, \nu \in \Lambda^k, a \in A_1\). Hence \(\mathcal{F}^l_k\) is finite dimensional.

(iv) There are two embeddings in \(\{\mathcal{F}^l_k\}_{k \leq l}\):

(iv-a) \(\iota_l: \mathcal{F}^l_k \subset \mathcal{F}^{l+1}_k\) through the embedding \(A_l \subset A_{l+1}\) and

(iv-b) \(\eta_k: \mathcal{F}^l_k \subset \mathcal{F}^{l+1}_k\) through the identity

\[
S^{\mu}_* a S^{\nu}_\nu = \sum_{j=1}^{n} S^{\mu}_j S^{\nu}_j a S^{\nu}_j S^{\nu}_j, \quad \mu, \nu \in \Lambda^k, a \in A_1.
\]

(v) Both \(\mathcal{F}^\infty_k = \lim_{l \to \infty} \mathcal{F}^l_k\) and \(\mathcal{F}^\infty_\Lambda = \lim_{k \to \infty} \mathcal{F}^\infty_k\) are \(AF\)-algebras.

In the preceding Hilbert space \(F_A\), the transformation \(e_\mu \to z^\mu e_\mu, \mu \in \Lambda^k, z \in \mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}\) on each base \(e_\mu\) yields a unitary representation which leaves \(\mathcal{N}(F_A)\) invariant. Thus it gives rise to an action \(\alpha\) of \(\mathbb{T}\) on the \(C^*\)-algebra \(\mathcal{O}_A\). It is called the gauge action and satisfies \(\alpha_z(S_i) = zS_i, i = 1, 2, \ldots, n\).
Each element $X$ of the *-subalgebra of $\mathcal{O}_A$ algebraically generated by $S_\mu, \mu \in \Lambda^*$ is written as a finite sum

$$X = \sum_{|\nu| \geq 1} X_{-\nu} S_\nu^* + X_0 + \sum_{|\mu| \geq 1} S_\mu X_\mu$$

for some $X_{-\nu}, X_0, X_\mu \in \mathcal{T}_A^\infty$ because of the relation (*). The map $E(X) = \int_{z \in \mathcal{T}} \alpha_\tau(X) dz, X \in \mathcal{O}_A$ defines a projection of norm one onto the fixed point algebra $\mathcal{O}_A^0$ under $\alpha$. We then have (cf. [Ma; Proposition 3.11])

**Lemma 2.2.** $\mathcal{T}_A^\infty = \mathcal{O}_A^0$.

We will next describe structure theorems for the $C^*$-algebra $\mathcal{O}_A$ proved in [Ma].

**Theorem A** ([Ma; Theorem 4.9 and 5.2]). Let $\mathcal{A}$ be a unital $C^*$-algebra. Suppose that there is a unital *-homomorphism $\pi$ from $A_A$ to $\mathcal{A}$ and there are $n$ partial isometries $s_1, \ldots, s_n \in \mathcal{A}$ satisfying the following relations

(a) $$\sum_{j=1}^n s_j s_j^* = 1, \quad s_\mu s_\nu = s_\nu s_\mu s_\mu, \quad \mu, \nu \in \Lambda^*$$

(b) $$s_\mu^* s_\mu = \pi(S_\mu^* S_\mu), \quad \mu \in \Lambda^*$$

where $s_\mu = s_{\mu_1} \cdots s_{\mu_k}, \mu = (\mu_1, \ldots, \mu_k)$. Then there exists a unital *-homomorphism $\tilde{\pi}$ from $\mathcal{O}_A$ to $\mathcal{A}$ such that $\tilde{\pi}(S_i) = s_i, i = 1, \ldots, n$ and its restriction to $A_A$ coincides with $\pi$. In addition, if $\mathcal{O}_A$ satisfy the condition $(I_A)$ below, this extended *-homomorphism $\tilde{\pi}$ becomes injective whenever $\pi$ is injective.

Let $\mathfrak{D}_A$ be the $C^*$-algebra generated by $S_\mu S_\mu^*, \mu \in \Lambda^*$ which is isomorphic to the $C^*$-algebra $C(X_A)$ of all continuous functions on the space of the one-sided subshift $X_A$ for $A$. Put

$$\phi_A(X) = \sum_{j=1}^n S_j X S_j^*, \quad X \in \mathfrak{D}_A$$

which corresponds to the shift $\sigma$ on the one-sided space $X_A$ of $A$.

Consider the following condition called $(I_A)$ in [Ma].

$(I_A)$: For any $l, k \in \mathbb{N}$ with $l \geq k$, there exists a projection $q_k^l$ in $\mathfrak{D}_A$ such that

(i) $q_k^l a \neq 0$ for any nonzero $a \in A_l$,

(ii) $q_k^l \phi_A^n(q_k^l) = 0$, $1 \leq m \leq k$.

Put
\[ \lambda_A(X) = \sum_{j=1}^{n} S_j X S_j, \quad X \in A_A. \]

We call \( \lambda_A \) the adjacency operator on \( A_A \). It is said to be irreducible if there is no \( \lambda_A \)-invariant ideal in \( A_A \). In addition, it is said to be aperiodic, if for any \( l \in \mathbb{N} \), there exists \( N \in \mathbb{N} \) such that \( \lambda_A^N(p) \geq 1 \) for any minimal projection \( p \) in \( A_l \). We thus have

**Theorem B** ([Ma; Theorem 6.3 and Corollary 7.4]). If the \( C^* \)-algebra \( \mathcal{O}_A \) satisfy the condition \((I_A)\) and \( \lambda_A \) is irreducible, then \( \mathcal{O}_A \) is simple. In addition, if \( \lambda_A \) is aperiodic (or if \( \mathcal{F}_A^{\infty} \) is simple), \( \mathcal{O}_A \) is purely infinite.

