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HARDY SPACES OF ANALYTIC MULTIFUNCTIONS

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Abstract.

In this paper we consider a generalization to analytic multifunctions of the classical Hardy space theory of analytic functions on the unit disc. With $|K(\lambda)| = \sup\big(|z|;\ z \in K(\lambda)\big)$ we define the Nevanlinna class N and the classes H^p by restricting the growth of the integral average of $|K|$ on circles approaching the boundary of D. It is observed that for $K \in N$, $|K(\lambda)|$ has radial boundary values (call them $|K|^{\mathrm{rad}}$) almost everywhere on T, and $\log|K|^{\mathrm{rad}} \in L^1(\mathsf{T})$. We prove that the radial maximal functions are in L^p even for $p < 1$ and with this we prove an inner-outer factorization theorem. We also define a multi-valued generalization B of the Blaschke product. With this every $K \in N$ can be written $K = fB$ where $f \in N$ is a zero-free function. Finally, using other functions on K than $|K|$, we develop criteria, in terms of a weak global condition and a strong boundary condition, for when an analytic multifunction is a function, and also when its values are of capacity zero.

1. Introduction.

The theory of analytic multifunctions has its origin in two parts of mathematics, spectral theory and several complex variables. In spectral theory it is natural to ask how the eigenvalues of an analytic family of matrices, i.e., a family of matrices whose coefficients depend analytically on a parameter, behave. In fact this is an old question; it was studied for the first time by A. Cauchy in the 1830's. From complex analysis one can think of many problems generating multifunctions, for instance the roots of an equation involving analytic expressions. Problems like these have led to the concept of an analytic multifunction. Their main applications are in spectral theory where one has found applications to, among other things, joint spectrum, spectral interpolation, local spectral theory and even to Jordan--Banach algebras. In other parts of mathemathics one has applications to, for instance, uniform algebras in connection with problems of analytic structure and to complex dynamics. We refer to [1] for a survey of the theory of analytic multifunctions and their applications.

In this paper we treat analytic multifunctions from a more function-theoretic point of view. A natural generalization of bounded analytic functions

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on the disc are functions fulfilling some controlled growth condition, for instance uniform bound on the L^p-mean on circles approaching the boundary. These so called H^p-spaces have a large number of interesting properties concerning among other things boundary values and different kinds of factorization. Considering the generalization to analytic multifunctions of these spaces and of the Nevanlinna space N we notice some similarities. We first notice that for $K \in N$, $|K(\lambda)| = \sup\left(|z|;\, z \in K(\lambda)\right)$ have radial boundary values almost everywhere (call them $|K|^{\mathrm{rad}}$) and that $\log|K|^{\mathrm{rad}} \in L^1(\mathsf{T})$. For $K \in H^p$ we get good behaviour on the radial maximal functions $K^\star(e^{it}) = \sup_{r\to 1}|K(re^{it})|$ meaning that $K^\star$ are in L^p. This follows from potential theory alone if $p > 1$, but to get it for $p \leq 1$ we use multifunctional analyticity in an essential way: The nth-root multifunction $R_n(\lambda) = \{z;\, z^n = \lambda\}$ is analytic across zero so we can compose K with this to wander between H^p-spaces.

Using this good behaviour we prove an inner-outer factorization theorem: given $K \in H^p$ there exists an inner multifunction M (i.e., M is a bounded analytic multifunction with $|M|^{\mathrm{rad}} = 1$ almost everywhere) and an outer function $q \in H^p$ such that $K = qM$. For multifunctions in N we prove that each $K \in N$ enjoys a representation as $(b_1/b_2)M$, where b_1 and b_2 are zero-free H^∞-functions and M is inner.

In section 5 we define and study a multi-valued generalization B of the Blaschke product. With this we can write every $K \in N$ as $K = fB$, where f is a zero-free function in N and we also get a structure theorem for inner multifunctions, similar to the one for ordinary inner functions: Every inner multifunction can be written as

$$M(\lambda) = B(\lambda)\exp\left\{-\int_{\mathsf{T}} \frac{e^{it}+\lambda}{e^{it}-\lambda}\,d\nu(t)\right\},$$

where ν is a positive finite measure on T which is singular with respect to the Lebesgue measure. These results helps us to prove that if the zeroes of the multifunction K fulfills the Blaschke condition, then there is a global selection to K, i.e., a function f such that $f(\lambda) \in K(\lambda)$ for all $\lambda \in \mathsf{D}$.

Finally in section 6, we notice that the fact that $\log|K|^{\mathrm{rad}} \in L^1$ is true not only for the radius $|K|$ but also for the nth-diameter $\delta_n(K)$ and the capacity $c(K)$. This helps us to draw strong global conclusions on the geometry of K from a weak global growth condition and a strong boundary condition.

2. Notation.

Let X and Y be sets. A function $K\colon X \to 2^Y$ (where 2^Y stands for the set of all subsets of Y) is called a multifunction. We fix the following notation:

$R_K = \{y \in Y;\ y \in K(x)\ \exists x \in X\}$ is the range of K.
$K^*(B) = \{x \in X;\ \emptyset \neq K(x) \subset B\}$ is the preimage of the set $B \subset Y$.
$K(A) = \{y \in Y;\ y \in K(x)\ \exists x \in A\}$ is the image of the set A under K.
$K^{-1}(y) = \{x \in X;\ y \in K(x)\}$ is the inverse of K.
$\operatorname{Ker} K = \{x \in X;\ 0 \in K(x)\} = K^{-1}(0)$ is the kernel of K.
$\Gamma_K = \{(x, y) \in X \times Y;\ x \in K(y)\}$ is the graph of K.
The following formula is immediate:

$$K^{-1}(Y \setminus B) = X \setminus K^*(B) = \{x \in X;\ K(x) \not\subset B\} \tag{2.1}$$

Let X and Y be topological spaces.

DEFINITION 2.1. $K: X \to 2^Y$ is
(i) upper semicontinuous if $K^*(B)$ is open (in X) for all open sets B (in Y).
(ii) locally bounded if for all $x \in X$ there exists a neighbourhood $N \ni x$ such that $K(N) \subset\subset Y$ (i.e., $K(N)$ is compactly enclosed in Y)

The formula (2.1) then yields

PROPOSITION 2.2. *K is upper semicontinuous if and only if $K^{-1}(F)$ is closed in R_K for all closed sets F in Y.*

For metric spaces X and Y we have the following characterization of upper semicontinuous compact-valued multifunctions:

PROPOSITION 2.3. *Let $K: X \to 2^Y$ be a multifunction between the metric spaces X and Y. Then K is upper semicontinuous and compact valued if and only if K is locally bounded and Γ_K is closed.*

The proof is easy.

3. Some facts about analytic multifunctions.

Following [8] we define an analytic multifunction.

DEFINITION 3.1. An upper semicontinuous, compact-valued map

$$K: D \to 2^{\mathbf{C}^n},$$

where D is an open subset of $\mathbf{C}^k$, is said to be analytic if for every open $D' \subset D$ and for every $\psi(\lambda, z)$, plurisubharmonic in a neighbourhood of $\{(\lambda, z) \in D' \times \mathbf{C};\ z \in K(\lambda)\}$, the function

$$\lambda \mapsto \sup\big(\psi(\lambda, z);\ z \in K(\lambda)\big)$$

is plurisubharmonic in D'.

