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EDGE OPERATIONS

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Abstract.

We introduce and describe a group $\mathscr{E}\mathscr{C}_2$ that acts on *dessins d'enfant* (a class of graphs on Riemann surfaces considered by Grothendieck) with one marked directed edge. The group is constructed with the help of a, presumably new, operation -- *semiflip* -- a "half" of a known operation *flip*. The group $\mathscr{E}\mathscr{C}_2$ is generated by the semiflip and the known operations of the cartographic group. Our main result: the action of $\mathscr{E}\mathscr{C}_2$ on a set of dessins with a marked directed edge with given number of edges and given genus is transitive.

§ 0. Introduction.

Our interest to *dessins d'enfant* has been inspired by Alexander Grothendieck's work [Gro], which suggests that there exists an equivalence between the category of dessins d'enfant (topological objects) and the category of *Belyi pairs* (see [ShVo]), which are objects of arithmetic geometry. Since the Galois group $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ acts on Belyi pairs, Grothendieck theory offers a possibility to visualize its action via dessins d'enfant. In the present paper we take a purely combinatorial approach and construct a group $\mathscr{E}\mathscr{C}_2$ that acts on dessins with one marked flag. To construct a group we introduce a new operation, *semiflip*, that resembles a "half" of the operation *flip* on triangulations (see [BKKM]).

The structure of this paper is as follows: in §1 we give the definitions, state the results, and explain intuitevely the ideas of proofs. This makes some of these statements look rather obvious; however, we have decided to include the formal combinatorial proofs, which can be found in §2. A brief discussion follows in §3.

We are indebted to Alexandre Zvonkin, Dimitry Zvonkin and Valentin Silantyev for stimulating discussions. We are grateful to the University of Stockholm and Dimitry Leites for hospitality at the final stages of the work. We would also like to thank Gavril Farkas for pointing to the reference [Gl].

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§ 1. Definitions and results.

1.1. Our main object, *dessin d'enfant* (or simply *dessin*) is a pair $X_2 \supset X_1$, where X_2 is a compact oriented surface and X_1 is a graph with a finite number of vertices such that $X_2 \backslash X_1$ is homeomorphic to a finite disjoins of disks. In other words, every 2-cell of $X_2 \backslash X_1$ must be topologically trivial. Graphic understanding of this object is sufficient for this paper.

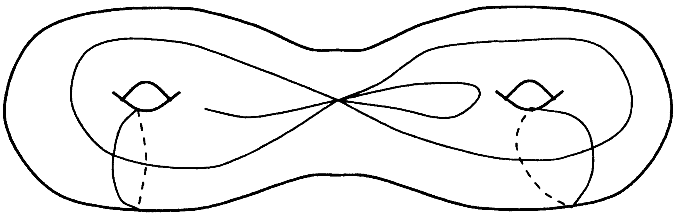

Figure 1. Dessin d'enfant

1.2. To deal with dessins combinatorially, we introduce a *map* -- a triple (M, a, i), where M is a finite set, $a, i \in \mathrm{Aut}(M)$, $i^2 = 1$, $< a, i > \subset \mathrm{Aut}(M)$ acts transitively on M and i has no fixed points in M. The elements of M will be called *flags*.

Each map defines a certain dessin. Take a map (M, a, i). To each orbit of $< a >$ we associate an oriented polygon such that its edges are in bijective correspondence with the flags of the orbit, and for any flag f of this orbit af is next to f in the counterclockwise direction.

On the set of polygons we obtain, the involution i defines the pairwise identification of edges. Now, we can glue these polygons together using the following rules:

(1) any flag f is glued to if,

(2) the orientation on the polygons must match each other after gluing.

