TOEPLITZ ALGEBRAS AND INFINITE SIMPLE C*-ALGEBRAS ASSOCIATED WITH REDUCED GROUP C*-ALGEBRAS

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Abstract

Assume that \( \Gamma \) is the free product of an arbitrary number of finite cyclic groups and any free group, and the generators of \( \Gamma \) are \( g_1, g_2, \ldots \). Let \( \Gamma_+ \) consist of the unit \( e \) of \( \Gamma \) and all those reduced words of the form \( g_1^{n_1} g_2^{n_2} \cdots g_k^{n_k} \) where \( n_1, n_2, \ldots, n_k \) are positive integers, and let \( R_+ \) be the projection onto the subspace \( L^2(\Gamma_+) \) of \( L^2(\Gamma) \). We prove that the C*-algebra \( C_r^*(\Gamma, R_+) \) generated by the reduced group C*-algebra \( C_r^*(\Gamma) \) and \( R_+ \) has either one or two non-trivial closed ideals which are stable and of real rank zero. This construction results some purely infinite simple C*-algebras.

1. Introduction.

Let \( \mathcal{F}_n \) be the free group on \( n \) generators (\( 1 \leq n \leq +\infty \)), i.e., the free product \( \bigotimes_{n} Z \ast Z \ast \ldots \ast Z \ast \ldots \) of \( n \) copies of the group \( Z \) of all integers, and let \( \Gamma_i \), be the free product of some finite cyclic groups \( Z_{n_i} := Z/n_i Z \), i.e.,

\[
\Gamma_i(m) := \underbrace{Z_{n_1} \ast Z_{n_2} \ast \ldots \ast Z_{n_k} \ast \ldots}_{m}, \quad \text{where} \quad 2 \leq n_i < +\infty, \ 2 \leq m \leq \infty.
\]

The groups considered in this article are the free product

\[
\Gamma := \Gamma_i \ast \mathcal{F}_n, \quad \text{or} \quad \mathcal{F}_\infty,
\]

The unit of \( \Gamma \) is denoted by \( e \), and the generators of \( \Gamma \) are denoted by \( \{g_1, g_2, \ldots, g_k, \ldots\} \). Each element of \( \Gamma \) is a reduced word \( w := g_1^{m_1} g_2^{m_2} \cdots g_k^{m_k} \) of finite length \( l(w) := \sum_{j=1}^{k} |m_j| \); the word "reduced" means that all factors of the forms \( gg^{-1} \) and \( g^{-1}g \) are canceled out. For any finite cyclic group \( Z_{n_i} \) we make a convention that each element in \( Z_{n_i} \) is uniquely expressed by \( g_i^m \) for some integer \( m \) with \( 0 \leq m \leq n_i - 1 \). In this way each element in \( \Gamma \) is uniquely represented by a reduced word of finite length.

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Let \( \{f_g : g \in \Gamma \} \) be a standard orthonormal basis of the Hilbert space \( L^2(\Gamma) \) of all complex valued, square-summable sequences indexed by \( \Gamma \). Let \( U : \Gamma \rightarrow L(L^2(\Gamma)) \) be the left regular representation of \( \Gamma \) on \( L(L^2(\Gamma)) \), defined by \( U(g)f_h := f_{g^{-1}h} \) for \( f_h \in L^2(\Gamma) \), where \( L(H) \) denotes the algebra of all bounded operators on a Hilbert space \( H \). Then \( U(g) \) is a unitary operator in \( L(L^2(\Gamma)) \) for any \( g \in \Gamma \). The reduced group C*-algebra \( C_r^*(\Gamma) \) is the norm closure of the group ring \( \mathbb{C}[\Gamma] \) consisting of all linear combinations: \( \sum_{i=1}^{n} \alpha_i U(h_i), h_i \in \Gamma, \alpha_i \in \mathbb{C}, \) and \( n \in \mathbb{N} \).