NOTE. For the purpose of this paper we can use the above definition of analytic multifunction. The definition of analyticity for multifuctions taking values in $\mathbf{C}^n$, $n \geq 2$ has however changed to a slightly smaller class. See for instance [9].

If K is a function, then K is analytic in the ordinary sense if and only if it is analytic in the above sense.

From [8, Proposition 5.1] we shall need

PROPOSITION 3.2.

(i) *If K is analytic on D, then the following are also analytic:*

$$\lambda \mapsto K(\lambda) \times \{\lambda\}$$

$$(z, \lambda) \mapsto \{z\} \times K(\lambda)$$

(ii) *If K is analytic on D and L is analytic on a neighbourhood of R_K then their composition*

$$L \circ K(\lambda) = \bigcup \{L(z);\ z \in K(\lambda)\}$$

is analytic on D.

With this we can build new analytic multifunctions out of old ones.

PROPOSITION 3.3. *Let K and L be analytic on D. Then the following are analytic on D:*

$$(K \times L)(\lambda) = \{(z, w);\ z \in K(\lambda),\ w \in L(\lambda)\};$$

$$(K + L)(\lambda) = \{z + w;\ z \in K(\lambda),\ w \in L(\lambda)\};$$

$$(K\,L)(\lambda) = \{zw;\ z \in K(\lambda),\ w \in L(\lambda)\}.$$

If $\operatorname{Ker} K = \emptyset$ *then* $1/K$ *is analytic, where*

$$(1/K)(\lambda) = \{1/z;\ z \in K(\lambda)\}.$$

PROOF. $\lambda \mapsto K(\lambda) \times L(\lambda)$ is the composition of $\lambda \mapsto K(\lambda) \times \{\lambda\}$ and $(z, \lambda) \mapsto \{z\} \times L(\lambda)$. Thus it is analytic. $\lambda \mapsto K(\lambda) + L(\lambda)$ is the composition of $\lambda \mapsto (K \times L)(\lambda)$ and $(z, \lambda) \mapsto z + \lambda$. Hence it is analytic. $K\,L$ works in the same way. For the last statement observe that $\operatorname{Ker} K = \emptyset$ says that $0 \notin R_K$ so we can find an open neighbourhood of R_K where $z \mapsto 1/z$ is analytic. Then we compose K with $z \mapsto 1/z$ to get the result. This ends the proof.

NOTE. If $1/K$ is analytic, it follows of course that $\operatorname{Ker} K = \emptyset$, since otherwise, $1/K$ would not even be bounded.

Just like for analytic functions, one can sometimes remove singularities. We get this from [5, Proposition 6.1], but first we introduce some notation concerning limits of multifunctions.

DEFINITION 3.4. Let K be a multifunction between two metric spaces. Then

$$\lim_{\mu\to\lambda}\operatorname{den} K(\mu) = \{z;\ \exists\, \lambda_n \to \lambda,\ z_n \to z,\ \text{with } z_n \in K(\lambda_n)\}$$

is the *thick limit (limes densus)* of the family $\{K(\mu)\}$ as $\mu \to \lambda$. We also introduce the *thin limit (limes rarus)* of K.

$$\lim_{\mu\to\lambda}\operatorname{rar} K(\mu) = \{z;\ \forall\lambda_n \to \lambda,\ \exists z_n \to z,\ \text{with } z_n \in K(\lambda_n)\}.$$

REMARK. lim den and lim inf are usually called lim sup and lim inf respectively, but this is, although more or less standard, not so good terminology: If one wishes to refer to the usual lim sup or lim inf for functions, then one has to call them something else.

PROPOSITION 3.5. *Let D' be open in $\mathbf{C}$, let P be a closed polar subset of D', and set $D = D' \setminus P$. Let $K{:}\, D \to 2^{\mathbf{C}}$ be an analytic multifunction. For $\lambda \in D'$, define*

$$K'(\lambda) = \begin{cases} K(\lambda), & \lambda \in D; \\ \lim\operatorname{den}_{D\ni\mu\to\lambda} K(\mu), & \lambda \in P. \end{cases}$$

Then if $\infty \notin K'(D')$, K' is analytic on D'.

We list a few properties of the thick and thin limits.

PROPOSITION 3.6. *Let K be a multifunction between two normed spaces and let $|K(\lambda)|$ denote* $\sup\big(|z|;\ z \in K(\lambda)\big)$. *Then*

(i)
$$\lim_{D\ni\mu\to\lambda}\operatorname{den} K(\mu) = \bigcap \overline{K(N \cap D)},$$

where the intersection is taken over all neighbourhoods N of λ;

(ii)
$$\Big|\lim_{\mu\to\lambda}\operatorname{den} K(\mu)\Big| \le \limsup_{\mu\to\lambda} |K(\mu)|,$$

with equality if the right-hand side is finite;

(iii)
$$\Big|\lim_{\mu\to\lambda}\operatorname{rar} K(\mu)\Big| \le \liminf_{\mu\to\lambda} |K(\mu)|,$$

if $\lim\operatorname{rar}_{\mu\to\lambda} K(\mu)$ *is non-empty.*

PROOF. (i) Let $z \in \lim\operatorname{den}_{D\ni\mu\to\lambda} K(\mu)$. Then there exist $D \ni \lambda_n \to \lambda$ and $z_n \to z$ with $z_n \in K(\lambda_n)$. For all neighbourhoods $N \ni \lambda$, $\lambda_n \in D \cap N$ if n is large enough. Thus $z_n \in K(N \cap D) \subset \overline{K(N \cap D)}$, which is closed, so $z \in \overline{K(N \cap D)}$ for all neighbourhoods $N \ni \lambda$. Conversely let $z \in \bigcap \overline{K(N \cap D)}$. Then $z \in \overline{K(B_{1/n} \cap D)}$ for all balls $B_{1/n}$ with radius $1/n$ centered at λ. Hence for any n we can find $z_n \in K(B_{1/n} \cap D)$ with $|z - z_n| < 1/n$, say. But $z_n \in K(B_{1/n} \cap D)$ means that $z_n \in K(\lambda_n)$ for some $\lambda_n \in B_{1/n} \cap D$, so we have found $\lambda_n \to \lambda$, $z_n \to z$ such that $z_n \in K(\lambda_n)$.

(ii) For $z \in \operatorname{lim\,den}_{\mu\to\lambda} K(\mu)$ it is clear that $|z| \leq \limsup_{\mu\to\lambda} |K(\mu)|$. Suppose then that the right hand-side is finite. Then we can find $\lambda_n \to \lambda$ such that $|K(\lambda_n)| \to \limsup_{\mu\to\lambda} |K(\mu)|$. Since $K(\lambda_n)$ is compact we can find $z_n \in K(\lambda_n)$ such that $|z| = |K(\lambda_n)|$. Thus the modulus of z_n converges so we can find a convergent subsequence $z'_n \to z$, for some z. This z is in ${}_{\mu\to\lambda}K(\mu)$ and $|z| = \lim_n |z'_n| = \limsup_{\mu\to\lambda} |K(\mu)|$.

(iii) This is clear, so the proof is complete.

REMARK. 1. If K is a function and lim rar K $\neq \emptyset$ then the ordinary limit of K exists and equals lim rar K, which is thus a singleton.

2. We see from (i) that lim den K is closed, and if $\limsup |K| < \infty$, (ii) implies, via the Cantor intersection theorem, that lim den K is nonempty and compact.