We notice the following proposition.

**Proposition C** ([Ma; Proposition 5.8], cf. [CK; 2.17 Proposition]). Let \((A_1, \sigma)\) and \((A_2, \sigma)\) be subshifts such that both the associated \( C^* \)-algebras \( \mathcal{O}_{A_1} \) and \( \mathcal{O}_{A_2} \) satisfy the condition \((I_A)\). If the associated one-sided subshifts \((X_{A_1}, \sigma)\) and \((X_{A_2}, \sigma)\) are topologically conjugate, then there exists an isomorphism from \( \mathcal{O}_{A_1} \) onto \( \mathcal{O}_{A_2} \) such that \( \Phi \circ \alpha_i^1 = \alpha_i^2 \circ \Phi, z \in T \) where \( \alpha_i \) is the gauge action on \( \mathcal{O}_{A_i}, i = 1, 2 \) respectively.

### 3. \( K_0(\mathcal{F}_A^{\infty}) \)

In this section, we will compute \( K_0 \)-group for the AF-algebra \( \mathcal{F}_A^{\infty} \).

Let \( m(l) \) be the dimension of the commutative finite dimensional \( C^* \)-algebra \( A_l, l \in \mathbb{N} \). Take a unique basis \( \{E_l^1, \ldots, E_l^{m(l)}\} \) of \( A_l \) as vector space consisting of minimal projections in \( A_l \) with orthogonal ranges so that \( \sum_{h=1}^{m(l)} E_l^h = 1 \).

We fix \( k \leq l \) for a while.

**Lemma 3.1.** \( \sum_{\mu \in A^k} S^*_\mu S_\mu \geq 1 \)

**Proof.** For any \( \nu \in A^e \), there is a block \( \mu \in A^k \) such that \( \mu \nu \in A^e \) and hence \( T^*_\mu T_\mu e_\nu = e_\nu \). Thus one has \( \sum_{\mu \in A^k} T^*_\mu T_\mu \geq 1 \) on the Hilbert space \( F_A \).

Hence we have

**Lemma 3.2.** For \( i = 1, 2, \ldots, m(l) \), there exists \( \mu \in A^k \) such that \( S_\mu E_i^l S^*_\mu \neq 0 \).

Let \( \mathcal{F}_k^{l,i} \) be the \( C^* \)-subalgebra of \( \mathcal{F}_k^l \) generated by elements \( S_\mu E_i^l S^*_\nu, \mu, \nu \in A^k \). Since \( \mathcal{F}_k^{l,i} \) is isomorphic to a full matrix algebra \( M_{m(k,l,i)}(\mathbb{C}) \), one has
\[ F_k^i \cong M_{n(k,l,1)}(C) \oplus \cdots \oplus M_{n(k,l,m(l))}(C). \]

Put
\[ \Lambda_i^{k,l} = \{ \mu \in \Lambda | E_i^j \leq S^\mu S^\mu \}. \]

Lemma 3.2 implies \( \Lambda_i^{k,l} \neq \emptyset \), \( i = 1, 2, \ldots, m(l) \) and \( n(k,l,i) = |\Lambda_i^{k,l}| \) the cardinal number of \( \Lambda_i^{k,l} \).

**Corollary 3.3.** \( K_0(F_k^i) \cong K_0(A_l) \cong \mathbb{Z}^{m(l)}. \)

The above isomorphism between \( K_0(F_k^i) \) and \( K_0(A_l) \) is given by the map
\[ \phi_k^i : [S^\mu E_i^j S^\mu] \in K_0(F_k^i) \rightarrow [E_i^j] \in K_0(A_l), \quad i = 1, 2, \ldots, m(l), \quad \mu \in \Lambda_i^{k,l}. \]

We next study \( K_0(F_k^\infty) \). We denote by \( \iota_l \) the inclusion from \( A_l \) into \( A_{l+1} \). It yields the inclusion from \( F_k^i \) into \( F_k^{i+1} \) which is also denoted by \( \iota_l \). One write \( E_i^j \) as
\[ E_i^j = \sum_{h=1}^{m(l+1)} \iota_l(i,h)E_{i+1}^h \]
for some \( \{0, 1\} \)-valued map \( \iota_l(i,h), i = 1, 2, \ldots, m(l), h = 1, 2, \ldots, m(l+1) \).

**Lemma 3.4.** The diagram
\[
\begin{array}{ccc}
K_0(F_k^i) & \xrightarrow{\iota_*} & K_0(F_k^{i+1}) \\
\phi_k^i \downarrow & & \downarrow \phi_k^{i+1} \\
K_0(A_l) & \xrightarrow{\iota_l} & K_0(A_{l+1})
\end{array}
\]
is commutative.

**Proof.** If \( S^\mu E_i^j S^\mu \neq 0 \) and \( \iota_l(i,h) \neq 0 \), then \( S^\mu E_{i+1}^h S^\mu \neq 0 \). Namely \( \Lambda_i^{k,l} \subset \Lambda_{i+1}^{k,l} \) if \( \iota_l(i,j) \neq 0 \). Hence the commutativity of the above diagram is clear.