The purpose of this article is to investigate the structure of the C*-algebra generated by \( C_r^*(\Gamma) \) and the projection \( R_+ \) onto the subspace \( L^2(\Gamma_+) \), denoted by \( C_r^*(\Gamma, R_+) \), and to investigate the structure of the Toeplitz algebra \( \mathcal{T}_+ \) generated by \( \{R_+ U(g)R_+ : h \in \Gamma_+ \} \), where \( \Gamma_+ \) consists of \( e \) and all those reduced words of the form \( g_{i_1}^{m_1}g_{i_2}^{m_2}...g_{i_k}^{m_k} (m_i \in \mathbb{N}) \). Briefly speaking, \( \mathcal{T}_+ \) is generated by some isometries and unitaries on \( L^2(\Gamma_+) \), as the reader will see later. The structures of \( \mathcal{T}_+ \) and \( C_r^*(\Gamma, R_+) \) depend on the generators of \( \Gamma \). One of main results, Theorem 3.1, asserts that if \( \Gamma = Z_{m_0} \ast \mathcal{F}_n \) where \( 2 \leq m_0 < +\infty \) and \( 1 \leq n < +\infty \), then \( C_r^*(\Gamma, R_+) \) contains exactly two nontrivial closed ideals, both are stable and of real rank zero; one is \( J_{R_+} \) generated by \( R_+ \), and the other is \( J_0 \) which is \(*\)-isomorphic to the algebra \( \mathcal{K} \) consisting of all compact operators on \( L^2(\Gamma) \). Furthermore, the quotient algebra \( J_{R_+}/J_0 \) is \(*\)-isomorphic to \( \mathcal{O}_{m_0n} \otimes \mathcal{K} \), where \( \mathcal{O}_{m_0n} \) is the Cuntz algebra generated by \( m_0n \) isometries. This compares with the case when \( \Gamma = \mathcal{F}_n \) \((2 \leq n < +\infty)\) that we studied in [48], for which \( C_r^*(\Gamma, R_+) \) has two stable, nontrivial, closed ideals of real rank zero, whose quotient is \(*\)-isomorphic to \( \mathcal{O}_n \otimes \mathcal{K} \). For all the following cases,

\[
\Gamma = \mathcal{F}_\infty, Z_{m_0} \ast \mathcal{F}_\infty, (Z_{n_1} \ast Z_{n_2} \ast ...)^m \ast \mathcal{F}_n
\]

(where \( 2 \leq m \leq +\infty \) and \( 1 \leq n \leq +\infty \)), the other main results, Theorem 4.1, concludes that the C*-algebra \( C_r^*(\Gamma, R_+) \) contains only one nontrivial closed ideal \( J_{R_+} \) generated by \( R_+ \) which is a non-unital, purely infinite, simple C*-algebra. Thus, by our earlier result in [44], \( J_{R_+} \cong J_{R_+} \otimes \mathcal{K} \). Here a C*-algebra is said to be simple, if \{0\} and itself are the only closed ideals. We remind the reader that a unital simple C*-algebra is purely infinite if and only if for any nonzero element \( X \) there exist two elements \( Y \) and \( Z \) such that \( YXZ = I \) ([16] and [29]), and that an arbitrary simple C*-algebra is purely infinite and simple if and only if each nonzero projection is infinite and it has real rank zero ([45,1,2]).

The classification of separable, purely infinite, simple C*-algebras has been under attack in recent years ([21], [37], [38]). With this classification problem in mind, we have lately put some efforts in dealing with the C*-al-
gebras of the form $C^*_r(\Gamma, P)$, since for many other choices of the projection $P$ it is a purely infinite, simple C*-algebra (see [47]). Indeed, the class of C*-algebras generated by $C^*_r\Gamma$ and one projection $P_\Omega$ onto a subspace $l^2(\Omega)$, i.e., \{$(C^*_r(\Gamma, P_\Omega), \Omega \subset \Gamma)$,\} contains some new types of C*-algebras. However, it remains a difficulty task to classify this class up to *-isomorphism.

2. Toeplitz operators and the Toeplitz algebra.

The article is essentially in a self-contained form. Most of the references in the list are relevant materials but not needed. We start with the following analyses on the construction of Toeplitz operators.

2.1 For each $h \in \Gamma$ one defines a Toeplitz operator associated with $R_+$ by

$$T_h := R_+U(h)R_+.$$ 

The Toeplitz algebra $T_+$ associated with $R_+$ is generated by $\{T_h : h \in \Gamma_+\}$, and the corner algebra associated with $R_+$, denoted by $R_+C^*_r(\Gamma, R_+)R_+$, is generated by $\{T_h : h \in \Gamma\}$.

To consider the structures of $T_+$ and $C^*_r(\Gamma, R_+)$, we start with the construction of each Toeplitz operator $T_h$. Let us collect as follows some obvious facts derived immediately from definition.

1. $U(h)^* = U(h^{-1})$ and $T_h^* = T_{h^{-1}}$ for all $h \in \Gamma$.

2. $U(h_1 h_2) = U(h_2)U(h_1)$ for $h_1, h_2 \in \Gamma$.

3. For $g \in \Gamma$ the projection $U(g)^*R_+U(g)$ is onto the subspace $l^2(\Gamma_+)$ and the projection $U(g)R_+U(g)^*$ is onto the subspace $l^2(\Gamma_+^{-1})$.

