EXPLICIT REPRESENTATION OF THE SOLUTION TO SOME BOUNDARY VALUE PROBLEM

S. MAZOUZI AND R. N. PEDERSON

Abstract.
In the half ray the unique solution to the boundary value problem

\[ L(D_t)u(t) = f(t), \quad t > 0 \]
\[ B_j(D_t)u(0) = \alpha_j, \quad \text{for} \quad j = 0, 1, \ldots, p - 1 \]

rapidly decreasing at infinity is shown to be explicitly represented in terms of Green’s function and some boundary kernels, namely,

\[ u(t) = \sum_{k=0}^{p-1} \mathcal{H}_k(t)\alpha_k + \int_0^\infty \mathcal{G}(t, s)f(s)\, ds \]

1. Introduction.

Let \( L(z) \) be a polynomial of degree \( m \geq 1 \), where the coefficient of \( z^m \) is equal to 1, and let \( \{B_j(z)\}_{j=0}^{p-1} \) be \( p \) polynomials of degrees \( \{m_j\}_{j=0}^{p-1} \) respectively, so that \( m_j < m \) for \( 0 \leq j \leq p - 1 \). We assume that \( L(z) \) has at most \( p \) roots having positive imaginary parts (counting multiplicities).

Throughout this paper we denote by \( D_t \) the differential operator \(-i\frac{d}{dt}\); and if \( J \subseteq \mathbb{R} \) then we denote by \( \mathcal{S}(J) \) the subspace of \( C^\infty(J) \) containing all the functions \( u(t) \) such that

\[ (1 + |t|)^p|u^{(q)}(t)| \]

are bounded for all \( p \) and \( q \) in \( \mathbb{N} \).

Consider the following boundary value problem

(2.1) \[ L(D_t)u(t) = f(t), \quad t > 0 \]

(2.2) \[ B_j(D_t)u(0) = \alpha_j, \quad \text{for} \quad j = 0, 1, \ldots, p - 1 \]
where the $\alpha_j$ are complex constants. Set

$$(2.3) \quad L_p(z) = (z - \tau_0)(z - \tau_1)\ldots(z - \tau_{p-1})$$

where the $\tau_j$'s are the $p$ roots of $L(z)$ having positive imaginary parts.

Firstly, we assume that $m_j < p$ for $j = 0, 1, \ldots, p - 1$, then

$$(2.4) \quad B_j(z) = \sum_{k=0}^{p-1} b_{jk} z^k$$

for $j = 0, 1, \ldots, p - 1$. Now, if the matrix $(b_{jk})$ is nonsingular, one can solve Eq. (2.4) for the unknown variables $z^k$ (for fixed $z$).

Hence, if we denote by $(b^{jk})$ the inverse matrix of $(b_{jk})$, we get

$$(2.5) \quad z^j = \sum_{k=0}^{p-1} b^{jk} B_k(z)$$

for $j = 0, 1, \ldots, p - 1$. So that, Eq. (2.2) is equivalent to

$$(2.6) \quad D^k u(0) = \sum_{j=0}^{p-1} b^{kj} \alpha_j$$

for $k = 0, 1, \ldots, p - 1$. Thus, any solution of the Cauchy problem (2.1), (2.2) is a solution of the problem

$$(2.7) \quad L(D_t)u(t) = f(t), \; t > 0$$

$$(2.8) \quad D^k u(0) = \sum_{j=0}^{p-1} b^{kj} \alpha_j$$

for $k = 0, 1, \ldots, p - 1$. Conversely, any solution of (2.7), (2.8) is a solution of (2.1), (2.2).

**Remarks.** If the matrix $(b_{jk})$ is singular, then there are constants $c_j$ not all zero so that

$$(2.9) \quad \sum_{j=0}^{p-1} c_j B_j(z) = 0$$

Consequently, a necessary condition for the problem (2.1), (2.2) to have a solution is the following

$$(2.10) \quad \sum_{j=0}^{p-1} c_j \alpha_j = 0$$
This shows that we can not hope to get a solution for all choices of the \( \alpha_j \)'s. Furthermore, even when we can solve, the solution is generally not unique.