3. In [5], lim den was defined by (i), but we prefer the definition above since it runs parallel with the definition of lim rar.

The thick and thin limits behave well when multiplying with a function having non-zero limit.

PROPOSITION 3.7. *Let K be a multifunction and f a function, both defined on a set D. If*

$$f(\lambda^0) =_{df} \lim_{D\ni\lambda\to\lambda^0} f(\lambda)$$

exists and is non-zero, then

$$\operatorname*{lim\,rar}_{D\ni\lambda\to\lambda^0} Kf(\lambda) = f(\lambda^0) \operatorname*{lim\,rar}_{D\ni\lambda\to\lambda^0} K(\lambda)$$

and

$$\operatorname*{lim\,den}_{D\ni\lambda\to\lambda^0} Kf(\lambda) = f(\lambda^0) \operatorname*{lim\,den}_{D\ni\lambda\to\lambda^0} K(\lambda).$$

PROOF. We prove the last statement. Let $z \in \operatorname{lim\,den}_{D\ni\lambda\to\lambda^0} Kf(\lambda)$. Then there exist $\lambda_n \in D$ so that $\lambda_n \to \lambda^0$, and $z_n \in Kf(\lambda_n)$ so that $z_n \to z$. But $z_n = w_n f(\lambda_n)$ for some $w \in K(\lambda_n)$, and since $f(\lambda^0) \neq 0$ it follows that $w_n \to z/f(\lambda^0)$. Therefore $z/f(\lambda^0) \in \operatorname{lim\,den}_{D\ni\lambda\to\lambda^0} K(\lambda)$ so

$$z = f(\lambda^0)\frac{z}{f(\lambda^0)} \in f(\lambda^0) \operatorname*{lim\,den}_{D\ni\lambda\to\lambda^0} K(\lambda).$$

This proves the inclusion "$\subset$". The other inclusion now follows since

$$\lim \operatorname{den} K(\lambda) = \lim \operatorname{den} Kf\frac{1}{f}(\lambda) \subset \frac{1}{f(\lambda^0)} \lim \operatorname{den} Kf(\lambda).$$

Finite analytic multifunctions have a nice structure. We shall need the following theorem. For the proof we refer to [1].

THEOREM 3.8 (Scarcity Theorem). *Let $K: D \to 2^{\mathbf{C}^n}$, $D \subset \mathbf{C}^k$, be analytic. Then either $\{\lambda \in D;\ \#K(\lambda) < \infty\}$ is pluripolar in D or there exists an integer m and a closed analytic subvariety F of D such that $\#K(\lambda) = m$ on $D \setminus F$ and $\#K(\lambda) < m$ on F. Moreover, in the last situation, for each $\lambda^0 \in D \setminus F$ there exist $h_1, , h_m$ holomorphic in a neighbourhood $U \ni \lambda^0$ so that $K(\lambda) = \{h_1(\lambda), , h_n(\lambda)\}$ in U.*

4. Hardy Spaces of analytic multifunctions.

Let $K: \mathbf{D} \to 2^{\mathbf{C}}$ be an analytic multifunction. Then, with

$$|K(\lambda)| = \sup\big(|z|;\ z \in K(\lambda)\big),$$

the function $\phi(\lambda) = |K(\lambda)|$ will become subharmonic, as will the functions ϕ^p and $\log^+ \phi$. This is clear from the definition of analytic multifunctions, since the functions $(z, \lambda) \mapsto |z|^p$ and $(z, \lambda) \mapsto \log|z|$ are plurisubharmonic.

Let us define an analogue of the H^p spaces but now for multifunctions. As a convention, capital letters will denote multifunctions and small letters ordinary functions. Thus if we say $K \in H^p$ we are talking about the multi-value analogue to H^p, but statements like $q \in H^p$ refers to the usual H^p space.

Let K be analytic on $\mathbf{D}$. For $0 \le r < 1$ and $t \in \mathbf{T}$ let $K_r(t) = K(re^{it})$. For $0 < p < \infty$ we define

$$\|K_r\|_p = \left(\int_{\mathbf{T}} |K_r(t)|^p \frac{dt}{2\pi}\right)^{1/p}.$$

For $p = \infty$ we let

$$\|K_r\|_\infty = \sup_t |K_r|.$$

We also introduce

$$\|K_r\|_0 = \exp \int_{\mathbf{T}} \log^+ |K_r| \frac{dt}{2\pi}.$$

DEFINITION 4.1. For $0 < p \le \infty$ we say that $K \in H^p$ if K is analytic on $\mathbf{D}$ and

$$\|K\|_p =_{\mathrm{df}} \sup\big(\|K_r\|_p;\ 0 \le r < 1\big) < \infty.$$

We say that $K \in N$ if K is analytic on D and

$$\|K\|_0 =_{\text{df}} \sup\big(\|K_r\|_0;\ 0 \le r < 1\big) < \infty.$$

THEOREM 4.2.

(i) $\|K_r\|$ *is increasing in* r *so*

$$\|K\|_p = \lim_{r\to 1} \|K_r\|_p.$$

(ii) *For all* $c \in \mathsf{C}$, $\|cK\|_p = |c|\,\|K\|_p$.

(iii) *For* $1 \le p \le \infty$, $\|\cdot\|_p$ *satisfies the triangle inequality,* i.e.,

$$\|K + L\|_p \le \|K\|_p + \|L\|_p.$$

PROOF. (i) For $p < \infty$, the subharmonicity of $|K(\lambda)|^p$ and $\log^+|K(\lambda)|$ gives the result. For $p = \infty$ the result follows from the subharmonicity of $|K(\lambda)|$ and the maximum principle for subharmonic functions.

(ii) Immediate.

(iii) We have

$$|(K+L)(\lambda)| = \sup_{z\in K(\lambda), w\in L(\lambda)} |z+w| \le \sup_{z\in K(\lambda)} |z| + \sup_{w\in L(\lambda)} |w| = |K(\lambda)| + |L(\lambda)|,$$

so for $1 \le p \le \infty$ Minkowski's inequality gives us that

$$\|(K+L)_r\|_p \le \|K_r\|_p + \|K_r\|_p$$

and the result follows. This finishes the proof.

The usual theory of H^p-spaces has three cornerstones:

(i) $|f|^p$ and $\log|f|$ are subharmonic.

(ii) radial maximal functions behave nicely, i.e., they belong to L^p if $f \in H^p$.

(iii) One can use Blaschke products to reduce problems to the case of a non-vanishing function.

As we saw above, we do have (i) for multifunctions so by well established potential theory, (ii) follows for $p > 1$. As we shall see below, (ii) follows in the case of analytic multifunctions even for $p \le 1$. Alas, there are problems with (iii). Let D be a bounded open subset of C and K an analytic multifunction on D. First note that there are two different kinds of zeroes of K. We have the kernel $\operatorname{Ker} K = \{\lambda;\ 0 \in K(\lambda)\}$ and also the zero-set of K, $Z(K) = \{\lambda;\ K(\lambda) = 0\}$. By Proposition 2.1, the kernel is always closed in D, but this is all we can say about it. In fact, given any closed subset F of D we can find an analytic multifunction on D with kernel equal to F:

EXAMPLE 4.3. Let F be a closed subset of D. Let

$$L(\lambda) = \overline{F} - \lambda.$$

Since D is bounded, $\overline{F}$ is compact, so L is analytic. Now $\operatorname{Ker} L = L^{-1}(0) = \{\lambda \in D; 0 \in \overline{F} - \lambda\}$ so $\lambda \in \operatorname{Ker} L$ if and only if $\lambda \in D$ and $\lambda \in \overline{F}$. But F is closed in D so $\lambda \in \operatorname{Ker} L$ if and only if $\lambda \in F$.