Thus one obtains an isomorphism \( \phi_k = \lim \phi_k^i \) from \( \lim K_0(F_k^i) = K_0(F_k^\infty) \) onto \( \lim K_0(A_l) = K_0(A_{\infty}) \). Namely, one has

**Proposition 3.5.** \( K_0(F_k^\infty) \cong K_0(A_{\infty}) \cong \lim (\mathbb{Z}^{m(l)}, \iota_l) \) where the inclusion \( \iota_l \) of \( \mathbb{Z}^{m(l)} \) into \( \mathbb{Z}^{m(l+1)} \) is given by
\[ [E_i^l] = \sum_{h=1}^{m(l+1)} t_l(i, h)[E_{l+1}^h], \quad i = 1, 2, \ldots, m(l) \]

and

\[ \mathbb{Z}^{m(l)} = \mathbb{Z}[E^1_i] \oplus \cdots \oplus \mathbb{Z}[E^{m(l)}_i]. \]

We denote by \( Z_A \) the above abelian group \( \lim_{\rightarrow} Z_m \) and so that

\[ Z_A \cong K_0(\mathcal{F}_k^\infty) \cong K_0(A), \quad k \in \mathbb{N}. \]

We next study \( K_0(\mathcal{F}_k^\infty) \) as the inductive limit \( \lim_{\rightarrow} K_0(\mathcal{F}_k^l) \).

The embedding \( \eta_k \) of \( \mathcal{F}_k^\infty \) into \( \mathcal{F}_{k+1}^\infty \) is given, through the embedding of \( \mathcal{F}_k^l \) into \( \mathcal{F}_{k+1}^l \), by the identity

\[ S_\mu E_i^l S_\nu^* = \sum_{j=1}^{n} S_{\mu ij}^* E_i^l S_j, \quad \mu, \nu \in A^k, \quad i = 1, 2, \ldots, m(l) \]

so that the induced homomorphism \( \eta_{k+} \) from \( K_0(\mathcal{F}_k^\infty) \) to \( K_0(\mathcal{F}_{k+1}^\infty) \) is given by

\[ \eta_{k+}[S_\mu E_i^l S_\nu^*] = \sum_{j=1}^{n} [S_{\mu ij}^* E_i^l S_j], \quad \mu \in A^{k}, \quad i = 1, 2, \ldots, m(l). \]

As the projection \( S_j^* E_i^l S_j \) belongs to \( A_{l+1} \), it can be written as

\[ S_j^* E_i^l S_j = \sum_{h=1}^{m(l+1)} A_l(i, j, h) E_{l+1}^h \]

for some \( \{0, 1\} \)-valued map \( A_l(i, j, h), \quad i = 1, 2, \ldots, m(l), \quad j = 1, 2, \ldots, n, \quad h = 1, 2, \ldots, m(l+1) \). Hence one has

\[ S_\mu E_i^l S_\nu^* = \sum_{j=1}^{n} \sum_{h=1}^{m(l+1)} A_l(i, j, h) S_{\mu ij}^* E_{l+1}^h S_{\nu j}, \quad \mu \in A^k, \quad i = 1, 2, \ldots, m(l). \]

**Lemma 3.6.** If \( S_\mu E_i^l S_\nu^* \neq 0 \), one has \( S_{\mu ij}^* E_{l+1}^h S_{\nu j}^* \neq 0 \) for \( A_l(i, j, h) \neq 0 \).

**Proof.** Since \( A_l(i, j, h) \neq 0 \), one has \( S_j^* E_i^l S_j \geq E_{l+1}^h \). We also have \( S_j^* a_\mu S_j \geq S_j^* E_i^l S_j \) because \( S_\mu E_i^l S_\nu^* \neq 0 \). Hence we obtain \( S_j^* a_\mu S_j \geq E_{l+1}^h \) which implies \( S_{\mu ij}^* E_{l+1}^h S_{\nu j}^* \neq 0 \).
Lemma 3.7. If $\Lambda_l(i,j_1,h) \neq 0$ and $\Lambda_l(i,j_2,h) \neq 0$, one has for $\mu \in A^k$

$$[S_{pj_1}E^h_{l+1}S^*_{pj_1}] = [S_{pj_2}E^h_{l+1}S^*_{pj_2}] \quad \text{in } K_0(\mathcal{F}_{k+1}^{l+1}).$$

Put

$$A_l(i,h) = \sum_{j=1}^n A_l(i,j,h) \in \mathbb{Z}_+, \quad i = 1,2,\ldots,m(l), \quad h = 1,2,\ldots,m(l+1).$$

We then define a homomorphism $\lambda_l$ from $K_0(A_l)$ to $K_0(A_{l+1})$ by

$$\lambda_l([E^l]) = \sum_{h=1}^{m(l+1)} A_l(i,h)[E^h_{l+1}]$$

where

$$K_0(A_l) = \sum_{i=1}^{m(l)} \oplus \mathbb{Z}[E^l_i], \quad K_0(A_{l+1}) = \sum_{h=1}^{m(l+1)} \oplus \mathbb{Z}[E^h_{l+1}].$$

We indeed have

Lemma 3.8. $\lambda_l([P]) = \sum_{j=1}^n [S^j_PS]$ for a projection $P$ in $A_l$.

Hence one has

Lemma 3.9. The diagram

$$\begin{array}{ccc}
K_0(A_l) & \xrightarrow{\lambda_l} & K_0(A_{l+1}) \\
\downarrow & & \downarrow \lambda_{l+1} \\
K_0(A_{l+1}) & \xrightarrow{\lambda_{l+1}} & K_0(A_{l+2})
\end{array}$$

is commutative.

Since $K_0(A_A) = \lim K_0(A_l)$, one can define a homomorphism $\lambda_A = \lambda_l$ on $K_0(A_A)$ induced by the sequence of homomorphisms $\lambda_l : K_0(A_l) \rightarrow K_0(A_{l+1}), l \in \mathbb{N}$. Namely, we obtain a homomorphism $\lambda_A$ on $Z_A(\cong K_0(A_A) \cong K_0(\mathcal{F}_k^\infty))$. We remark that it is exactly regarded as the induced homomorphism on $K_0(A_A)$ from the adjancy operator $\lambda_A$ defined in the previous section. Hence we use the same notation $\lambda_A$ without confusion.
Lemma 3.10. The diagram
\[ K_0(\mathcal{F}_k^\infty) \xrightarrow{\eta_k} K_0(\mathcal{F}_{k+1}^\infty) \]
\[ \Phi_k \downarrow \quad \Phi_{k+1} \]
\[ K_0(A_A) \xrightarrow{\lambda_A} K_0(A_A) \]
is commutative.