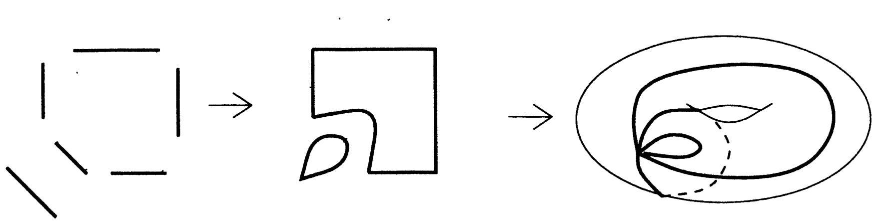

Figure 2. Restoring a dessin from a map

As a result, we get the graph X_1 (its edges are the glued edges of polygons) on the oriented surface X_2 (see [Gl], p.136, for a definition of oriented manifold). It is a dessin.

Conversely, given a dessin, we can restore the corresponding map. To do this, we represent each edge of a dessin as a pair of flags bordering the adjacent cells (the involution i maps them onto each other). The permutation a maps each flag to the next one bordering the same cell (in the counterclockwise direction).

We focus our attention on *dessins with one marked flag*. When drawing a marked flag we draw an arrow in place of the corresponding edge, pointed counterclockwise with respect to the cell that the marked flag borders.

1.3. We now define the new unary operation on the set of dessins with one marked flag. This operation will be called *semiflip* (and denoted ψ). Graphically, it will look as in Fig. 3:

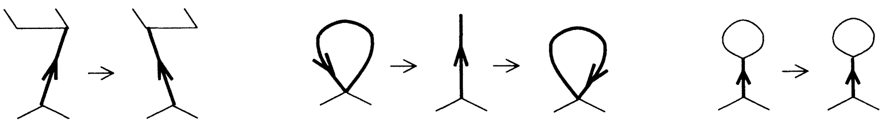

Figure 3. Semiflip

This operation is "local", meaning it only changes the area around the marked flag without changing the rest of the dessin; for this reason we draw only those parts of the dessin that are changed.

Now, let us define semiflip in terms of maps. Let (M, a, i) be a map and $f \in M$ be a marked flag. Take two transpositions: $A, B \in \mathrm{Aut}(M)$, where A swaps f with $a^{-1}if$, and B swaps f with af. Then

$$\psi(M, a, i) := (M, aAB, i)$$

with f as a marked flag.

Several remarks below motivate this definition.

Take a dessin (M, a, i) with a marked flag f. Let us denote by $Y(f)$ the set $\{f, af, a^{-1}if\} \subset M$.

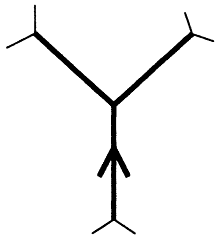

Figure 4. $Y(f)$

STATEMENT. *$Y(f)$ consists of two or three elements.*

PROOF. Suppose it consists of just one element, which means $f = af = a^{-1}if$; then $f = af = a(a^{-1}if) = if$ which is impossible because i has no fixed points in M by definition.

Now, we classify dessins with one marked flag by the position of the marked flag. If $\#Y(f) = 3$, the dessin will be called *regular*; if $\#Y(f) = 2$, it will be called *degenerate*. The degenerate cases may be classified (see Fig.5):

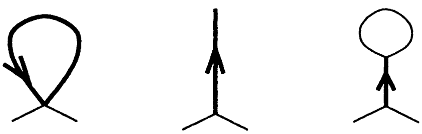

Figure 5: Degenerate dessins with marked flag

1) if $af = f$, this case will be called a *loop*;
2) if $a^{-1}if = f$, this case will be called a *tail*;
3) if $a^{-1}if = af$, this case will be called a *bubble*.

The semiflip has to be defined separately for the degenerate cases.

Take a bijection $\nabla : M \longrightarrow M$ such that it is the identity outside $Y(f)$, and inside $Y(f)$ it acts like this:

if the marked flag f is nondegenerate, it is a cyclic permutation $f \to af \to a^{-1}if \to f$;

if the marked flag f is a loop or a tail, it permutes two elements of $Y(f)$;

if the marked flag f is a bubble, it is the identity.

Now, if we put $\psi(M, a, i) := (M, a\nabla, i)$, we get exactly the operation displayed in Figure 3. In the first definition of semiflip we represented ∇ as a product of two operators so that the definition is invariant.