To study the zero-set of $K \not\equiv 0$, let $\phi(\lambda) = \log|K(\lambda)|$. Then ϕ is subharmonic and $K(\lambda) = 0$ if and only if $\phi(\lambda) = -\infty$. Conversely let ϕ be subharmonic and define $L(\lambda) = \{z; |z| \leq e^{\phi(\lambda)}\}$. This is analytic (see for instance [1]) and the zero-set of L equals the set where ϕ is $-\infty$. Thus zero-sets of analytic multifunctions are exactly sets where subharmonic functions are $-\infty$, so they are G_δ-sets of logaritmic capacity zero. Therefore we see that no "Blaschke-condition" can be fulfilled for the zeroes of an analytic multifunction. Moreover, the order of a zero need not be a natural number, like for analytic functions which have zeroes of order 1, 2 et cetera. Thus it is not trivial what should be meant by "dividing away a zero".

EXAMPLE 4.4. Let n be a natural number and let $K(\lambda) = \{z; z^n = \lambda\}$. Then K is the multifunction inverse of $P(z) = \{z^n\}$. P is a polynomial, hence proper. By [6, Theorem 3.2] the inverse of any proper analytic function is an analytic multifunction, so K is analytic. But what is the order of the zero at the origin? If anything, then it should be $1/n$.

The above example, although negative in character, is in fact what we need to get good behaviour on the radial maximal function even for $p \leq 1$ since it allows us to get from H^p to H^{np} just by taking the nth root, an operation, by the example and using composition, allowed even if K has zeroes. So for multifunctions there is much less need to consider zero-free functions, and this gives us hope to do some H^p-theory using just cornerstones (i) and (ii) above.

We first recall some facts from potential theory. For details, see for instance [2].

Given $K\colon \mathsf{D} \to 2^{\mathsf{C}}$ we define the radial maximal function $K^\star$ on T by

$$K^\star(e^{it}) = \sup_{r \to 1} |K(re^{it})|.$$

If μ is a complex measure on T we denote by $P[d\mu]$ the Poisson integral of μ, i.e.,

$$P[d\mu](\lambda) = \int_{\mathsf{T}} \frac{1 - |\lambda|^2}{|e^{it} - \lambda|^2}\, d\mu(t),$$

and if $f \in L^1(\mathsf{T})$, we set $P[f] = P\left[f\frac{dt}{2\pi}\right]$. Thus if f is real,

$$P[f](\lambda) = \Re e \int_{\mathsf{T}} \frac{e^{it} + \lambda}{e^{it} - \lambda} f(e^{it}) \, \frac{dt}{2\pi}.$$

Let μ be a positive measure on D such that the Green's potential,

$$U_\mu(z) = \int \log\left|\frac{1 - \bar{\zeta} z}{\zeta - z}\right| d\mu(\zeta)$$

exists at some point $z \in \mathsf{D}$. Then Littlewood [4] proved that

$$\lim_{r \to 1} U_\mu(re^{it}) = 0 \text{ almost everywhere.} \tag{4.2}$$

Let now $\phi \not\equiv -\infty$ be a subharmonic function on D having a harmonic majorant, and let $\mu = \Delta\phi$. Then for $|z| < r < 1$ we get from the Riesz decomposition theorem that

$$\phi(z) = -\int_{r\mathsf{D}} \log\left|\frac{1 - \bar{\zeta} z}{\zeta - z}\right| \frac{d\mu(\zeta)}{2\pi} + h_r(z)$$

where h_r is harmonic in $r\mathsf{D}$. We also know that the least harmonic majorant of ϕ is

$$h(z) = \lim_{r \to 1} h_r(z) = \lim_{r \to 1} P[\phi(re^{it})](z)$$

and

$$\phi(z) = h(z) - \frac{1}{2\pi} U_\mu(z). \tag{4.3}$$

If $\phi(z) > -\infty$, the potential therefore converges, so in particular if $\phi \not\equiv -\infty$, (4.2) holds.

If

$$\sup_r \int \phi^+(re^{it}) \, dt < \infty, \tag{4.4}$$

then ϕ has a harmonic majorant

$$h = P[f \, dt/2\pi + d\sigma], \tag{4.5}$$

where $f \in L^1(\mathsf{T})$ and $d\sigma$ is singular, so if $\phi(z) > -\infty$ for some $z \in \mathsf{D}$ we get from (4.3) and (4.2) that ϕ has radial limit $f(e^{it})$ almost everywhere on T.

We will apply this to multifunctions in N and H^p, but first we introduce some notation.

DEFINITION 4.5. Given any multifunction K we let the peripheral part of K equal

$$PK(\lambda) = \{z \in K(\lambda);\ |z| = |K(\lambda)|\}.$$

In general for a multifunction K, PK inherits very little structure from K. It is not upper semicontinuous in general if K is, not even when K is analytic. It is certainly not analytic if K is analytic (although its convex circled hull is, but this is just saying that $\log|PK(\lambda)|$ is subharmonic which is no news, since $|PK(\lambda)| = |K(\lambda)|$), and the graph is not closed in general. We can see this in an example.

EXAMPLE 4.6. Let ϕ be subharmonic in D and such that $\phi(\lambda_n) = -\infty$ for some sequence $\lambda_n \to 0$ whilst $\phi(0) = 0$, and let $K(\lambda) = \{z;\ |z| \leq e^{\phi(\lambda)}\}$. Then K is analytic, $PK(\lambda_n) = 0$ and $PK(0) = \mathsf{T}$. This implies that PK is not upper semicontinuous: Let $U = \{z;\ 1/2 < |z| < 3/2\}$. Then $\lambda_n \notin (PK)^*(U)$ and $0 \in (PK)^*(U)$ so $(PK)^*(U)$ is not open. Since PK is locally bounded it follows from Proposition 2.3 that the graph cannot be closed.

If $PK(\lambda)$ is constant, and of modulus c, say, then certainly $|K(\lambda)| \equiv c$. The converse is of course not true in general, but for analytic multifunctions it is true.

PROPOSITION 4.7. *Let K be an analytic multifunction on a domain D. Then $|K|$ is constant if and only if PK is constant.*

PROOF. The "if"-part is trivial. Conversely suppose that $|K|$ is constant, say equal to c. If PK is nonconstant, then we can find z with $|z| = c$ and λ_1, $\lambda_2 \in \mathsf{D}$ so that $z \in K(\lambda_1)$ but $z \notin K(\lambda_2)$. Let $L(\lambda) = K(\lambda) + az$, for some $a > 0$. Then L is analytic and $|L(\lambda)| \leq (1+a)|K|$. But the point $(1+a)z \in L(\lambda_1)$ so $|L(\lambda_1)| \geq (1+a)|K|$ and it follows from the maximum principle for subharmonic functions that $|L|$ is constant. However for $w + az \in L(\lambda_2)$ we have, since $|w| \leq |z|$ and $w \neq z$, $|w+az|^2 = |w|^2 + a^2|z|^2 + 2a\Re e\langle w, z\rangle < (1+a)^2|K|^2 \leq |L(\lambda_1)|^2$. This contradiction shows us that PK is constant, and the theorem is proved.