Proof. By Lemma 3.7, it follows that
\[ \Phi_{k+1} \circ \eta_k \cdot ([S_{j,i}E_l^jS_{l,j}^*]) = \Phi_{k+1} \left( \sum_{j=1}^{n} \left( S_{j,i} \left( \sum_{h=1}^{m(l+1)} A_t(i,j,h)E_{i+1}^j \right) \right) \right) \]
\[ = \sum_{h=1}^{m(l+1)} \Phi_{k+1} \left( \sum_{j=1}^{n} A_t(i,j,h)[S_{j,i}E_{i+1}^jS_{l,j}^*] \right) \]
\[ = \sum_{h=1}^{m(l+1)} A_t(i,h)[E_{i+1}^j] = \lambda_A[E_{i+1}^j] = \lambda_A \circ \Phi_k([S_{j,i}E_l^jS_{l,j}^*]). \]

Therefore we conclude

Theorem 3.11. \( K_0(\mathcal{F}_A^\infty) = \lim \rightarrow (Z_A, \lambda_A). \)

Corollary 3.12. If \( A \) is a sofic subshift, \( K_0(\mathcal{F}_A^\infty) = \lim \rightarrow (Z^{m(l)}, \lambda_l). \)

Proof. Let \( j_t \) be the canonical inclusion of \( Z^{m(l)}(= K_0(A_l)) \) into \( Z_A(= K_0(A_A)) \), which is induced by the natural inclusion of \( A_l \) into \( A_A \). Since the following diagram
\[ Z_A \xrightarrow{\lambda_A} Z_A \]
\[ j_t \downarrow \quad j_{l+1} \downarrow \]
\[ Z^{m(l)} \xrightarrow{\lambda_l} Z^{m(l+1)} \]
is commutative, there is a homomorphism \( \pi \) from \( \lim \rightarrow (Z^{m(l)}, \lambda_l) \) to \( \lim \rightarrow (Z_A, \lambda_A) \). It is easy to see that it is indeed a surjective isomorphism because \( Z_A = Z^{m(l)} \) for some large enough \( l \) by [Ma; Proposition 8.2]

Before ending this section, we define the dimension group \( DG(A) \) for a general subshift \( (A, \sigma) \) as the dimension group for the AF-algebra \( \mathcal{F}_A^\infty \), namely,
The notion of the dimension group for a topological Markov shift \((A, \sigma)\) determined by a matrix \(A\) with entries in \(\{0, 1\}\) has been introduced by W. Krieger in [Kr1] and [Kr2]. It is realized as the dimension group for the canonical AF-algebra \(\mathcal{F}_A\) appeared inside of the Cuntz-Krieger algebra \(\mathcal{O}_A\) associated with the topological Markov shift \((A, \sigma)\). If we restrict our construction of \(C^*\)-algebras \(\mathcal{O}_A\) and \(\mathcal{F}_A\) to a topological Markov shift \((X, \sigma)\), they coincide with the Cuntz-Krieger algebra \(\mathcal{O}_A\) and the canonical AF-algebra \(\mathcal{F}_A\) respectively. Hence our above definition of the dimension group for general subshifts is a generalization of the case of topological Markov shifts. By Proposition C, we see

**Proposition 3.13.** The dimension group \(DG(A)\) for subshift \((A, \sigma)\) is an invariant under topological conjugacy for the associated one-sided subshift \((X, \sigma)\) among the class of all subshifts such that the associated \(C^*\)-algebra \(\mathcal{O}_A\) satisfies the condition \((I_\alpha)\).

4. \(K_0(\mathcal{O}_A)\).

We will, in this section, present K-theory formula for the \(C^*\)-algebra \(\mathcal{O}_A\). We denote by \(\mathcal{K}\) the \(C^*\)-algebra of all compact operators on a separable infinite dimensional Hilbert space. We will notice that the crossed product \(\mathcal{O}_A \times_{\alpha} T\) of \(\mathcal{O}_A\) by the gauge action \(\alpha\) of \(T\) is stably isomorphic to the associated AF-algebra \(\mathcal{F}_A^\infty\). Since \(\mathcal{O}_A\) is stably isomorphic to the crossed product \((\mathcal{O}_A \times_{\alpha} T) \times_{\hat{\alpha}} Z\) of \(\mathcal{O}_A \times_{\alpha} T\) by the dual action \(\hat{\alpha}\), it will be possible to present K-theory formula for the \(C^*\)-algebra \(\mathcal{O}_A\) by using the previous K-theory formula for the AF-algebra \(\mathcal{F}_A^\infty\) and by applying the Pimsner-Voiculescu’s six-term exact sequence of the K-theory for the crossed products by \(Z\) ([PV]).

We will first see that the crossed product \(\mathcal{O}_A \times_{\alpha} T\) is stably isomorphic to the AF-algebra \(\mathcal{F}_A^\infty\).

Let \(p_0 : T \to \mathcal{O}_A\) be the constant function whose value everywhere is the unit 1 of \(\mathcal{O}_A\). Hence \(p_0\) belongs to the algebra \(L^1(T, \mathcal{O}_A)\) and hence to the crossed product \(\mathcal{O}_A \times_{\alpha} T\). By [Ro], the fixed point algebra \(\mathcal{O}_A^{\alpha}\) is canonically isomorphic to the algebra \(p_0(\mathcal{O}_A \times_{\alpha} T)p_0\). The isomorphism between them is given by the correspondence \(x \in \mathcal{O}_A^{\alpha} \to \hat{x} \in L^1(T, \mathcal{O}_A) \subset \mathcal{O}_A \times_{\alpha} T\) where the function \(\hat{x}\) is defined by \(\hat{x}(t) = x, t \in T\).