Note on notation. Because the set of flags M and the involution i do not change under semiflip as well as under other operations we consider, we shall omit, where possible, the notation for M and i. The dessin with a marked flag will be denoted as (a,f). The operator ∇ depends on both f and a, so we write $\nabla(a,f)$.

Thus, the definition of the semiflip can be written as

$$\psi(a,f) = (a\nabla(a,f),f).$$

To denote permutations we use cyclic notation: for example, in the regular case $\nabla(a,f) = (f, af, a^{-1}if)$.

1.4. We are going to use the semiflip to construct a group, so we should prove its invertibility. The formal combinatorial proof is given further in section 2.1; its idea can, however, be well seen in the pictures (Fig. 6): if we look at the application of ψ "in the mirror", what we actually see is the application of ψ^{-1}. More precisely, if we introduce the reflection operator μ, then $(\psi\mu)^2 = 1$, hence, $\psi^{-1} = \mu\psi\mu$.

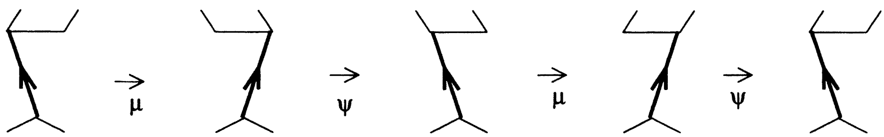

Figure 6: Semiflip's invertibility

1.5. The *cartographic group* $\mathscr{C}_2^+$ is the free product $\mathsf{Z} * \mathsf{Z}/(2)$ with marked generators ρ_2 and ρ_1 in Z and $\mathsf{Z}/(2)$, respectively (the elements of this group are often dubbed "Grothendieck operators"). It acts on the set of dessins with marked flags (via their maps) by changing the marked flag: ρ_2 marks the flag which is next in the counterclockwise direction to the one previously marked, and ρ_1 marks the opposite to the previously marked. In our notations we may write this as:

$$\rho_2(a,f) = (a, af), \quad \rho_1(a,f) = (a, if).$$

1.6. Let us denote the set of dessins with marked flags by D'; we have defined the operators ρ_1, ρ_2 and ψ that act on D'.

DEFINITION. The subgroup $< \rho_1, \rho_2, \psi >$ Aut(D') will be called the *edge group* and denoted $\mathscr{E}\mathscr{C}_2$.

1.7. The semiflip (and, hence, the edge group) does not change the genus of the dessin. This follows from the Euler formula:

(number of vertices) − (number of edges) + (number of faces) $= 2 - 2 \cdot$ genus.

The semiflip only changes the number of vertices and number of faces in cases of loop and tail, but the value in the left part remains constant.

1.8. MAIN THEOREM. *The action of the edge group on the set of dessins with one marked flag with a given number of edges and a given genus is transitive.*

For the proof see 2.2. It states that $\mathcal{EC}_2$ can transform a dessin with given genus and given number of edges into a certain canonical form. This is done in two stages: first, $\mathcal{EC}_2$ transforms the dessin into a dessin with one face. The dessin is a polygon with its edges glued together to form a surface and, possibly, with some edges drawn on it. In the second stage we prove $\mathcal{EC}_2$-equivalency of the dessins with one face using Gauss words to describe them; the canonical form is one of the dessins with one face.

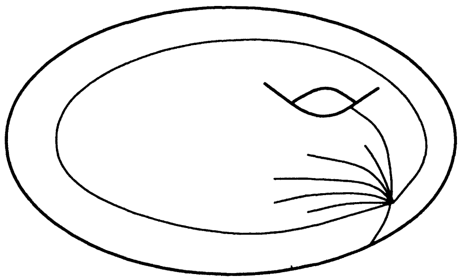

Figure 7 : The canonical form of the dessin

1.9. The investigation of the structure of the edge group has so far led us to find two relations its elements satisfy (see also 3.3). Consider the operators $T := \rho_2^{-1}\rho_1\psi$ and $S := \rho_2^{-2}\rho_1\psi^2$.