DEFINITION 4.8. Given $K\colon \mathsf{D} \to 2^{\mathsf{C}}$, we define

$$K^{\mathrm{rad}}(e^{it}) = \lim_{r\to 1} \operatorname{den} K(re^{it})$$

and

$$K_{\mathrm{rad}}(e^{it}) = \lim_{r\to 1} \operatorname{rar} K(re^{it}).$$

THEOREM 4.9. *If $K \in N$ then $|K|$ has radial limit almost everywhere, and this limit equals $|K^{\mathrm{rad}}|$. Moreover if $K \not\equiv 0$ then*

$$\log |K^{\mathrm{rad}}| \in L^1(\mathsf{T}).$$

PROOF. Assume $K \not\equiv 0$. Then putting $\phi(\lambda) = \log |K(\lambda)|$, $\phi \not\equiv -\infty$. Moreover since $K \in N$, (4.4) holds for ϕ so we get from (4.5) that ϕ has radial limit $f(e^{it})$ almost everywhere. Hence $e^{\phi} = |K|$ has radial limit almost everywhere. Thus

$$\lim_{r \to 1} |K(re^{it})|$$

is finite, so

$$|\lim_{r \to 1} \operatorname{den} K(re^{it})| = \limsup_{r \to 1} |K(re^{it})| = \lim_{r \to 1} |K(re^{it})|.$$

This shows the first statement and also that $\log |K^{\mathrm{rad}}| = f$ almost everywhere. This gives the second statement since $f \in L^1(\mathsf{T})$. Thus the theorem is proved.

The fact that $\log |K^{\mathrm{rad}}| \in L^1(\mathsf{T})$ has an immediate consequence which is worth stating separately:

THEOREM 4.10. *If $K \in N$ and $K \not\equiv 0$, then $|K^{\mathrm{rad}}(e^{it})| \neq 0$ almost everywhere.*

An interesting question in this context is in what sense K itself has radial limits. This question was studied by S. R. Harbottle in [3] where the following nice result was proved.

THEOREM 4.11. *Let $K \colon \mathsf{D} \to 2^{\mathsf{C}}$ be an analytic multifunction such that*

$$\lim_{r \to 1} \int_{\mathsf{T}} \log^{+} |K(re^{it})| \, dt < \infty.$$

Then there is a set $A \subset \mathsf{T}$ of measure 2π such that, for all $e^{it} \in A$,

$$[K^{\mathrm{rad}}(e^{it})]^{\hat{}} = [K_{\mathrm{rad}}(e^{it})]^{\hat{}},$$

where the hats denote polynomially convex hulls.

Actually the author, unaware of Harbottle's result, has proved something along these lines, although weaker, namely that the peripheral parts of K^{rad} and K_{rad} agree almost everywhere on T.

By Harbottle's result it is clear that $|K^{\mathrm{rad}}| = |[K^{\mathrm{rad}}]^{\hat{}}| = |[K_{\mathrm{rad}}]^{\hat{}}| = |K_{\mathrm{rad}}| =$ the radial limit of $|K|$, so in many theorems that follow, one can use these interchangeably.

We now turn our attention to H^p-multifunctions and shall see that their radial maximal functions are in $L^p(\mathsf{T})$. Again, this follows from the theory of subharmonic functions if $p > 1$ but to prove it for $p \leq 1$ we use multi-functional analyticity in an essential way. The integrability of the maximal functions are also the key step in the inner-outer factorization of ordinary H^p-functions, and in fact we shall prove something very much like this for H^p-multifunctions.

THEOREM 4.12. *If $0 < p \leq \infty$ and $K \in H^p$ then*
(i) *The radial maximal functions $K^\star$ are in $L^p(\mathsf{T})$;*
(ii) *$|K|$ has radial limit $|K^{\mathrm{rad}}|$ almost everywhere, and $K^{\mathrm{rad}} \in L^p(\mathsf{T})$.*

LEMMA 4.13. *If ϕ is a positive subharmonic function on D such that*

$$\sup_r \int_{\mathsf{T}} \phi^p(re^{it}) \frac{dt}{2\pi} < \infty \tag{4.6}$$

and $1 < p < \infty$, then the least harmonic majorant h of ϕ enjoys the same property.

PROOF. For $R < 1$ let

$$h_R = P[\phi(Re^{it})].$$

Since $p > 1$ it follows that

$$\sup_r \int_{\mathsf{T}} \phi(re^{it}) \frac{dt}{2\pi} < \infty.$$

Hence if $r < R$,

$$h_R(0) = \int_{\mathsf{T}} h_R(re^{it}) \frac{dt}{2\pi} \leq \int_{\mathsf{T}} \phi(Re^{it}) \frac{dt}{2\pi} \leq \sup_r \int_{\mathsf{T}} \phi(re^{it}) \frac{dt}{2\pi} < \infty,$$

so $h_R(0)$ stays bounded. By subharmonicity, h_R is increasing in R so by Harnack's theorem $h_R \to h$ for some harmonic function h, uniformly on compact subsets. Clearly, h is a harmonic majorant of ϕ so it suffices to prove the estimate (4.6) for h. Let therefore $r < 1$ be given. Since $h_R \to h$ uniformly on compact subsets, we can find an $R > r$ so that $|h_R(re^{it}) - h(re^{it})|^p \leq 1$. Then

$$\int_{\mathsf{T}} h^p(re^{it}) \frac{dt}{2\pi} \leq \int_{\mathsf{T}} (h - h_R)^p(re^{it}) \frac{dt}{2\pi} + \int_{\mathsf{T}} h_R^p(re^{it}) \frac{dt}{2\pi} \leq$$

$$\leq 1 + \sup_R \int_{\mathsf{T}} \phi^p(Re^{it}) \frac{dt}{2\pi} < \infty.$$

DEFINITION 4.14. Let $n \geq 1$, $n \in \mathsf{N}$ and define $R_n(\lambda) = \{z;\, z^n = \lambda\}$. As we saw in Example 4.4 this is analytic on C. Given an analytic multifunction K we understand by $K^{1/n}$ the composition $R_n \circ K$.

It is clear that $|K^{1/n}| = |K|^{1/n}$, so that if $K \in H^p$, then $K^{1/n} \in H^{np}$, and also $|(K^{1/n})^{\text{rad}}| = |K^{\text{rad}}|^{1/n}$, if the limit exists.

Proof of Theorem 4.12. Let h be the least harmonic majorant of $|K|$. First assume $1 < p < \infty$. Then by Lemma 4.13

$$\sup_{0<r<1} \|h(re^{it})\|_{L^p(\mathsf{T})} < \infty$$

so by standard theorems on harmonic functions, $h^\star \in L^p(\mathsf{T})$. Hence the same holds for $K^\star$. If $0 < p \leq 1$ we choose an n so that $np > 1$ and let $L = K^{1/n}$. Then $L \in H^{np}$, so $L^\star \in L^{np}$, i.e., $(L^\star)^{np} \in L^1$. But $|L|^{np} = |K|^p$, so $(K^\star)^p = (L^\star)^{np} \in L^1$. This proves (i), since the case $p = \infty$ is trivial. The first part of (ii) has already been proved since we have seen in Theorem 4.9 that radial limits of $|K|$ exist and equal $|K^{\text{rad}}|$ almost everywhere. Obviously, $|K^{\text{rad}}| \leq K^\star$, so by (i), $K^{\text{rad}} \in L^p$. The theorem is proved.