4. $T_g(f_h) = R_+f_{g^{-1}h} = \begin{cases} f_{g^{-1}} & \text{if } h \in g\Gamma_+ \cap \Gamma_+ \\ 0 & \text{if } h \notin g\Gamma_+ \cap \Gamma_+. \end{cases}$ Thus, $T_g$ is a partial isometry in $\mathcal{L}(l^2(\Gamma_+))$ whose initial projection $T_g^*T_g$ is onto the subspace $l^2(g\Gamma_+ \cap \Gamma_+)$ and whose final projection $T_gT_g^*$ is onto the subspace $l^2(g^{-1}\Gamma_+ \cap \Gamma_+)$. 

From now on the notation $\Gamma_+(h)$ is reserved for the subset of $\Gamma_+$ consisting all reduced words of the form $hh_1$, where $hh_1$ is an irreducible product in the sense that the last word of $h$ of length one is not the inverse of the rst word of $h_1$. The notion $R_h$ denotes the projection onto the subspace $l^2(\Gamma_+(h))$. Here we point out that $\Gamma_+(h) \neq h\Gamma_+$ does not hold in general.

2.2. Proposition.

i) $\Gamma_+ = \{e\} \cup \Gamma_+(g_1) \cup \Gamma_+(g_2) \cup ... \cup \Gamma_+(g_k) \cup ....$

ii) If $g_i \neq g_j$, then $g_i\Gamma_+(g_j) = \Gamma_+(g_jg_i)$.

iii) If the order $n_i$ of $g_i$ is finite, then $g_i\Gamma_+(g_i) = \Gamma_+(g_i^{n_i})$.

iv) If the order $n_i$ of $g_i$ is finite, then $g_i\Gamma_+ = \Gamma_+$; furthermore,

$$g_i\Gamma_+(g_i^{n_i-1}) = \Gamma_+ \backslash \Gamma_+(g_i) \quad \text{and} \quad g_i(\Gamma_+ \backslash \Gamma_+(g_i^{n_i-1})) = \Gamma_+(g_i).$$
2.3. Proposition. These facts are readily checked.

(i) If a generator $g_i$ of $\Gamma$ is of finite order (i.e., $g_i^{-n} = e$ for some $2 \leq n_i < +\infty$), then $U(g_i) R_+ = R_+ U(g_i)$. Consequently, if $h_0 \in \Gamma$, then $T_{h_0}$ is a unitary operator on $L^2(\Gamma_+)$.

(ii) If $g_i$ is of infinite order, then $U(g_i) R_+ \neq R_+ U(g_i)$, and $T_{g_i}$ is a co-isometry such that $T_{g_i}^* T_{g_i} = R_{g_i}$ and $T_{g_i} T_{g_i}^* = R_+$.

Proof. (i) Assume that $g_i$ is of finite order. To show $R_+ U(g_i) = U(g_i) R_+$, it is equivalent to show $g_i \Gamma_+ \subset \Gamma_+ \text{ and } g_i \Gamma_+ \setminus \Gamma_+ \subset \Gamma_+ \setminus \Gamma_+$.

The first inclusion is trivial by the definition of $\Gamma_+$. Let $h$ be any reduced word in $\Gamma_+ \setminus \Gamma_+$. Then $h$ contains a factor of the form $g_j^{-1}$ for some generator $g_j$ of infinite order. If $h$ is an irreducible product $g_j^{-1} h_1$ for some reduced word $h_1$, then $g_i h \in \Gamma_+ \setminus \Gamma_+$. If $h$ is an irreducible product of the form $g_k h_1 g_j^{-1} h_2$ for some reduced words $h_1$ and $h_2$ and a generator $g_k$, then $g_i h$ is again in $\Gamma_+ \setminus \Gamma_+$. Thus, the second inclusion also holds. If $h_0 \in \Gamma$, write $h_0 = g_{j_1} g_{j_2} \ldots g_{j_k}$ for some generators of $\Gamma$ with finite order. Then $U(h_0) = U(g_{j_k}) U(g_{j_{k-1}}) \ldots U(g_{j_1})$, and hence $T_{h_0} = R_+ U(h_0) R_+ = T_{g_{j_k}} T_{g_{j_{k-1}}} \ldots T_{g_{j_1}}$, which is a unitary operator on $L^2(\Gamma_+)$. 