THEOREM. $T^6 = S^{12} = 1$.

A proof is outlined in 2.3. The restrictions of these operators to different cases have different orders. In the regular case these relations are displayed on Fig. 8.

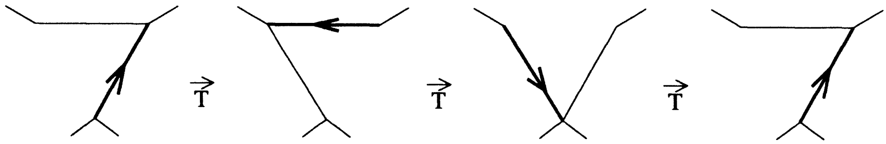

Figure 8a: $T^6 = 1$

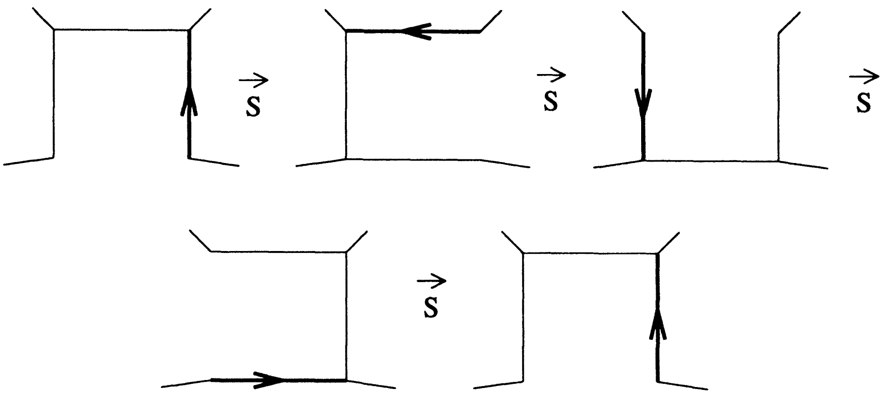

Figure 8b: $S^{12} = 1$

§ 2. Proofs.

2.1. *Semiflip's invertibility.*

To formally prove semiflip's invertibility, we introduce a reflection operator $\mu : (a,f) \to (a^{-1}, if)$.

THEOREM $(\mu\psi)^2 = 1$.

PROOF. Consider $R = \mu\psi$. Because it includes a semiflip we must consider various positions of the marked flag in the dessin R is applied to. It turns out that:

(a,f) is a loop $\Longrightarrow$ $R(a,f)$ is a tail;
(a,f) is a tail $\Longrightarrow$ $R(a,f)$ is a loop;
(a,f) is a bubble $\Longrightarrow$ $R(a,f)$ is a bubble;
(a,f) is regular $\Longrightarrow$ $R(a,f)$ is regular.

We first check this for a degenerate (a,f) and then see that if (a,f) is regular, $R(a,f)$ cannot be degenerate.

By definition, $\psi(a,f) = (a\nabla(a,f),f)$; hence, $\psi(a,f) = R(a,f) = (\nabla^{-1}(a,f)a^{-1}, if)$.

If (a,f) is a loop: $af = f$, $\nabla(a,f) = (f, a^{-1}if)$ (transposition);

$R(a,f) = ((f, a^{-1}if)a^{-1}, if)$ is a tail: $(f, a^{-1}if)a^{-1}if = f = iif$.

If (a,f) is a tail: $af = if$, $\nabla(a,f) = (af, a^{-1}if)$;

$R(a,f) = ((af, a^{-1}if)a^{-1}, if)$ is a loop: $(af, a^{-1}if)a^{-1}if = af = if$.

If (a,f) is a bubble: $af = a^{-1}if$, $\nabla(a,f) = 1$,

$R(a,f) = (a^{-1}, if)$ is a bubble: $a^{-1}if = aiif = af$.

Now, if (a,f) if regular: $\nabla(a,f) = (f, af, a^{-1}if)$, $R(a,f) = ((f, a^{-1}if, af)a^{-1}, if)$.