We shall now see that there is an inner-outer factorization for H^p-multifunctions, and it turns out that the outer factor is a function. First we recall the definition of an outer function, together with a few properties. For details see [7].

If ϕ is a positive measurable function on T such that $\log \phi \in L^1(\mathsf{T})$, and if

$$q(\lambda) = c \exp\left\{ \int_{\mathsf{T}} \frac{e^{it} + \lambda}{e^{it} - \lambda} \log \phi(e^{it}) \frac{dt}{2\pi} \right\}$$

for $\lambda \in \mathsf{D}$, then q is called an *outer function*. Here c is a constant of modulus 1.

Theorem 4.15. *Suppose q is the outer function related to ϕ as above. Then*
(i) $\lim_{r \to 1} |q(re^{it})| = \phi(e^{it})$ *almost everywhere on* T;
(ii) $q \in H^p$ *if and only if* $\phi \in L^p(\mathsf{T})$.

In analogy with the definition of an inner function we define an inner multifunction.

Definition 4.16. K analytic on D is said to be inner if $K \in H^\infty$ and $|K^{\text{rad}}| = 1$ almost everywhere on T.

Theorem 4.17. *If $0 < p \leq \infty$, $K \in H^p$ and K is not identically zero, then* $\log |K^{\text{rad}}| \in L^1(\mathsf{T})$, *the outer function*

$$q_K(\lambda) = \exp \int_{\mathsf{T}} \frac{e^{it} + \lambda}{e^{it} - \lambda} \log |K^{\text{rad}}(e^{it})| \frac{dt}{2\pi}$$

is in H^p, and there is an inner multifunction M_K such that

$$K = q_K M_K.$$

Furthermore,

$$\log|K(0)| \leq \int_{\mathsf{T}} \log|K^{\mathrm{rad}}(e^{it})| \frac{dt}{2\pi}. \tag{4.7}$$

Equality holds in (4.7) *if and only if PM_K is constant.*

PROOF. From Theorem 4.9 we get $\log|K^{\mathrm{rad}}| \in L^1$ so that q_K is a well defined outer function. Since by Theorem 4.12, $|K^{\mathrm{rad}}| \in L^p$, it follows that $q_K \in H^p$.

Next, since $K \in H^p$, $\log|K(\lambda)|$ has a harmonic majorant, and the least harmonic majorant is $\lim_{r\to 1} P[\log|K(re^{it})|]$. Thus

$$\log|K(\lambda)| \leq \lim_{r\to 1} P[\log|K(re^{it})|](\lambda).$$

Since $\log|K(re^{it})| \to \log|K^{\mathrm{rad}}(e^{it})|$ almost everywhere as $r \to 1$ and these functions are bounded above by the function $(1/p)(K^\star(e^{it}))^p$, which is integrable by Theorem 4.12, we can use dominated convergence to conclude

$$P[\log^+|K(re^{it})|](\lambda) \to P[\log^+|K^{\mathrm{rad}}(e^{it})|](\lambda).$$

Thus using Fatou's lemma we get

$$\lim_{r\to 1} P[\log|K(re^{it})|] = \lim \left(P[\log^+|K(re^{it})|] - P[\log^-|K(re^{it})|]\right)$$

$$\leq P[\log^+|K^{\mathrm{rad}}(e^{it})|] - \liminf P[\log^-|K(re^{it})|] \leq P[\log^+|K^{\mathrm{rad}}(e^{it})|]$$

$$-P[\log^-|K^{\mathrm{rad}}(e^{it})|] = P[\log|K^{\mathrm{rad}}(e^{it})|].$$

But $P[\log|K^{\mathrm{rad}}(e^{it})|](\lambda) = \log|q_K(\lambda)|$, so $\log|K(\lambda)| \leq \log|q_K(\lambda)|$ i.e.,

$$|K(\lambda)| \leq |q_K(\lambda)|. \tag{4.8}$$

Therefore, if we define

$$M_K(\lambda) = \frac{K(\lambda)}{q_K(\lambda)},$$

M_K will be an analytic multifunction of modulus at most 1, hence bounded. Since $\lim_{r\to 1} |q_K(re^{it})| = |K^{\mathrm{rad}}(e^{it})|$ almost everywhere and $\log|K^{\mathrm{rad}}(e^{it})| \in L^1(\mathsf{T})$ implies that $|K^{\mathrm{rad}}(e^{it})| > 0$ almost everywhere, we get

$$\lim_{r\to 1}|M_K(re^{it})| = \lim_{r\to 1}\left|\frac{K(re^{it})}{q_K(re^{it})}\right| = \lim_{r\to 1}\frac{|K(re^{it})|}{|q_K(re^{it})|} = 1$$

almost everywhere. Hence M_K is inner.

If we put $\lambda = 0$ in (4.8) we obtain (4.7) and equality holds if and only if $|K(0)| = |q_K(0)|$, i.e., if and only if $|M_K(0)| = 1$. This can only happen when $|M_K|$ is constant. Proposition 4.7 now gives the result. This finishes the proof.

For a function $f \in N$ it is well known that

$$f = \frac{b_1}{b_2},$$

where $b_1, b_2 \in H^\infty$ and b_2 is non-zero. For a multifunction in the Nevanlinna class we have an analogous statement.

THEOREM 4.18. *For every $K \in N$ there corresponds two functions b_1 and b_2 in H^∞, both zero-free, and an inner multifunction M such that*

$$K = \frac{b_1}{b_2}M.$$

PROOF. Let h be the least harmonic majorant of $\log|K|$. Then according to (4.5),

$$h(\lambda) = P[d\mu](\lambda)$$

for some real finite measure μ. If $\mu^+ - \mu^-$ is the Jordan decomposition of μ, then

$$|K(\lambda)| \le \exp h(\lambda) = \exp\left\{\Re e\int_{\mathsf{T}}\frac{e^{it}+\lambda}{e^{it}-\lambda}\,d\mu\right\} \tag{4.9}$$

$$= \exp\left\{-\Re e\int_{\mathsf{T}}\frac{e^{it}+\lambda}{e^{it}-\lambda}\,d\mu^-\right\}\bigg/\exp\left\{-\Re e\int_{\mathsf{T}}\frac{e^{it}+\lambda}{e^{it}-\lambda}\,d\mu^+\right\}.$$

Put

$$b_2(\lambda) = \exp\left\{-\int_{\mathsf{T}}\frac{e^{it}+\lambda}{e^{it}-\lambda}\,d\mu^+\right\}.$$

Then b_2 is non-zero and $b_2 \in H^\infty$. Moreover $|Kb_2|$ will be bounded by the numerator in (4.9) so $Kb_2 \in H^\infty$. Let b_1M be the inner-outer factorization of Kb_2. Then b_1 is zero-free and since b_2 also is non-zero we get

$$K = \frac{b_1}{b_2}M.$$

5. Blaschke products.

It is well known that for functions analytic on the disc, the property

$$\lim_{r\to 1}\int_{\mathsf{T}} \left|\log|f(re^{it})|\right| = 0 \tag{5.1}$$

is equivalent to f being a Blaschke product. This is easy to see: That every Blaschke product satisfies (5.1) is just Theorem 15.24 in [7]. Conversely, if f satisfies (5.1) then

$$\lim_{r\to 1}\int_{\mathsf{T}} \log^+ |f(re^{it})| = 0,$$

so by subharmonicity of $|f|$, $\log^+ |f| = 0$. Thus $|f| \le 1$ and we can factor f as $f = bg$, with g zero-free, $|g| \le 1$ and b a Blaschke product. Then $\left|\log|1/g|\right| = \left|\log|b/f|\right| = \left|\log|b| - \log|f|\right| = \log|b| + \left|\log|f|\right|$ and (5.1) holds with $1/g$ instead of f. By the first argument, $|1/g| \le 1$, so g is just a rotation and f is indeed a Blaschke product.