(ii) If $g_i$ is of infinite order, then $g_i^{-1} h \in \Gamma_+ \setminus \Gamma_+$ and $g_i (g_i^{-1} h) = h \in \Gamma_+$ as long as $h$ is a reduced word in $\Gamma_+$ starting with another generator different from $g_i$. Thus, $L^2(\Gamma_+) \setminus \Gamma_+$ is not a reduced subspace of $U(g_i)$. By definition $T_{g_i}^* T_{g_i}$ is the projection onto the subspace $L^2(g_i \Gamma_+)$. Since $g_i \Gamma_+ = \Gamma_+(g_i)$ in case $g_i$ is of infinite order, one sees that $T_{g_i}^* T_{g_i} = R_{g_i}$.

2.4. Proposition. Let $h \in \Gamma$. Then $T_h \neq 0$ if and only if $h$ can be uniquely written as an irreducible product $h' h_0 h''^{-1}$, where $h_0 \in \Gamma$, and $h', h'' \in \Gamma_+$ such that the last words of $h'$ and $h''$ with length one are some generators of infinite order whenever $h' \neq e, h'' \neq e$.

Proof. First, each reduced word $h \in \Gamma$ can be written uniquely as a product $g_{i_1}^{n_1} g_{i_2}^{n_2} \ldots g_{i_k}^{n_k}$ for some generators $g_{i_1}, g_{i_2}, \ldots, g_{i_k}$, where $g_j \neq g_{j+1}$ for $1 \leq j \leq k - 1$, and all $n_j$ are integers. By our convention, $n_j > 0$ whenever $g_{i_j}$ is a generator of finite order. Our attention will be on the powers of those generators of infinite order in the above product. If all $g_{i_1}, \ldots, g_{i_k}$ are of finite order, then $h = h_0$, and hence $T_h \neq 0$. Assume that there is at least one generator involved is of infinite order. Select all generators of infinite order in the ordered tuple $(g_{i_1}, g_{i_2}, \ldots, g_{i_k})$, with the order kept, and write them as an ordered tuple $(g_{i_{j_1}}, g_{i_{j_2}}, \ldots, g_{i_{j_l}})$ (where $j_1 < j_2 < \ldots < j_l$). Clearly, the assertion of this proposition is equivalent to the following: $T_h \neq 0$ if and only if the
signs of the corresponding powers \( n_{j_1}, n_{j_2}, \ldots, n_{j_k} \), as an ordered tuple, have patterns \(+, +, \ldots, +\) (for the case \( h = h' h_0 \); i.e., \( h'' = e \)), or \(-, -, \ldots, -\) (for the case \( h = h_0 h''^{-1} \); i.e., \( h' = e \)), or \(+, +, \ldots, +, -, \ldots, -\) (for the case \( h = h' h_0 h''^{-1} \); i.e., \( h' \neq e \) and \( h'' \neq e \)). These patterns are exactly all possibilities for which \( h^{-1} \Gamma_+ \cap \Gamma_+ \neq \emptyset \), that is, the final projection of \( T_h \) is non-zero.

2.5. Proposition.
(i) If \( T_h \neq 0 \), write \( h = h' h_0 h''^{-1} \) as in Proposition 2.4, then \( T_h = T_{h'}^* T_h T_{h''} \), which is a partial isometry whose final projection is \( T_{h'}^* T_{h''} \) and whose initial projection is \( T_h^* T_{h'} \); both are independent of \( h_0 \in \Gamma_+ \).

(ii) If \( h_1 \in \Gamma_+ \), write \( h = g_{i_1}^{n_1} g_{i_2}^{n_2} \cdots g_{i_k}^{n_k} \) for some generators \( g_{i_1}, g_{i_2}, \ldots, g_{i_k} \), then \( T_{h_1} = (T_{g_{i_1}})^{n_1} \cdots (T_{g_{i_2}})^{n_2} (T_{g_{i_1}})^{n_1} \).

Proof. (i) Obviously, \( T_{h' h_0 h''^{-1}} = R_+ U(h'')^* U(h_0) U(h') R_+ \). It is easily seen by definition that

\[
R_+ U(h'')^* U(h_0) (I - R_+) U(h') R_+ = 0.
\]

Thus, \( T_h = T_{h'}^* T_h T_{h''} \); here we use the fact that \( U(h_0) R_+ = R_+ U(h_0) \). It is obvious that \( T_h \) is a partial isometry.