Suppose $R(a,f)$ is a loop: $(f, a^{-1}if, af)a^{-1}if = if \Longrightarrow af = if \Longrightarrow (a,f)$ is a tail.

Suppose $R(a,f)$ is a tail: $(f, a^{-1}if, af)a^{-1}if = iif \Longrightarrow af = f \Longrightarrow (a,f)$ is a loop.

Suppose $R(a,f)$ is a bubble: $(f, a^{-1}if, af)a^{-1}if = a(f, af, a^{-1}if)iif \Longrightarrow af = a^2f \Longrightarrow (a,f)$ is a loop.

It remains to check that $R^2 = 1$ in all these four cases.

If (a,f) is regular, $R^2(a,f) = ((if, a^2f, af)a(f, af, a^{-1}if), f)$. To see that $R^2(a,f) = (a,f)$, one must simply check that $(if, a^2f, af)a(f, af, a^{-1}if)a^{-1} = 1$.

If (a,f) is a loop, $R^2(a,f) = ((f, if)a(f, a^{-1}if), f) = (a,f)$ because $(f, if)a(f, a^{-1}if)a^{-1} = 1$.

If (a,f) is a tail, $R^2(a,f) = ((if, af)a(f, af), f) = (a,f)$ because $(if, af)a(f, af)a^{-1} = 1$.

If (a,f) is a bubble, $R^2(a,f) = (a,f)$.

2.2. *Transitivity theorem.*

We shall prove that all the dessins with a given number of edges on a surface of given genus are $\mathscr{E}\mathscr{C}_2$-equivalent, i.e., belong to the same $\mathscr{E}\mathscr{C}_2$-orbit.

STATEMENT. *Any dessin is $\mathscr{E}\mathscr{C}_2$-equivalent to a dessin with one face.*

PROOF. In a dessin with more than one face, there exists a flag f such that if does not belong to the orbit $< a > f$. Then if we mark f (via the cartographic group) and apply semiflip to (a,f), the orbit $< a > f$ becomes shorter; repeating this, we obtain a loop and after one more semiflip the number of faces will decrease.

To work with dessins with one face we use Gauss words, which are a special kind of cyclic words. Let us recall that a *cyclic word* is a class of equivalence of words in an alphabet, where the equivalence relation is defined as follows: two words W_1 and W_2 are equivalent, if there exist words A and B such that $W_1 = AB$, $W_2 = BA$. For each letter of a Gauss word, *next* letter is correctly defined: take a representative in which the given letter is not the last (this is always possible since the length of a Gauss word is at least two), and then take the next letter in the usual sense.

Now, we define a *Gauss word* as a cyclic word in which every letter occurs exactly twice. Each Gauss word defines a unique map as follows: the set of letters occuring in a Gauss word is taken for the set of flags. Now, we can take a permutation which maps a letter into the next one and the involution which permutes two occurencies of the same letter, and clearly we get a map. Conversely, each map defines a Gauss word, uniquely up to the renaming of letters in an alphabet. We shall enclose Gauss words in round brackets.

On the set of Gauss words consider the following operation: mark one of the two occurencies of some letter (we shall do this by typing it boldface), then cut out the letter following it and paste it right in front of the other occurency of the marked letter:

$$(\ldots \mathbf{f}g \ldots f \ldots) \to (\ldots f \ldots gf \ldots).$$

This operation corresponds to a semiflip on a dessin, applied to the flag f. There is one restriction to it --- the second occurency of the marked letter should not be the letter next to it (the semiflip should not be applied to a tail).

A subword of a Gauss word will be called a *block* (of course it is not a cyclic word by itself). By the *equivalence* of two blocks we mean that one can be transformed into the other one using semiflip without changing the remaining part of the Gauss word. The blocks will be enclosed in square brackets.

A block B will be called *transparent* if the blocks Bx and xB (where x is some letter) are equivalent. When speaking of transparent block we do not need to state its location in the Gauss word explicitly because it can be moved through the Gauss word to any location.

A block will be called *isolated* if each of its letters is included in it twice. We will also need to consider two special kinds of blocks: a *handle* is a block of type $[fgfg]$ and a *branch* is a block of type $[gg]$.