Now if we allow to the \( m_j \)'s to be greater than \( p \), then, by partial fractions we can write

\[
(2.11) \quad B_j(z) = Q_j(z) L_p^+ (z) + B'_j(z)
\]

Where the degree of \( B'_j \) is less than \( p \).

When \( p < m \) we define the polynomial

\[
(2.12) \quad L^-(z) = \frac{L(z)}{L_p^+(z)}
\]

We conclude by the following Lemma [5]:

**Lemma.** Let \( L(D_t) \) be any constant coefficient differential operator, and let \( f(t) \) be any function in \( \mathcal{S}(\mathbb{R}) \). Then, there exists a function \( u(t) \in \mathcal{S}(\mathbb{R}) \) which satisfies the differential equation

\[
(2.13) \quad L(D_t)u(t) = f(t), \text{ for } t > 0
\]

that the differential equation

\[
(2.14) \quad L^-(D_t)v(t) = f(t), \quad t > 0
\]

has always a solution \( v(t) \) belonging to \( \mathcal{S}(\mathbb{R}) \) for any choice of \( f(t) \) in \( \mathcal{S}(\mathbb{R}) \). Thus, the boundary value problem (2.1), (2.2) is equivalent to the following

\[
(2.15) \quad L_p^+(D_t)u(t) = v(t), \quad t > 0
\]

\[
(2.16) \quad B'_j(D_t)u(0) = \alpha_j - Q_j(D_t)v(0), \quad j = 0, 1, ..., p - 1
\]

which is the form just treated. If we write

\[
(2.17) \quad B'_j(z) = \sum_{k=0}^{p-1} b'_{jk} z^k, \quad 0 \leq j \leq p - 1
\]

then the problem (2.15), (2.16) has a unique solution for any given \( v(t) \) in \( \mathcal{S}(\mathbb{R}) \) and \( \{\alpha_j\} \subset \mathbb{C} \) if and only if the matrix \( (b'_{jk}) \) is nonsingular.

We observe from Eq. (2.9) that the matrix \( (b_{ij}) \) is nonsingular if and only if the polynomials \( B_j(z) \) are linearly independent. Similarly, the matrix \( (b'_{ij}) \) is nonsingular if and only if the polynomials \( B'_j(z) \) are linearly independent.

We say that the \( \{B_j(z)\} \) are linearly independent modulo \( L_p^+(z) \) if the \( \{B'_j\} \) are linearly independent.
2. Main results.

Now we are able to give explicitly the solution to the boundary value problem (2.1), (2.2) in terms of Green’s function; we shall follow closely Pederson’s work after redefining in a convenient manner the functions $u^+_j(t)$ (see [3], [4]).

**Theorem 1.** Let $L(z)$ be a polynomial of degree $m$, having at most $p$ roots $\tau_0, \tau_1, \ldots, \tau_{p-1}$ with positive imaginary parts and no real roots. Let $\{B_j(z)\}_{j=0}^{p-1}$ be $p$ polynomials of degrees $\{m_j\}$ with $m_j < m$, which are linearly independent modulo

$$L^+_{p}(z) = (z - \tau_0)(z - \tau_1) \cdots (z - \tau_{p-1})$$

Then, for any $f \in \mathcal{S}(\mathbb{R}^+)$ and for any choice of the constants $\alpha_0$, $\alpha_1$, $\ldots$, $\alpha_{p-1}$, there exists a unique solution $u(t) \in \mathcal{S}(\mathbb{R}^+)$ satisfying the boundary value problem (2.1), (2.2). Furthermore, this solution can be represented as follows

$$(2.18) \quad u(t) = \sum_{j=0}^{p-1} f_j(t) \alpha_j + \int_0^\infty g(t,s)f(s) \, ds$$

**Proof.** Let us first consider the case $f = 0$; then, the general solution of the equation

$$(2.19) \quad L(D_t)u(t) = 0$$

has the form

$$u(t) = \sum_{k=0}^{m-1} \beta_k \exp(i \tau_k)$$

where the $\tau_k$’s are the roots of $L(z) = 0$. It is worth to recall that the coefficients $\beta_k$ become polynomials in $t$ whenever there are multiple roots. Now, in order for $u(t)$ to be in $L^2(0, \infty)$, the coefficients $\beta_k$ must vanish for any $k$ such that $\Im \tau_k \leq 0$ otherwise $u(t)$ could not be in $L^2(0, \infty)$. Therefore, the solution of (2.19) which belongs to the space $L^2(0, \infty)$ is

$$(2.20) \quad u(t) = \sum_{k=0}^{p-1} \beta_k \exp(i \tau_k)$$

with $\Im \tau_k > 0$.