(ii) For each generator \( g_i \) of \( \Gamma \) one has \( R_+ U(g_i) R_+ = 0 \) (where \( R_+ = I - R_+ \)). Thus, with respect to the decomposition \( R_+ \oplus R_+^\perp = I \) one can write \( U(g_i) \) as a \( 2 \times 2 \) matrix

\[
\begin{pmatrix}
T_{g_i} & 0 \\
R_+^\perp U(g_i) R_+ & R_+^\perp U(g_i) R_+^\perp
\end{pmatrix}.
\]

It follows that \( T_{g_i g_{j_1}} = T_{g_{j_1}} T_{g_i} \), and hence (by induction)

\[
T_{h_1} = (T_{g_{i_1}})^{n_1} \cdots (T_{g_{i_2}})^{n_2} (T_{g_{i_1}})^{n_1}.
\]

2.6. Proposition.
(i) If \( h \in \Gamma_+ \), then \( h \) can be uniquely written as an irreducible product of the form \( h = (h_1 g_{i_1})(h_2 g_{i_2}) \cdots (h_k g_{i_k}) h_0 \), where all \( g_{i_1}, g_{i_2}, \ldots, g_{i_k} \), are generators of \( \Gamma \) of infinite order, and \( h_0, h_1, h_2, \ldots, h_{k+1} \) are elements of \( \Gamma_+ \).

(ii) \( T_{h g_i} T_{h g_{j}}^* = R_+ \) and \( T_{h g_i}^* T_{h g_{j}} = R_{h g_i} \). Consequently, \( T_h \) is a co-isometry.

Proof. (i) is trivial. (ii) is also straightforward. In fact,

\[
T_{h g_i} T_{h g_{j}}^* = T_{g_i} T_{h} T_{h} T_{g_{j}}^* = R_+,
\]

\[
T_{h g_i}^* T_{h g_{j}} = T_{h}^* T_{g_{j}} T_{h} R_{g_i} T_{h} = T_{h}^* R_{g_{j}} T_{h} = R_{h g_i}.
\]

Clearly, \( T_h T_h^* = R_+ \), and \( T_h^* T_h = R_h \); i.e., \( T_h \) is a co-isometry.
2.7. Corollary. The Toeplitz algebra $\mathcal{T}_+$ coincides with $R_+ C^*_r(\Gamma, R_+) R_+$; both are generated by $\{T_h : h \in \Gamma_+\}$, and in turn, by $\{T_{g_i}\}$.

Proof. By the fact each nonzero Toeplitz operator $T_h$ can be written as a product $T_{h''} T_{h_0} T_{h'}$, where $h_0, h', h''$ are in $\Gamma_+$, one sees that the corner algebra is also generated by $\{T_h : h \in \Gamma_+\}$ as $\mathcal{T}_+$ is. Furthermore, Proposition 2.5(ii) asserts that each element $T_h \in \{T_h : h \in \Gamma_+\}$ is a product of Toeplitz operators in $\{T_{g_i}\}$ where $\{g_i\}$ is the set of all generators of $\Gamma$.

2.8. Remark. Proposition 2.3 and Corollary 2.7 combined tell the following:

(a) If $\Gamma = \mathcal{F}_\infty$, then $\mathcal{T}_+$ is generated by a sequence $\{T_{g_i}\}$ of isometries.

(b) If $\Gamma$ is any other group among the ones considered in this article, then $\mathcal{T}_+$ is generated by some isometries and unitaries, these isometries are $T_{g_i}$, where $g_i$ are of infinite order and these unitaries are $T_{g_i}$, where $g_i$ is of finite order.

2.9. Proposition The closed ideal $\mathcal{J}_{R_+}$ of $C^*_r(\Gamma, R_+)$ generated by $R_+$ is nontrivial.

Proof. We proved in [48] that the projection $P_\Omega$ onto the subspace $l^2(\Omega)$ (where $\Omega \subset \Gamma$) generates a nontrivial closed ideal of $C^*_r(\Gamma, R_+)$ if and only if there is no finite subset $\{h_1, h_2, ..., h_m\}$ of $\Gamma$ such that $\bigcup_{k=1}^m h_k \Omega = \Gamma$. Clearly, there is no finite subset $\{h_1, h_2, ..., h_m\}$ of $\Gamma$ such that $\bigcup_{k=1}^m h_k \Gamma_+ = \Gamma$. Thus, $\mathcal{J}_{R_+}$ is non-trivial.

3. The case $\Gamma = \mathbb{Z}_{n_0} * \mathcal{F}_n$.

In [48] we have dealt with the case $\Gamma = \mathcal{F}_n$ for $2 \leq n < +\infty$. As a result, $C^*_r(\Gamma, R_+)$ contains exactly two nontrivial (closed) ideals, the ideal $\mathcal{K}(l^2(\Gamma))$ of all compact operators and the ideal $\mathcal{J}_{R_+}$ generated by $R_+$, both are stable C*-algebras with real rank zero and $\mathcal{J}_{R_+} / \mathcal{K}(l^2(\Gamma)) \cong \mathcal{O}_n \otimes \mathcal{K}$. The structures of $C^*_r(\Gamma, R_+)$ for other cases turn out to be different. We will apply two different techniques to deal with the following separate cases:

(i) $\Gamma = \mathbb{Z}_{n_0} * \mathcal{F}_n$, where $2 \leq n_0 < +\infty$ and $1 \leq n < +\infty$.