With this in mind we are tempted to define the multi-valued analogue to a Blaschke product by the property (5.1).

DEFINITION 5.1. Let B be an analytic multifunction on D. We say that B is a Blaschke multifunction if

$$\lim_{r\to 1}\int_{\mathsf{T}} \left|\log|B(re^{it})|\right| = 0. \tag{5.2}$$

PROPOSITION 5.2. *Every Blaschke multifunction is inner.*

PROOF. Since $\log^+ |B|$ is subharmonic it follows from (5.2), that $\log^+ |B| = 0$. Thus $|B| \le 1$. To show that $|B^{\mathrm{rad}}| = 1$ almost everywhere, let $B = qM$ be the inner-outer factorization of B. Then $|q^{\mathrm{rad}}| = |B^{\mathrm{rad}}| \le 1$ almost everywhere so $|q(\lambda)| \le 1$. Moreover, by (4.8), $|B(\lambda)| \le |q(\lambda)|$ so $\left|\log|q|\right| = \log^- |q| \le \log^- |B| = \left|\log|B|\right|$. Thus q fulfills condition (5.1) and is therefore a Blaschke product. Hence $|q^{\mathrm{rad}}| = 1$ almost everywhere and the proof is complete.

THEOREM 5.3. *Suppose $K \in N$. Then K can be factored, in a unique way up to rotations, as*

$$K = fB,$$

where B is a Blaschke multifunction, $f \in N$ is zero-free, and $\|f\|_0 = \|K\|_0$. Moreover, if $K \in H^p$, then $f \in H^p$ and $\|f\|_p = \|K\|_p$.

PROOF. First we prove the existence. We know that $\log|K| = h - U_\mu$, where h is the least harmonic majorant of $\log|K|$ and $\mu = \Delta \log|K|$. According to (4.5), $h = P[d\nu]$ for some real finite measure ν on T. Put

$$f(\lambda) = \exp\left\{ \int_{\mathsf{T}} \frac{e^{it} + \lambda}{e^{it} - \lambda}\, d\nu \right\},$$

and define $B = K/f$. Then $\log|B| = \log|K/f| = h - U_\mu - h = -U_\mu$. Let for $\rho < 1$ μ_ρ be the restriction of μ to $\mathsf{D} \setminus (1-\rho)\overline{\mathsf{D}}$. The support of $\mu - \mu_\rho$ is contained in $(1-\rho)\overline{\mathsf{D}}$ so $U_{\mu-\mu_\rho}(re^{it}) = -U_{\mu_\rho}(re^{it}) + U_\mu(re^{it}) \to 0$ uniformly as $r \to 1$. Thus

$$\lim_{r\to 1} \int_{\mathsf{T}} -U_{\mu_\rho}(re^{it})\, \frac{dt}{2\pi} = \lim_{r\to 1} \int_{\mathsf{T}} -U_\mu(re^{it})\, \frac{dt}{2\pi}.$$

We get

$$-U_{\mu_\rho}(0) \le \lim_{r\to 1} \int_{\mathsf{T}} -U_{\mu_\rho}(re^{it})\, \frac{dt}{2\pi} = \lim_{r\to 1} \int_{\mathsf{T}} -U_\mu(re^{it})\, \frac{dt}{2\pi} \le 0.$$

But $\mu_\rho \to 0$ weakly as $\rho \to 1$ so the left hand side tends to zero. This proves that B is a Blaschke multifunction.

Next we prove the statements about the norms. First, since $|B| \le 1$ we get $|f| \ge |K|$, so we only have to prove $\|f\|_0 \le \|K\|_0$ and $\|f\|_p \le \|K\|_p$. We have $\log^+|f| \le \log^+|K| - \log|B|$, so by (5.2), $\|f\|_0 \le \|K\|_0$. This proves the first statement. For the second statement, let for $\rho < 1$, μ_ρ be the restriction of μ to $\rho\overline{\mathsf{D}}$, put $s_\rho = \log|K| + U_{\mu_\rho}$ and let $f_\rho = \exp(s_\rho)$. The support of μ_ρ is contained in $\rho\overline{\mathsf{D}}$, so $U_{\mu_\rho}(re^{it}) \to 0$ uniformly as $r \to 1$. Therefore, $f_\rho(re^{it}) \to |K(re^{it})|$ as $r \to 1$, so, writing

$$\|f_\rho\|_p = \left(\lim_{r\to 1} \int (f_\rho(re^{it}))^p \right)^{1/p},$$

we get $\|f_\rho\|_p = \|K\|_p$. Moreover, $s_\rho = h - U_\mu + U_{\mu_\rho} = \log|f| - U_{\mu-\mu_\rho}$, so f_ρ increases to $|f|$ and we get $\|f_r\|_p = \lim \|(f_\rho)_r\|_p \le \lim \|f_\rho\|_p = \|K\|_p$. Thus $\|f\|_p \le \|K\|_p$. The other inequality is trivial.

For the uniqueness, let $K = fB = f_1B_1$, where f and B are from above, f_1 is zero-free and B_1 a Blaschke multifunction. Then $\log|K| = \log|f_1| + \log|B_1|$, with $\log|f_1|$ harmonic and $\log|B_1| \le 0$, so since h is the least harmonic majorant, $\log|f| = h \le \log|f_1|$. We thus get $\log|B| = \log|f_1B_1/f| = \log|B_1| + \big|\log|f_1/f|\big|$, and it follows that f_1/f satisfies (5.1). Hence f_1/f is a Blaschke product, but it is also non-zero, thus it is constant. This proves the uniqueness of the non-zero factor of K, and *a fortiori* the uniqueness of the Blaschke factor, since $B_1 = (f/f_1)B$. This completes the proof.

COROLLARY 5.4. *Suppose* $0 < p < \infty$, $K \in H^p$, $K \not\equiv 0$ *and* B *is the Blaschke* multifunction associated with K. Then there is a zero-free function $h \in H^2$ such that

$$K = Bh^{2/p}.$$

THEOREM 5.5. *Suppose B is a Blaschke multifunction, ν is a finite positive Borel measure on* T *which is singular with respect to Lebesgue measure, and*

$$M(\lambda) = B(\lambda) \exp\left\{ - \int_{\mathsf{T}} \frac{e^{it} + \lambda}{e^{it} - \lambda}\, d\nu \right\}. \tag{5.3}$$

Then M is inner and every inner multifunction is of this form.