Following AGMON, DOUGLIS and NIRENBERG [1], we define

$$(2.21) \quad L^+_{k}(\tau) = \sum_{j=0}^{k} a_{j}^+ \tau^{k-j}, \quad k = 0, 1, \ldots, p - 1$$
where the constants $a_j^+$ are defined through the expression

\begin{equation}
L_p^+(\tau) = \sum_{j=0}^{p} a_j^+ \tau^{p-j}
\end{equation}

Let $\Gamma^+$ and $\Gamma^-$ be rectifiable Jordan contours in the upper and the lower half plane enclosing the roots of $L_p^+(z)$ and $L_p^-(z)$ respectively.

Define the functions

\begin{equation}
u_j^+(t) = \frac{1}{2\pi i} \oint_{\Gamma^+} \frac{L_{p-j-1}^+(\tau)}{L_p^+(\tau)} \exp(it\tau) \, d\tau
\end{equation}

We claim that

\begin{equation}
L(D_t)u_j^+(t) = 0, \quad t > 0
\end{equation}

and

\begin{equation}
D_k^t u_j^+(0) = \delta_{jk}
\end{equation}

for $j, k = 0, 1, ..., p - 1$, where $\delta_{jk}$ is the Kronecker Delta. Indeed, by differentiation under the integral sign (which is of course allowed), we get

\begin{equation}
D_k^t u_j^+(t) = \frac{1}{2\pi i} \oint_{\Gamma^+} \frac{L_{p-j-1}^+(\tau)}{L_p^+(\tau)} \tau^k \exp(it\tau) \, d\tau
\end{equation}

Now, if we take $\Gamma^+$ to be a large circle about the origin with radius $n \in \mathbb{N}^*$ so that $\Gamma^+$ encloses $\tau_0$, $\tau_1$, ..., $\tau_{p-1}$, then,

\begin{equation}
D_k^t u_j^+(0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{L_{p-j-1}^+(n e^{i\omega})}{L_p^+(n e^{i\omega})} n^{k+1} e^{i\omega(k+1)} \, d\omega
\end{equation}

\begin{equation}
= \frac{1}{2\pi} \int_0^{2\pi} \frac{Q(n e^{i\omega})}{L_p^+(n e^{i\omega})} \, d\omega
\end{equation}

Where $Q$ is a polynomial of degree $p + k - j$ in $n$. Since $L_p^+$ is of degree $p$, then, by letting $n$ go to infinity, we obviously get

\begin{equation}
D_k^t u_j^+(0) = \delta_{jk}, \text{ for } k - j \leq 0
\end{equation}

For the case $k - j > 0$, we note that the polynomial $\tau^k L_{p-j-1}^+$ differs from $\tau^{k-j-1} L_p^+$ by a polynomial $Q$ of degree at most equals to $k - 1$. Thus,
By the same argument as before, since the degree of $Q(z)$ is equal to $k - 1 < p - 1$, we can observe that the last integral is zero. As a consequence, we obtain

$$D^k_t u_j^+(0) = \delta_{jk}, \quad j, k = 0, 1, ..., p - 1$$

On the other hand we have

$$L(D_t) u_j^+(t) = \frac{1}{2\pi i} \oint_{\Gamma^+} \frac{L_{p-j-i}(\tau)L_p^-(\tau)}{L_p^+(\tau)} e^{i\tau t} d\tau = 0.$$ 

It follows that the set $\{u_j^+(t)\}$ spans the negative exponential solutions of the homogeneous boundary value problem associated to (2.1), (2.2). Now, in order to obtain a solution to the inhomogeneous boundary value problem (2.1), (2.2), we define the functions

$$v^+(t) = \frac{1}{2\pi} \oint_{\Gamma^+} e^{i\tau t} L(\tau) d\tau$$

It follows from the fact that the contour $\Gamma^+ \cup \Gamma^-$ can be deformed into a large circle that

$$D^k_t (v^+(0) + v^-(0)) = i\delta_{m-1,k} \quad \text{where} \quad k = 0, ..., m - 1.$$ 

As a consequence, the function

$$\int_0^t (v^+(t - s) + v^-(t - s)) f(s) \, ds$$

is a solution of the Eq. (2.1) with zero Cauchy Data.