(ii) $\Gamma = \mathcal{F}_\infty$.

(iii) $\Gamma = \mathbb{Z}_{n_0} * \mathcal{F}_\infty$.

(iv) $\Gamma = (\mathbb{Z}_{n_1} * \mathbb{Z}_{n_2} * \ldots) * \mathcal{F}_n$, where $2 \leq m < +\infty$ and $1 \leq n < +\infty$.

(v) $\Gamma = (\mathbb{Z}_{n_1} * \mathbb{Z}_{n_2} * \ldots) * \mathcal{F}_\infty$.

In this section we consider the case (i), $\Gamma = \mathbb{Z}_{n_0} * \mathcal{F}_n$ where $1 \leq n < +\infty$. 
Assume that $g_0$ is the generator of $\mathbb{Z}_{n_0}$, and $g_1, g_2, \ldots, g_n$ are the generators of $\mathcal{F}_n$.

The structure of $C_r^*(\Gamma, R_+)$ is summarized as in the following theorem:

**3.1. Theorem.** Assume $\Gamma = \mathbb{Z}_{n_0} \ast \mathcal{F}_n$ where $1 \leq n < +\infty$. Then the following hold:

(i) The Toeplitz algebra $\mathcal{T}_+$ contains only one nontrivial closed ideal $\mathcal{I}_0$ which is $*$-isomorphic to $\mathcal{K}$.

(ii) $\mathcal{T}_+ / \mathcal{I}_0 \cong \mathcal{O}_{n_0}$, where $\mathcal{O}_{n_0}$ the Cuntz algebra with $n_0$ generators.

(iii) $C_r^*(\Gamma, R_+)$ contains a chain of exactly two nontrivial closed ideals which are stable and of real rank zero; one is $\mathcal{I}$ generated by $Q_0$ which is $*$-isomorphic to $\mathcal{K}$, and the other is $\mathcal{I}_{R_+}$ generated by $R_+$ which is $*$-isomorphic to $\mathcal{T}_+ \otimes \mathcal{K}$; furthermore, $\mathcal{I}_{R_+} / \mathcal{I} \cong \mathcal{O}_{n_0} \otimes \mathcal{K}$.

(iv) $\text{RR}(\mathcal{I}_{R_+}) = \text{RR}(\mathcal{T}_+) = 0$.

Before proving Theorem 3.1 we state the following immediate corollary.

**3.2. Corollary.** Assume that $\Gamma$ is as in Theorem 3.1. Then the following short sequences are exact:

$$
0 \rightarrow \mathcal{I}_{R_+} \rightarrow C_r^*(\Gamma, R_+) \rightarrow C_r^*(\Gamma) \rightarrow 0,
$$

$$
0 \rightarrow \mathcal{I}_0 \rightarrow \mathcal{T}_+ \rightarrow \mathcal{O}_{n_0} \rightarrow 0,
$$

$$
0 \rightarrow \mathcal{I} \rightarrow \mathcal{I}_{R_+} \rightarrow \mathcal{O}_{n_0} \otimes \mathcal{K} \rightarrow 0.
$$

We now turn to the proof of Theorem 3.1.

**3.3. Lemma.** Let $Q_0$ be the projection onto the subspace spanned by $f_e, f_{g_0}, \ldots, f_{g_0^{n_0-1}}$ and $P_n$ be the projection onto the subspace $\mathcal{F}(\Gamma_+(g_1) \cup F_+(g_2) \cup \ldots \cup F_+(g_n))$, where $1 \leq n < +\infty$. Then

(i) $P_n = \sum_{i=1}^{n_0} T_{g_i} * T_{g_i}, \quad Q_0 \in \mathcal{T}_+, \quad R_+ = P_n + \sum_{i=1}^{n_0} U(g_0)^* P_n U(g_0^i) + Q_0$,

(ii) $\mathcal{T}_+$ is generated by $\{T_{g_i} : 0 \leq i \leq n_0 - 1, \ 1 \leq j \leq n\} \cup \{T_{g_0}\}$, and

(iii) the closed ideal $\mathcal{I}_0$ of $\mathcal{T}_+$ generated by $Q_0$ is $*$-isomorphic to $\mathcal{K}$.