**Lemma 4.1.** The projection \(p_0\) is full in \(\mathcal{O}_A \times_{\alpha} T\).
**Proof.** Suppose that there exists a nondegenerate representation \( \pi \) of \( \mathcal{O}_A \times_\alpha \mathbb{T} \) such that \( \pi(p_0) = 0 \). For any element \( S \) in \( \mathcal{O}_A \), put \( \hat{S}(z) = S, z \in \mathbb{T} \), which belongs to \( L^1(\mathbb{T}, \mathcal{O}_A) \). We denote by \( * \) the \( \alpha \)-twisted convolution product in \( L^1(\mathbb{T}, \mathcal{O}_A) \) (the usual product as elements of \( \mathcal{O}_A \times_\alpha \mathbb{T} \)). It then follows that \( \hat{S} \ast p_0 = \hat{S} \). Hence \( \hat{S} \) belongs to the ideal \( \ker(\pi) \) in \( \mathcal{O}_A \times_\alpha \mathbb{T} \). For \( S, T \in \mathcal{O}_A \), one has \( (\hat{S} \ast \hat{T})(z) = S\alpha(z)(T^*) \) by using the identity \( (\hat{T})(z) = \alpha(z)(T^*) \). For any \( X \in \mathcal{O}_A \) and \( \mu \in \Lambda^k \), we have

\[
(XS^*_\mu \ast \hat{S}^*_\mu)(z) = z^{-k}XS^*_\mu S^*_\mu
\]

and hence

\[
\left( \sum_{|\mu|=k} XS^*_\mu \ast \hat{S}^*_\mu \right)(z) = z^{-k}X, \quad k \in \mathbb{N}.
\]

We denote by \( B_k \) the commutative \( C^* \)-algebra generated by \( a_\mu, \mu \in \Lambda^k \). Let \( F_i, i = 1, 2, \ldots, n(k) \) be the set of all minimal projections in \( B_k \). Since one sees for \( \mu \in \Lambda^k \),

\[
(XF_i^*S^*_\mu \ast \hat{S}^*_\mu)(z) = z^kXF_i^*,
\]

one has

\[
\left( \sum_{i=1}^{n(k)} XF_i^*S^*_\mu \ast \hat{S}^*_\mu \right)(z) = z^kX, \quad k \in \mathbb{N}.
\]

Hence any \( \mathcal{O}_A \)-valued function of the form

\[
z \in \mathbb{T} \rightarrow z^kX \in \mathcal{O}_A, \quad k \in \mathbb{Z}, \quad X \in \mathcal{O}_A
\]

is contained in the ideal \( \ker(\pi) \). Thus we conclude \( \pi \equiv 0 \) on \( \mathcal{O}_A \times_\alpha \mathbb{T} \). This implies that \( p_0 \) is a full projection in \( \mathcal{O}_A \times_\alpha \mathbb{T} \).

Since the AF-algebra \( \mathcal{F}_A^\infty \) is realized as the fixed point algebra \( \mathcal{O}_A^\alpha \), one sees, by [Bro; Corollary 2.6]

Corollary 4.2. \( \mathcal{O}_A \times_\alpha \mathbb{T} \) is stably isomorphic to \( \mathcal{F}_A^\infty \).

The Pimsner-Voiculescu’s six term exact sequence of the K-theory for the crossed product \( (\mathcal{O}_A \times_\alpha \mathbb{T}) \times_\alpha \mathbb{Z} \) says that the following sequence becomes exact:
Since the inclusion \( (\mathcal{O}_A \times_\alpha T) \) follows from Lemma 4.1 and [Ri; Proposition 2.4], the next lemma follows from Lemma 4.1 and [Ri; Proposition 2.4].

**Lemma 4.3.**

(i) \( K_0(\mathcal{O}_A) \cong K_0(\mathcal{O}_A \times_\alpha T)/(\text{id} - \hat{\alpha}_e^{-1})K_0(\mathcal{O}_A \times_\alpha T) \)

(ii) \( K_1(\mathcal{O}_A) \cong K_0(\mathcal{O}_A \times_\alpha T)/\text{Ker}(\text{id} - \hat{\alpha}_e^{-1}) \)

We will next study the group \( K_0(\mathcal{O}_A \times_\alpha T) \) and the action \( \hat{\alpha}_e \) on it. The next lemma follows from Lemma 4.1 and [Ri; Proposition 2.4].

**Lemma 4.4.** The inclusion \( \iota: p_0(\mathcal{O}_A \times_\alpha T)p_0 \to \mathcal{O}_A \times_\alpha T \) induces an isomorphism \( \iota_*: K_0(p_0(\mathcal{O}_A \times_\alpha T)p_0) \to K_0(\mathcal{O}_A \times_\alpha T) \) on \( K \)-theory.

Under the identification, \( \mathcal{F}_A^\infty = \mathcal{O}_A^\alpha = p_0(\mathcal{O}_A \times_\alpha T)p_0 \), we define an isomorphism \( \beta \) on \( K_0(\mathcal{F}_A^\infty) \) as \( \beta = \iota_*^{-1} \circ \hat{\alpha}_e \circ \iota_* \). Namely the diagram

\[
\begin{array}{ccc}
K_0(\mathcal{O}_A \times_\alpha T) & \xrightarrow{\alpha_*} & K_0(\mathcal{O}_A \times_\alpha T) \\
\downarrow \iota_* & & \downarrow \iota_* \\
K_0(\mathcal{F}_A^\infty) & \xrightarrow{\beta} & K_0(\mathcal{F}_A^\infty)
\end{array}
\]

is commutative.