LEMMA. *Handles and blocks are transparent.*

PROOF.

$$[fgf\mathbf{g}x] \to [\mathbf{f}xgfg] \to [f\mathbf{g}xfg] \to [fg\mathbf{f}xg] \to [xfgfg]; \quad [g\mathbf{g}x] \to [xgg].$$

THEOREM. *Any isolated block is equivalent to a block that consists of handles and branches (the order is not important because handles and branches are transparent).*

PROOF. It is sufficient to show that from any isolated block it is possible to extract either a handle or a branch. Suppose a block contains no handles or branches. Then it is of one of two kinds:

$[fg \ldots f \ldots g \ldots]$ or $[fg \ldots g \ldots f \ldots]$.

In the first case, we can proceed as follows (note that we will put one arrow which actually represents the application of semiflip several times) :

$[f\mathbf{g} \ldots f \ldots g \ldots] \to [fg\mathbf{f} \ldots\ldots g \ldots] \to [\ldots\ldots fgfg \ldots]$

and we obtain a handle.

In the second case, we have the following possibilities:

a) $[fg \ldots x \ldots g \ldots \mathbf{x} \ldots f \ldots] \to [fg \ldots\ldots fx \ldots g \ldots x \ldots]$ (back to the first case).

b) $[fg \ldots x \ldots g \ldots f \ldots x \ldots] \to [fg \ldots \mathbf{x} \ldots f \ldots\ldots gx \ldots]$ (back to the first case).

c) If we cannot find such x, it means the block between two g's is isolated. Here we need to use induction: suppose the theorem has already been proven for all the shorter isolated blocks, then the block between two g's is equivalent to a collection of handles and branches; hence, it is transparent and we can move it aside. Thus, we get $[fg \ldots g \ldots f \ldots] \to [\ldots fgg \ldots f \ldots]$.

We have obtained a branch.

COROLLARY. *An arbitrary Gauss word is equivalent to a word that consists of handles and branches* (*which will be called its canonical form*).

To deduce the transitivity theorem from the Corollary, it remains to observe that the number of handles in the canonical form is precisely equal to the genus of the corresponding dessin; this follows directly from Euler's formula.

2.3. *Relations within the edge group.*

Proofs of the relations within the edge group is indeed a combinatorial verification of the facts that we have noticed and checked on the pictures. If one trusts the pictures it will be worth it just to ensure, by combinatorics, that no cases were missed. We shall just outline the proofs for a few examples.

Consider the operators $T := \rho_2^{-1}\rho_1\psi$ and $S := \rho_2^{-2}\rho_1\psi^2$.

THEOREM. $T^6 = 1$.

PROOF. The semiflip is applied several times, so we must check the possible combinations of cases. Two of the regular cases here should be looked at separately: if $iaf = a^{-1}f$, then (a,f) will be called bubble-1 and if $iaf = aif$, then (a,f) will be called bubble-2. It turns out that

(a,f) is a loop $\Longrightarrow$ $T(a,f)$ is a tail;

(a,f) is a tail $\Longrightarrow$ $T(a,f)$ is a loop;

(a,f) is a bubble $\Longrightarrow$ $T(a,f)$ is a bubble-1;
(a,f) is a bubble-1 $\Longrightarrow$ $T(a,f)$ is a bubble-2;
(a,f) is a bubble-2 $\Longrightarrow$ $T(a,f)$ is a bubble;
(a,f) is regular $\Longrightarrow$ $T(a,f)$ is regular ("regular" here does not include bubble-1 and bubble-2).

The above can be checked by straightforward calculation.

It remains to check that $T^6 = 1$ in all these six cases. Here and further ahead, we encounter a new situation: $T^6(a,f) = (cac^{-1}, cf)$ where c is some permutation of flags. This shows that the resulting dessin is isomorphic to the original one but flags have changed their places.