**Proof.** (i) First, $T_{g_k} * T_{g_k} = R_{g_k}$ is a projection in $\mathcal{T}_+$ for any $1 \leq k \leq n$. Hence $P_n = \sum_{k=1}^n R_{g_k} \in \mathcal{T}_+$. Clearly, $R_+ - P_n = R_{g_0} + P(e)$, where $P(e)$ is the one dimensional projection onto the subspace spanned by $f_e$ ($R_{g_0}$ and $P(e)$ may not in $\mathcal{T}_+$). It is easily seen that

$$
\Gamma_+ = (\bigcup_{k=1}^n \Gamma_+(g_k)) \cup \Gamma_+(g_0) \cup \{e\}
$$

$$
= (\bigcup_{k=1}^n \Gamma_+(g_k)) \cup (\bigcup_{i=1}^{n_0-1} g_0^i \cup \bigcup_{k=1}^{n_0-1} \Gamma_+(g_k)) \cup \{e, g_0, g_0^2, \ldots, g_0^{n_0-1}\}.
$$

Then $R_+ = P_n + \sum_{i=1}^{n_0-1} U(g_0^i)^* P_n U(g_0^i) + Q_0$. It follows that $Q_0 \in \mathcal{T}_+$. The conclusion (ii) follows from Proposition 2.5 and (iii) is obvious.
3.4. Proof of Theorem 3.1. (i) has been proved in Lemma 3.3 (ii).
(ii) For any $X \in \mathcal{T}_+$ let $\tilde{X}$ denote the image of $X$ in the quotient algebra $\mathcal{T}_+/\mathcal{I}_0$. Since $\mathcal{T}_+$ is generated by the following set
\[ \{T_{g_0^{k}g_j} : 0 \leq k \leq n_0 - 1, \; 1 \leq j \leq n \} \cup \{T_{g_j}\}, \]
of course $\mathcal{T}_+/\mathcal{I}_0$ is generated by the set
\[ \{\tilde{T}_{g_0^{k}g_j} : 0 \leq k \leq n_0 - 1, \; 1 \leq j \leq n \} \cup \{\tilde{T}_{g_j}\}. \]
We further claim, for the special case $\Gamma = \mathbb{Z}_{n_0} \ast \mathcal{F}_n$ only, that $\mathcal{T}_+/\mathcal{I}_0$ is generated by the following set of $n_0n$ isometries:
\[ \{\tilde{T}_{g_0^{k}g_j} : 0 \leq k \leq n_0 - 1, \; 1 \leq j \leq n \}. \]
In fact, one has
\[
\sum_{k=0}^{n_0-1} \sum_{j=1}^{n} T_{g_0^{k}g_j}^* T_{g_0^{k}g_j} = \sum_{k=0}^{n_0-1} \sum_{j=1}^{n} T_{g_0}^* T_{g_0}^{-1} g_j T_{g_0}^{-1} g_j = T_{g_0}^* (R_+ - Q_0),
\]
the last equality is due to the following:
\[
R_+ - Q_0 = \sum_{k=0}^{n_0-1} \sum_{j=1}^{n} T_{g_0^{k}g_j}^* T_{g_0^{k}g_j} = \sum_{k=0}^{n_0-1} \sum_{j=1}^{n} T_{g_0}^* T_{g_0}^{-1} g_j T_{g_0}^{-1} g_j;
\]
note here $g_0^{-1} = g_0^{n_0-1}$. Thus, $T_{g_0}^* (R_+ - Q_0)$ is in the $*$-algebra generated by
\[ \{T_{g_0^{k}g_j} : 0 \leq k \leq n_0 - 1, \; 1 \leq j \leq n \}. \]
It follows that $\tilde{T}_{g_0}$ is in the $*$-algebra generated by $\{\tilde{T}_{g_0^{k}g_j} : 0 \leq k \leq n_0 - 1, \; 1 \leq j \leq n \}$. Therefore, $\mathcal{T}_+/\mathcal{I}_0$ is generated by
\[ \{\tilde{T}_{g_0^{k}g_j} : 0 \leq k \leq n_0 - 1, \; 1 \leq j \leq n \}, \]
and hence, is $*$-isomorphic to $\mathcal{O}_{n_0n}$ ([17]).
(iii) Clearly, the closed ideal $\mathcal{I}$ of $C^*_\Gamma (\mathcal{F}, R_+)$ generated by $Q_0$ is stably isomorphic to $\mathcal{I}_0$. Thus, $\mathcal{I} \cong \mathcal{K}$. The closed ideal $\mathcal{I}_{R_+}$ generated by $\mathcal{R}_+$ is stably isomorphic to $\mathcal{T}_+$ (i.e., $\mathcal{I}_{R_+} \otimes \mathcal{K} \cong \mathcal{T}_+ \otimes \mathcal{K}$). Since $\mathcal{T}_+/\mathcal{I}_0 \cong \mathcal{O}_{n_0n}$, one can show that $\mathcal{I}_{R_+}/\mathcal{I} \cong \mathcal{O}_{n_0n} \otimes \mathcal{K}$ by using exactly the same argument as in the proof of [48, Lemma 3.3]. Furthermore, again by exactly the same arguments as in the proof of [47, 3.5 and 3.7], one can show that
\( \mathcal{I}_{R_+} \cong \mathcal{I}_{R_+} \otimes \mathcal{K} \). It is clear that \( \mathcal{I}_{R_+} \) and \( \mathcal{I} \) are the only two nontrivial closed ideals of \( C^*_r(\Gamma, R_+) \).