The following lemma is a key.

**Lemma 4.5.** For a projection \( P \) in \( \mathcal{F}_A^\infty \) and a partial isometry \( S \) in \( \mathcal{O}_A \) with \( \alpha_\circ (S) = \alpha(zT) = \alpha(S) = zS, z \in T \) and \( P \leq S^*S \), we have \( \beta[P] = [SPS^*] \) in \( K_0(\mathcal{F}_A^\infty) \).

**Proof.** Let \( j: \mathcal{F}_A^\infty \to p_0(\mathcal{O}_A \times_\alpha T)p_0 \) be the canonical isomorphism and \( \iota: p_0(\mathcal{O}_A \times_\alpha T)p_0 \to \mathcal{O}_A \times_\alpha T \) the inclusion. For \( P \in \mathcal{F}_A^\infty \), we denote by \( \tilde{P}(z) = P, z \in T \). As \( SPS^* \in \mathcal{F}_A^\infty \), we similarly denote by \( \tilde{SPS}^* = \iota \circ j(SP) \in L^1(T, \mathcal{O}_A) \) the constant \( \mathcal{F}^\infty \)-valued function. It suffices to show \( [SPS^*] = \alpha(\tilde{P}) \) in \( K_0(\mathcal{O}_A \times_\alpha T) \). Let \( \hat{S} \in L^1(T, \mathcal{O}_A) \) be the constant \( S \)-valued function : \( \hat{S}(z) = S, z \in T \). We denote by \( * \) the twisted convolution product (usual product) in \( \mathcal{O}_A \times_\alpha T \). It then follows that \( (\hat{S} * \tilde{P})(z) = SP, z \in T \) and \( (\hat{S} * \tilde{P} * \hat{S})(z) = z^{-1}SPS^*, z \in T \). Thus we have \( \alpha(\tilde{P} * \hat{S})(z) = z^{-1}SPS^*, z \in T \). Since the inclusion \( \alpha(\iota_* j: \mathcal{F}_A^\infty = \mathcal{O}_A \times_\alpha T \) is a homomorphism, one has \( \tilde{P} \leq \hat{S}^* \hat{S} \) because \( P \leq S^*S \). Thus one sees...
\[
\left[ \hat{S} \ast \hat{P} \ast \hat{S}^* \right] = \left[ \hat{P} \right] \quad \text{in} \quad K_0(\mathcal{O}_A \times \alpha \times T)
\]
so that we conclude
\[
\hat{\alpha}_* \left[ \hat{P} \right] = \left[ SPS^* \right] \quad \text{in} \quad K_0(\mathcal{O}_A \times \alpha \times T) \quad \text{and} \quad \beta \left[ P \right] = \left[ SPS^* \right] \quad \text{in} \quad K_0(\mathcal{F}_A^\infty).
\]

**Lemma 4.6.** For a nonzero projection \( S_\mu E_i^j S_\nu^* \) in \( \mathcal{F}_k^l \) with \( \mu = j\nu \in \Lambda^k \), one has \( \beta^{-1} \left[ S_\mu E_i^j S_\nu^* \right] = \left[ S_i E_j^k S_\nu^* \right] \) in \( K_0(\mathcal{F}_k^l) \).

**Proof.** Since \( S_\mu E_i^j S_\nu^* \neq 0 \), we see that \( S_\mu E_i^j S_\nu^* \leq S_\nu^* S_j \) because of the identity \( S_\nu^* S_j S_i E_i^j S_\nu^* = S_\nu^* a_\mu E_i^j S_\nu^* = S_i E_j^k S_\nu^* \). Hence we have the conclusion by the previous lemma.

**Corollary 4.7.** The homomorphism \( \beta^{-1} : K_0(\mathcal{F}_A^\infty) \to K_0(\mathcal{F}_A^\infty) \) corresponds to the shift \( \sigma \) in \( \lim K_0(\mathcal{F}_k^l) = \lim Z_A \). Namely, if \( x = (x_1, x_2, \ldots) \) is a sequence representing an element of \( \lim K_0(\mathcal{F}_k^l) \), then \( \beta^{-1}x \) is represented by \( \sigma(x) = (x_2, x_3, \ldots) \).

Since the diagram
\[
\begin{array}{ccc}
K_0(\mathcal{F}_A^\infty) & \xrightarrow{id - \beta^{-1}} & K_0(\mathcal{F}_A^\infty) \\
\phi \downarrow & & \downarrow \phi \\
\lim Z_A & \xrightarrow{id - \sigma} & \lim Z_A
\end{array}
\]
is commutative, one has

**Corollary 4.8.**

(i) \( K_0(\mathcal{O}_A) \cong \lim Z_A / (id - \sigma) \lim Z_A \)

(ii) \( K_1(\mathcal{O}_A) \cong \ker (id - \sigma) \) on \( \lim Z_A \).

Let \( j \) be the homomorphism from \( K_0(\mathcal{F}_0^\infty) = Z_A \) to \( K_0(\mathcal{F}_A^\infty) = \lim Z_A \) induced by the inclusion \( : \mathcal{F}_0^\infty \hookrightarrow \mathcal{F}_A^\infty \).