If (a,f) is regular, $T^3(a,f) = (cac^{-1}, cf)$ where $c = (f, iaf, if, af)$. Indeed, $T^3(a,f) = (a\nabla_1\nabla_2\nabla_3, iaf)$, where

$$\nabla_1 = (f, af, a^{-1}if), \quad \nabla_2 = (af, if, \nabla_1^{-1}a^{-1}iaf), \quad \nabla_3 = (if, iaf, \nabla_2^{-1}\nabla_1^{-1}a^{-1}f).$$

The equation $a\nabla_1\nabla_2\nabla_3 = cac^{-1}$ can be verified manually, checking consecutively all the possible cases of inclusion or noninclusion of certain flags into the area changed by the nablas.

If (a,f) is a loop, $T^2(a,f) = (cac^{-1}, cf)$, where $c = (f, if)$. Indeed, $T^2(a,f) = (a\nabla_1\nabla_2, if)$, where $\nabla_1 = (f, a^{-1}if)$, $\nabla_2 = (f, if)$.

There is a case in which the equation has to be proven separately: $if = a^{-1}if$ -- a dessin with one edge (a loop) on the sphere. But the verification is obvious.

In other cases, we have $\nabla_1\nabla_2 = (f, if, a^{-1}if) = a^{-1}cac^{-1}$.

Continuing this process in the same manner for other cases we will get a complete proof of the theorem.

THEOREM. $S^{12} = 1$.

PROOF. Let us see the possible layouts. Here we need to consider, separately, the cases *tail* $-$ 1 (in which $a^2f = f$) and *tail* $-$ 2 (in which $iaf = a^2f$). We have:

(a,f) is a loop $\Longrightarrow$ $S(a,f)$ is a loop,
(a,f) is a bubble $\Longrightarrow$ $S(a,f)$ is a bubble,
(a,f) ia a tail $\Longrightarrow$ $S(a,f)$ is a tail-1,
(a,f) is a tail-1 $\Longrightarrow$ $S(a,f)$ is a tail-2,
(a,f) is a tail-2 $\Longrightarrow$ $S(a,f)$ is a tail,
(a,f) is regular $\Longrightarrow$ $S(a,f)$ is regular.

We leave the proof of these assertions and and of the theorem itself to the reader. The technique of the proof is similar to that of the previous theorem.

§ 3. Discussion.

3.1. The definition of semiflip presented in this paper is not the only possible one. One could define it differently in degenerate cases (see Fig.9).

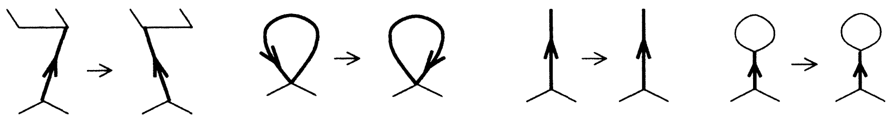

Figure 9. Alternative definition of semiflip.

The operation displayed in Fig.9 is invertible for the same reason the semiflip is. A kind of transitivity theorem for it also exists: the "small edge group", defined using this operation, acts transitively on the set of dessins with given number of edges, given genus and given number of faces. Because of the transitivity of the edge group the small edge group is a subgroup of the edge group.

3.2. It will be useful for applications of the edge group to Grothendieck theory to know what the *effective edge group* (the group of automorphisms of the set of dessins with one marked flag with given genus and given number of edges induced by the edge group) is. At this point we have calculated this group for one- and two-edged spherical dessins, two- and three- edged dessins on the torus and four-edged dessins on the surface of genus 2. In all these cases it turned out to be the symmetric group. It will be interesting to see if this always holds.

3.3. Because the two relations within the edge group we have found look alike, one might suppose there exists a generalization of a kind $(\rho_2^{-n}\rho_1\psi^n)^{6n} = 1$. This does not hold for $n = 3$. We offer, however, a conjecture derived from observations:

CONJECTURE. *The operator $\rho_2^{-n}\rho_1\psi^n$ has a finite order for any n.*

3.4. The groups similar to the ones defined in the present paper could be used in the theory of cell decompositions of moduli spaces of curves. We shall address this topic in our future works.

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