Thus, the general solution of the inhomogeneous boundary value problem (2.1), (2.2) which is bounded must have the form:
Explicit representation of the solution...

(2.30) \[ u(t) = \sum_{j=0}^{p-1} \beta_j u_j^+(t) + \int_0^t (v^+(t-s) + v^-(t-s))f(s) \, ds \]
\[ - \int_0^\infty v^-(t-s)f(s) \, ds \]

This is a consequence of the facts that \( v^+(t) \) is a sum of the negative exponentials when \( t > 0 \), and \( v^-(t) \) is a sum of negative exponentials when \( t < 0 \). Now, by virtue of the complementing condition (linear independence of \( B_j(s) \)), we conclude that

\[ B_k(D_t) \int_0^t (v^+(t-s) + v^-(t-s))f(s) \, ds \bigg|_{t=0} = 0 \]

Since the function (2.30) is a formal solution of (2.1), (2.2) then, it must satisfy the following

\[ \alpha_k = B_k(D_t)u(0) = \sum_{j=0}^{p-1} \beta_j B_k(D_t)u_j^+(0) + \]
\[ + B_k(D_t) \int_0^t (v^+(t-s) + v^-(t-s))f(s) \, ds \bigg|_{t=0} - \]
\[ - \int_0^\infty f(s)B_k(D_t)v^-(t-s) \, ds \bigg|_{t=0} \]
\[ = \sum_{j=0}^{p-1} \beta_j \{ Q_k(D_t)L_p^+(D_t)u_j^+(0) + B'_k(D_t)u_j^+(0) \} - \]
\[ - \int_0^\infty f(s)B_k(D_t)v^-(t-s) \, ds \bigg|_{t=0} \]
\[ = \sum_{j=0}^{p-1} \beta_j b_{kj}^+ - \int_0^\infty f(s)B_k(D_t)v^-(t-s) \, ds \bigg|_{t=0} \]

where \( B_k(z) = Q_k(z)L_p^+(z) + B'_k(z) = B_k(z) \mod(L_p^+) \) and
We get an algebraic system of equations with unknown variables $\beta_j$

$$B'_k(z) = \sum_{j=0}^{p-1} b_{kj}^+ z^j$$

We deduce from the complementing condition that the determinant of the matrix $(b_{kj}^+)$ is not zero; so, as a consequence, the above set of equations has a unique solution $\{\beta_0, \ldots, \beta_{p-1}\}$.

Define the inverse matrix

$$(b^+_{kj}) = (b_{kj}^+)^{-1}$$

Hence,

$$\beta_j = \sum_{k=0}^{p-1} b_{kj}^+ \alpha_k + \int_0^\infty \sum_{k=0}^{p-1} b_{kj}^+ f(s) B_k(D_t) v^-(t - s) \left|_{t=0}^\infty \right., \quad j = 0, 1, \ldots, p - 1$$

and upon substitution of the $\beta_j$'s into (2.30) we obtain the bounded solution of the given BVP,

$$u(t) = \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} b^+_{kj} \alpha_k u^+_j(t) +$$

$$+ \left\{ \int_0^\infty \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} b^+_{kj} B_k(D_t) v^-(t - s) \left|_{t=0}^\infty \right. f(s) ds \right\} u^+_j(t) +$$

$$+ \int_0^t v^+(t - s)f(s) ds - \int_t^\infty v^-(t - s)f(s) ds$$

If we set

$$\mathcal{H}_k(t) = \frac{1}{2\pi i} \oint_{L^+_{p-j-1}(\tau)} \frac{\sum_{j=0}^{p-1} b^+_{kj} L^+_{p-j-1}(\tau)}{L^+_p(\tau)} e^{i\tau\tau} d\tau$$