As in the proof of [C2; 3.1 Proposition], we see that every element in \( \lim Z_A \) is equivalent modulo \( (id - \sigma) \lim Z_A \) to an element in \( Z_A \). Since the diagram
\[
\begin{array}{ccc}
Z_A & \xrightarrow{id - \lambda_A} & Z_A \\
j \downarrow & & \downarrow j \\
\lim Z_A & \xrightarrow{id - \sigma} & \lim Z_A
\end{array}
\]
is commutative and \( j(x) \in (id - \sigma) \lim Z_A, x \in Z_A \) implies \( x \in (id - \lambda_A)Z_A \), we then have
Similarly as in the same argument in [C2; 3.1.Proposition], we have
\[ K_1(\mathcal{O}_A) \cong \text{Ker} \ (\text{id} - \lambda_A) \text{ on } \mathbb{Z}. \]

Thus we present the K-theory formula for the \( C^* \)-algebra \( \mathcal{O}_A \)

**Theorem 4.9.**

(i) \( K_0(\mathcal{O}_A) \cong \mathbb{Z}/(\text{id} - \lambda_A) \mathbb{Z} \cong \lim_{\rightarrow} (\mathbb{Z}^{m(l+1)}/(t_l - \lambda_l) \mathbb{Z}^{m(l)}) \)

(ii) \( K_1(\mathcal{O}_A) \cong \text{Ker} \ (\text{id} - \lambda_A) \text{ in } \mathbb{Z} \cong \lim_{\rightarrow} \text{Ker} \ (t_l - \lambda_l) \text{ in } \mathbb{Z}^{m(l)} \)

where
\[ Z_A = \lim_{\rightarrow} (Z^{m(l)}, t_l), \quad m(l) = \dim A_l \]

and
\[ \lambda_A = \lim_{\rightarrow} \lambda_l, \quad \lambda_l : Z^{m(l)} = K_0(A_l) \to Z^{m(l+1)} = K_0(A_{l+1}) \]

is defined by
\[ \lambda_l([P]) = \sum_{j=1}^{n} [S_j^* P S_j] \quad \text{for a projection } P \text{ in } A_l. \]

More precisely, for the minimal projections \( E_l^1, \ldots, E_l^{m(l)} \) of \( A_l \) with \( \sum_{i=1}^{m(l)} E_i = 1 \) and the canonical basis \( e_l^1, \ldots, e_l^{m(l)} \) of \( Z^{m(l)} \), the map \([E_l^1] \to e_l^1\) extends to an isomorphism of \( K_0(\mathcal{O}_A) \) onto \( \lim_{\rightarrow} (Z^{m(l+1)}/(t_l - \lambda_l) Z^{m(l)}) \).

Before ending this section, we note the following lemma.

**Lemma 4.10.** The \( C^* \)-algebra \( \mathcal{O}_A \) is nuclear and satisfies the Universal Coefficient Theorem in the sense of Rosenberg and Schochet.

**Proof.** Since the double crossed product \( (\mathcal{O}_A \times_\alpha T) \times_\alpha \mathbb{Z} \) is stably isomorphic to \( \mathcal{O}_A \), the assertion is immediate from Corollary 4.2 (cf. [RS], [Bl; p. 287]).

Hence, as in Theorem B, one sees by [Ki] and [Ph]

**Corollary 4.11.** If the \( C^* \)-algebra \( \mathcal{O}_A \) satisfies the condition \((I_A)\) and the adjency operator \( \lambda_A \) is aperiodic, then \( \mathcal{O}_A \) is a separable nuclear purely infinite simple \( C^* \)-algebra satisfying the Universal Coefficient Theorem. Thus, these \( C^* \)-algebras are completely classified by their own K-theory up to isomorphism.
5. Sofic subshifts and examples.

There is a class of subshifts called sofic subshifts. It is truly wider, up to conjugate, than the class of subshifts of finite type and hence that of topological Markov shifts. Hence the $C^*$-algebras associated with sofic subshifts which are not conjugate to topological Markov shifts can not be dealt with within the Cuntz-Krieger’s approach. For a subshift $(\Lambda, \sigma)$ and words $\mu, \nu \in \Lambda^*$, we write $\mu \sim \nu$ if

$$\{ \gamma \in \Lambda^* | \mu \gamma \in \Lambda^* \} = \{ \gamma \in \Lambda^* | \nu \gamma \in \Lambda^* \}. $$

If the cardinality of the equivalence classes $\Lambda^*/\sim$ is finite, the subshift $(\Lambda, \sigma)$ is said to be sofic (cf. [DGS], [W]). Hence a subshift $(\Lambda, \sigma)$ is sofic if and only if the commutative $C^*$-subalgebra $A_\Lambda$ of $\mathcal{O}_\Lambda$ is finite dimensional (cf. [Ma; Proposition 8.2]).

Suppose that a subshift $(\Lambda, \sigma)$ is sofic. Put $N = \dim A_\Lambda < \infty$. Hence the adjacency operator $\lambda_\Lambda$ on $A_\Lambda$ is realized as an $N \times N$ matrix with entries in non-negative integers. We then notice that $\lambda_\Lambda$ is irreducible (resp. aperiodic) in the sense of Section 2 if and only if it is irreducible (resp. aperiodic) in the sense of non-negative matrix.

It is well-known that if $\lambda_\Lambda$ is aperiodic, the AF-algebra $\mathcal{F}_\Lambda^\infty$ is simple and has a unique tracial state $\tau_\Lambda$ (cf. [Bra], [Ef], [Ev2]). Thus we can summarize the previous discussions on K-theory for the $C^*$-algebras $\mathcal{O}_\Lambda$ and $\mathcal{F}_\Lambda^\infty$ as in the following way.

**Proposition 5.1.** Suppose that a subshift $(\Lambda, \sigma)$ is sofic. If the $C^*$-algebra $\mathcal{O}_\Lambda$ satisfies the condition $(I_\Lambda)$ and the adjacency operator $\lambda_\Lambda$ is aperiodic, then we have

(i) $\mathcal{O}_\Lambda$ is simple and purely infinite.