PROOF. Suppose (5.3) holds. It is clear that $|M| \leq 1$. Let

$$f(\lambda) = \exp\left\{ - \int_{\mathsf{T}} \frac{e^{it} + \lambda}{e^{it} - \lambda}\, d\nu \right\}.$$

Then $\log|f|$ is the Poisson integral of $-d\nu$, and since ν is singular, $\log|f^{\mathrm{rad}}| = 0$, i.e., $|f^{\mathrm{rad}}| = 1$ almost everywhere. Since $|B^{\mathrm{rad}}| = 1$ almost everywhere it follows that M is inner.

Conversely, let M be inner. By Theorem 5.3, $M = fB$, where $\log|f|$ is the least harmonic majorant of M. By (4.5), $\log|f| = P[g\,dt/2\pi + d\xi]$, where $g \in L^1(\mathsf{T})$ and ξ is singular with respect to the Lebesgue measure. Both $|M|$ and $|B|$ have radial boundary values equal to 1 almost everywhere, so it is clear that $\log|f^{\mathrm{rad}}| = 0$ almost everywhere. But the radial boundary value of $\log|f|$ equals g almost everywhere, so $h = P[d\xi]$. Since h is real and $|h| \leq 1$, it is clear that $h = P[-d\nu]$ for some positive singular measure ν on T. This proves the theorem.

By these theorems, we see that the Blaschke multifunction has many properties in common with the ordinary Blaschke product. However, it is not determined by the zero-set of K, as is the case for functions. Moreover, we cannot divide by B and hope to get something analytic, simply because B is a multifunction: $K = fB$ does not imply $f = K/B$; factoring and dividing are not equivalent when we deal with multifunctions.

We close this section with a theorem concerning the existence of global selections to certain multifunctions in N.

THEOREM 5.6. *Suppose that $K \in N$ is such that $Z(K)$ fulfills the Blaschke condition, i.e., $\mu = \Delta \log|K| = \sum a_j \delta_{\lambda_j}$, with the a_j's being integers and $\sum a_j(1 - |\lambda_j|) < \infty$, Then there is a function $f \in N$ which is a global selection for K, i.e., $f(\lambda) \in K(\lambda)$ for all $\lambda \in$* D.

PROOF. Write $K = fB$ with f zero-free. Since $Z(K)$ fulfills the Blaschke condition we can form the corresponding Blaschke product b. Then

$\log|b| = -U_\mu$, and we also know that $\log|K| = -U_\mu$, so it follows that $|K| = |b|$. Thus defining

$$\frac{B}{b}(\lambda) = \begin{cases} (B/b)(\lambda), & \text{if } b(\lambda) \neq 0 \\ \lim \operatorname{den}_{\xi \to \lambda} (B/b)(\xi), & \text{if } b(\lambda) = 0, \end{cases}$$

it is clear from Proposition 3.5 that B/b is analytic. But it is also of constant modulus so by Proposition 4.7., $P(B/b) \equiv C$ and it follows that, for some constant c, $cb(\lambda) \in B(\lambda)$ for all $\lambda \in \mathsf{D}$. Therefore $c(fb)(\lambda) \in K(\lambda)$ for all $\lambda \in \mathsf{D}$. The theorem is proved.

6. Criteria for zero-capacity and finiteness of an analytic multifunction.

Given a compact set $K \subset \mathsf{C}$ we recall that the nth-diameter is defined by

$$\delta_n(K) = \sup\left(\prod_{i \neq j} |z_i - z_j|\right)^{\binom{n}{2}^{-1}},$$

taken over $n+1$ points z_i in K. For $n = 0$, with the convention that $\binom{0}{k} = 1$, this is just the above defined $|K|$ and for $n = 1$ the usual diameter of K, $\operatorname{diam} K$

If K is an analytic multifunction, we know by Proposition 3.3 that

$$L(\lambda) = \underbrace{K(\lambda) \times \cdots \times K(\lambda)}_{n+1}$$

is analytic. Thus since

$$\psi(\lambda, z_1, \ldots, z_{n+1}) = \binom{n}{2}^{-1} \sum \log|z_i - z_j|$$

is plurisubharmonic it follows that

$$\lambda \mapsto \log \delta_n\big(K(\lambda)\big) = \sup\big(\psi(\lambda, z_1, \ldots, z_{n+1});\ (z_1, \ldots, z_{n+1}) \in L(\lambda)\big)$$

is subharmonic. Thus also

$$\lambda \mapsto \delta_n\big(K(\lambda)\big)$$

is subharmonic. We state this in the following

PROPOSITION 6.1. *If $K: D \to 2^{\mathsf{C}}$, where $D \subset \mathsf{C}$, is an analytic multifunction, then the functions*

$$\lambda \mapsto \delta_n\big(K(\lambda)\big)$$

and

$$\lambda \mapsto \log \delta_n\big(K(\lambda)\big)$$

are subharmonic on D.

It is easy to see that for $n \geq 0$, $\delta_n(cK) = |c|\,\delta_n(K)$ for all $c \in \mathsf{C}$ and that for $n = 0, 1$, $\delta_n(K + L) \leq \delta_n(K) + \delta_n(L)$. This is not true for $n \geq 2$.

If $\delta_n(K)$ satisfies the growth condition in Theorem 4.9, we can apply the theorem and in this way get a test for when $\delta_n(K) \equiv 0$, namely if the radial limits are zero on a non-null subset of T, the logarithm of these cannot be in L^1 and thus the function is identically zero. Noting that for $n \geq 1$, $\delta_n(K) = 0$ if and only if $\#K \leq n$ and using the Scarcity Theorem we get

THEOREM 6.2. *Let $K \in N$. If for some $n \geq 1$, $\liminf_{r\to 1} \delta_n\big(K(re^{it})\big) = 0$ on a non-null subset of T, then $\#K(\lambda) \leq n$. Thus there is a closed analytic subvariety F in D such that for any $\lambda^0 \in \mathsf{D} \setminus F$ there exist $h_1, \dots, h_n$ holomorphic in a neighbourhood $U \ni \lambda^0$ so that $K(\lambda) = \{h_1(\lambda), \dots, h_n(\lambda)\}$ in U.*

PROOF. We notice that $\delta_n(K) \leq 2|K|$ so if $K \in N$ the growth condition of Theorem 4.9 is fulfilled for all $\delta_n(K)$ and the theorem applies. In particular if $\delta_n(K) \not\equiv 0$, then $\log\big(\delta_n(K)^{\mathrm{rad}}\big) \in L^1(\mathsf{T})$, which is a contradiction if the assumptions in the theorem are fulfilled.

The case $n = 1$ is interesting enough to state separetely.

THEOREM 6.3. *Let K be analytic on D. If*

$$\sup_r \int_{\mathsf{T}} \log^+ \operatorname{diam} K(re^{it}) \frac{dt}{2\pi} < \infty$$

and

$$\liminf_{r\to 1} \operatorname{diam} K(re^{it}) = 0$$

on a non-null subset of T, then K is a holomorphic function on D.

If $c(K)$ denotes the (logarithmic) capacity of K, it is known (M. Fekete, G. Szeg, see [1]) that $c(K)$ is the decreasing limit of $\delta_n(K)$ as $n \to \infty$. Thus $c(K)$ is subharmonic and using Theorem 4.9 again we get

THEOREM 6.4. *Let K be analytic on D. If*

$$\sup_r \int_{\mathsf{T}} \log^+ c\big(K(re^{it})\big) \frac{dt}{2\pi} < \infty$$

and

$$\liminf_{r \to 1} c\big(K(re^{it})\big) = 0$$

on a non-null subset of T, *then* K *is of capacity zero on* D.

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