for $k = 0, \ldots, p - 1$
\[ G_1(t) = \frac{1}{2\pi} \int_{\Gamma^+} e^{i\tau} \frac{d\tau}{L(\tau)} , \text{ if } t > 0 \]

\[ = -\frac{1}{2\pi} \int_{\Gamma^-} e^{i\tau} \frac{d\tau}{L(\tau)} , \text{ if } t < 0 \]

\[ G_2(t,s) = -\frac{i}{4\pi} \int_{\Gamma^+} \int_{\Gamma^-} \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} L^+_{p-j-k}(\tau) B_k(\delta) \frac{d\tau}{L^+(\tau)L(\delta)} e^{i(\tau-s\delta)} d\tau d\delta \]

and

\[ G(t,s) = G_1(t-s) + G_2(t,s) \]

then, the solution of the boundary value problem (2.1), (2.2) takes the final form

\[ u(t) = \sum_{j=0}^{p-1} \mathcal{H}_j(t) \alpha_j + \int_0^\infty G(t,s)f(s) ds \]

An immediate computation shows that the above kernels satisfy the following estimates

\[ |D^k H_j(t)| \leq C_0 \exp(-r_0|t|), \quad \forall t > 0, \forall k = 0, 1, \ldots \]

\[ |D^k G_1(t)| \leq C_1 \exp(-r_1|t|), \quad \forall t \in \mathbb{R}^+, \forall k = 0, 1, \ldots \]

\[ |D^k G_2(t,s)| \leq C_2 \exp(-r_2(t+s)), \quad \forall t > 0, \forall s > 0, \forall k = 0, 1, \ldots \]

for some positive constants \( C_0, C_1, C_2, r_0, r_1, \) and \( r_2, \) depending only on \( L(z) \) and \( \{B_j\}. \)

Now to see that the expression (2.18) is rapidly decreasing at infinity it suffices to show that the function

\[ v_j(t) = t^j \int_0^\infty |f(s)| (C_1 \exp(-r_1|t-s|) + C_2 \exp(-r_2(t+s))) ds \]

is bounded for each nonnegative integer \( j. \) Since \( f \in \mathcal{S}(\mathbb{R}^+) \) there is a constant \( C > 0 \) such that

\[ |f(s)| \leq \frac{C}{(1+s)^{j+2}}, \forall s > 0 \]

it then follows that
\[ v_j(t) \leq C_3 J \left( \frac{2}{t} \right)^j \int_0^t \frac{ds}{(1+s)^{j+2}} + C_1 \frac{(2t)^j}{(2+t)^j} \int_0^t \frac{ds}{(1+s)^j} + C_3 t^j \int_0^\infty \exp(-r_2(t+s)) \, ds \]

\[ \leq C(j) \int_0^\infty \frac{ds}{(1+s)^2} + C_3 t^j \exp(-r_2t) < +\infty \]

Hence \( v_j(t) \) is bounded in \( \mathbb{R}^+ \) and consequently \( u(t) \in \mathcal{S}(\mathbb{R}^+) \).

Finally, using classical techniques we can easily prove the uniqueness of this solution. This establishes the proof of the given theorem.

Let us denote by \( H^k(\mathbb{R}^+), k \geq 0 \) the completion of the space \( \mathcal{S}(\mathbb{R}^+) \) with respect to the norm

\[ \|u\|_k^2 = \sum_{j=0}^k \int_0^\infty |u^{(k)}(t)|^2 \, dt \]

and we define the subspace

\[ V^k = H^k(\mathbb{R}^+) \cap C^k[0, \infty] \]

As a consequence of the previous representation theorem and Theorem 6–9 [5] we obtain the estimate of the solution to the problem (2.1)–(2.2) in terms of the Data \( f \) and \( \alpha_0, \ldots, \alpha_{p-1} \):