(ii) $K_0(\mathcal{O}_\Lambda) \cong \mathbb{Z}^N/(1-\lambda_\Lambda)\mathbb{Z}^N$ and $K_1(\mathcal{O}_\Lambda) \cong \text{Ker} (1-\lambda_\Lambda)$ in $\mathbb{Z}^N$.

(iii) $DG(\Lambda) \cong \lim\text{sup} (\mathbb{Z}^N, \lambda_\Lambda) \cong \tau_\Lambda(\mathcal{F}_\Lambda^\infty)$ in $\mathbb{R}$.

where $\tau_\Lambda$ is a unique tracial state on $\mathcal{F}_\Lambda^\infty$.

Thus by Corollary 4.11 we see that if a subshift $(\Lambda, \sigma)$ is sofic, the $C^*$-algebra $\mathcal{O}_\Lambda$ is stably isomorphic to some Cuntz-Krieger algebra $\mathcal{O}_{\lambda_\Lambda}$ associated with a matrix $\lambda_\Lambda$ with entries in non-negative integers.

We present examples of the $C^*$-algebras associated with sofic subshifts.

**Example 1** (Cuntz algebras $\mathcal{O}_n$, [C], [C2], [C3]).

Let $(\Lambda_n, \sigma)$ be the full shift over $\Sigma = \{1, 2, \ldots, n\}$. The $C^*$-algebra $\mathcal{O}_{\Lambda_n}$ associated with it is the Cuntz algebra $\mathcal{O}_n$ of order $n$. Then the commutative $C^*$-algebras $\Lambda_l$ are reduced to the scalar $\mathbb{C}$ so that $m(l) = 1, l \in \mathbb{N}$. It is easy
to see that the adjancy operator $\lambda_A$ is the $n$-multiplication on $Z = K_0(A_l) = K_0(C)$. Hence we see

$$K_0(\mathcal{F}_A) = Z[\frac{1}{n}], \quad K_0(\mathcal{C}_n) = Z/(1-n)Z, \quad K_1(\mathcal{C}_n) = 0.$$ 

**Example 2 (Cuntz-Krieger algebras $C_A$, [CK], [C2], [C3]).**

Let $(A_A, \mathcal{O})$ be the topological Markov shift determined by an $n \times n$-matrix $A$ with $\{0,1\}$-entries. The $C^*$-algebra $\mathcal{O}_A$ associated with it is the Cuntz-Krieger algebra $\mathcal{O}_A$. Suppose that $A$ is an irreducible but not permutation matrix with rank $n$. Hence one sees that $A_l^* A_l C_S^1 S_l^* \in Z^{n \times n}$; $K_0(\mathcal{O}_A) = \text{Ker}(1-A^l)$ in $Z^n$. Hence we see

$$K_0(\mathcal{F}_A) = \lim_{\rightarrow} Z^n, \quad K_0(\mathcal{O}_A) = Z^n/(1-A^l)Z^n,$$

$$K_1(\mathcal{O}_A) = \text{Ker}(1-A^l) \text{ in } Z^n.$$

**Example 3.**

Suppose $\Sigma = \{1,2\}$. Let $Y$ be the subshift in $\Sigma^Z$ defined by the condition that all blocks of 2’s which have maximal length have even length, which is called the even shift (cf. [DGS; p. 251]). It is a sofic subshift but not conjugate to a topological Markov shift. One easily sees for $\mu = (\mu_1, \ldots, \mu_k) \in Y^*$

$$S_{\mu}^* S_{\mu} = \begin{cases} 1 & \text{if } \mu = (2, \ldots, 2), \\ S_1^* S_1 & \text{if } \mu = (*, \ldots, *, 1) \text{ or } \mu = (*, \ldots, *, 1, 2, \ldots, 2) \text{ even} \\ S_2^* S_1^* S_1 S_2 & \text{if } \mu = (*, \ldots, *, 1, 2, \ldots, 2). \end{cases}$$

Put

$$P_1 = S_1^* S_1 - P_2, \quad P_2 = S_1^* S_1 \cdot S_2^* S_1 S_2 \quad \text{and} \quad P_3 = S_2^* S_1^* S_1 S_2 - P_2$$

so that one has $P_1 + P_2 + P_3 = 1$. Hence one sees

$$A_l = A_Y = C P_1 \oplus C P_2 \oplus C P_3, \quad l \geq 2$$

and hence $m(l) = 3, l \geq 2$. This means that

$$Z_Y = K_0(A_Y) = Z[P_1] \oplus Z[P_2] \oplus Z[P_3] \cong Z^3.$$
It is easy to see that the adjacency operator $\lambda_A (= \lambda_I)$ is the homomorphism on $\mathbb{Z}^3$ given by the matrix
\[
\begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}.
\]
Thus we have
\[
K_0(\mathcal{F}_Y) \cong \lim_{\rightarrow} \left( \mathbb{Z}^3, \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right) \cong \mathbb{Z} \oplus \mathbb{Z} + \frac{1 + \sqrt{5}}{2} \mathbb{Z}
\text{ in } \mathbb{R} \oplus \mathbb{R},
\]
\[
K_0(\mathcal{O}_Y) \cong \left( 1 - \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right) \mathbb{Z}^3 \cong \mathbb{Z},
\]
\[
K_1(\mathcal{O}_Y) \cong \text{Ker} \left( 1 - \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right) \text{ in } \mathbb{Z}^3 \cong \mathbb{Z}.
\]

Other concrete examples which are not sofic subshifts will be dealt with in some papers (cf. [KMW]).

**REFERENCES**


K-THEORY FOR $C^*$-ALGEBRAS ASSOCIATED WITH SUBSHIFTS