**Theorem 2.** Under the same assumptions of Theorem 1, we conclude that for each \( k \in \mathbb{N} \), there is a constant \( C > 0 \) (depending only on \( L(z), B_j(z) \) and \( k \)) such that, for each \( f \in V^k \) and \( \alpha_0, \ldots, \alpha_{p-1} \in C^0 \), the solution \( u \in V^{m+k} \) to the BVP (2.1) – (2.2) satisfies the estimate

\[ \|u\|_{m+k} \leq C \left( \sum_{j=0}^{p-1} |\alpha_j| + \|f\|_k \right) \]

and has the representation

\[ u(t) = \sum_{j=0}^{p-1} \mathcal{H}_j(t) \alpha_j + \int_0^\infty \mathcal{G}(t,s) f(s) \, ds \]

(where \( \mathcal{H}_j(t) \) and \( \mathcal{G}(t,s) \) are the same as in Theorem 1.).
Proof. We deduce from the density of $s_{\mathbb{R}^+}$ in $V^k$ that there is a sequence $(f_n) \subset s_{\mathbb{R}^+}$ converging to $f$ in $V^k$. On the other hand there corresponds to each $f_n$ at most one solution $u_n \in s_{\mathbb{R}^+}$ satisfying:

$$L(D_t)u_n(t) = f_n(t), \ (t > 0)$$

$$B_j(D_t)u_n(0) = \alpha_j, \ j = 0, \ldots, p - 1$$

and given by

$$(2.33) \quad u_n(t) = \sum_{j=0}^{p-1} \mathcal{H}_j(t)\alpha_j + \int_0^{\infty} \mathcal{G}(t,s)f_n(s) \, ds$$

We conclude by Theorem 6–9 [5] that there is a constant $C_0 > 0$ depending only on $L(z)$ and $k$ such that

$$\|u_n - \sum_{j=0}^{p-1} \mathcal{H}_j(t)\alpha_j\|_{m+k} \leq C_0 \|f_n\|_k$$

and

$$\|u_n - u\|_{m+k} \leq C_0 \|f_n - f\|_k$$

Hence,

$$\|u\|_{m+k} \leq \sum_{j=0}^{p-1} \|\mathcal{H}_j\|_{m+k} \cdot |\alpha_j| + C_0 \|f\|_k \leq C \left( \sum_{j=0}^{p-1} |\alpha_j| + \|f\|_k \right)$$

where $C = \max\{C_0, \|\mathcal{H}_j\|_{m+k}; \ j = 0, \ldots, p - 1\}.$

The estimate (2.36) shows that the isomorphism

$$\Psi : (\alpha_0, \ldots, \alpha_{p-1}, f) \rightarrow u$$

is continuous from $C^p \times V^k$ onto $V^{m+k}$. Consequently, by letting $n \rightarrow +\infty$ in (2.33) we obtain

$$u(t) = \sum_{j=0}^{p-1} \mathcal{H}_j(t)\alpha_j + \int_0^{\infty} \mathcal{G}(t,s)f(s) \, ds, \ (t > 0)$$

$$= \Psi(\alpha_0, \ldots, \alpha_{p-1}, f)$$

This proves the theorem.

Remarks. 1) If $L(z)$ admits a real root then we cannot hope to get an
estimate of the form (2.36) even under smooth Data as shows the following example:

\[
\frac{du}{dt} = \frac{1}{t+1} \in L^2(\mathbb{R}^+) \cap C^\infty(\mathbb{R}^+)
\]

\[u(0) = \alpha_0\]

whose unique solution is

\[u(t) = \alpha_0 + \ln(1+t), \quad (t > 0)\]

which is not in \(L^2(0, \infty)\) whatsoever the value of the constant \(\alpha_0\).

2) The best constant \(C\) in (2.36) is equal to the norm of the isomorphism \(\mathfrak{P}\) defined by

\[
\sup \left| \sum_{j=0}^{p-1} \mathcal{H}_j(t)\beta_j + \int_0^\infty \mathcal{G}(t,s)h(s) \, ds \right|
\]

where the supremum is taken over all \(h \in V^k\) and \(\beta_0, \ldots, \beta_{p-1} \in C\) such that

\[
\sum_{j=0}^{p-1} |\beta_j| + \|h\|_k = 1
\]

REFERENCES