(iv) Using a general lifting result ([7,3.14] and [45,2.4]), we conclude that

\[
\text{RR}(\mathcal{I}_+) = 0 \quad \text{and} \quad \text{RR}(\mathcal{I}_{R_+}) = 0,
\]

based on the fact \( \text{RR}(\mathcal{O}_{n,n}) = 0 \) ([44]).

4. The other cases.

In this section we will investigate, by using a different technique, the structure of \( C^*_r(\Gamma, R_+) \) for the remaining cases, i.e., \( \Gamma \) is any of the following groups

\[
\mathcal{F}_\infty, \quad Z_{n_0} \ast \mathcal{F}_\infty, \quad \left( \underbrace{Z_{n_1} \ast Z_{n_2} \ast \ldots}_{m} \right) \ast \mathcal{F}_n, \quad \left( \underbrace{Z_{n_1} \ast Z_{n_2} \ast \ldots}_{m} \right) \ast \mathcal{F}_\infty;
\]

where \( 2 \leq m \leq +\infty \) and \( 1 \leq n < +\infty \).

4.1. Theorem. Assume that \( \Gamma = \mathcal{F}_\infty \), or \( Z_{n_0} \ast \mathcal{F}_\infty \), or \( \left( \underbrace{Z_{n_1} \ast Z_{n_2} \ast \ldots}_{m} \right) \ast \mathcal{F}_n \), or \( \left( \underbrace{Z_{n_1} \ast Z_{n_2} \ast \ldots}_{m} \right) \ast \mathcal{F}_\infty \), where \( m \geq 2 \) but \( 1 \leq n < +\infty \) is arbitrary. Then

(i) \( \mathcal{I}_+ \) is a purely infinite simple \( C^* \)-algebra (and hence \( \text{RR}(\mathcal{I}_+) = 0 \)), and

(ii) \( C^*_r(\Gamma, R_+) \) contains only one nontrivial closed ideal \( \mathcal{I}_{R_+} \), which is generated by \( R_+ \) and \( * \)-isomorphic to \( \mathcal{I}_+ \otimes \mathcal{K} \) (and hence, is a stable, purely infinite, simple \( C^* \)-algebra).

4.2. Corollary. Assume that \( \Gamma \) is as in Theorem 4.1. Then the \( * \)-isomorphism \( \mathcal{I}_{R_+} \cong \mathcal{I}_+ \otimes \mathcal{K} \) induces a short sequence:

\[
0 \rightarrow \mathcal{I}_+ \otimes \mathcal{K} \rightarrow C^*_r(\Gamma, R_+) \rightarrow C^*_r(\Gamma) \rightarrow 0.
\]

Two reduced words \( h_1, h_2 \in \Gamma \) are said to be comparable, if either \( h_1 \in \Gamma(h_2) \), denoted by \( h_1 \prec h_2 \), or \( h_2 \in \Gamma(h_1) \), denoted by \( h_2 \prec h_1 \) (cf. [47,4.1]). If neither \( h_1 \prec h_2 \) nor \( h_2 \prec h_1 \), we say that \( h_1 \) and \( h_2 \) are incomparable. Obviously, \( h_1 \) and \( h_2 \) are incomparable if and only if \( \Gamma(h_1) \cap \Gamma(h_2) = \emptyset \).

To prove Theorem 4.1, we first prove two key lemmas.

4.3. Lemma. Assume that \( \Gamma \) is as in the statement of Theorem 4.1. If \( h_1, h_2, \ldots, h_n \) are distinct reduced words in \( \Gamma_+ \) and \( k_1, k_2, \ldots, k_m \) are distinct reduced words in \( \Gamma \setminus \Gamma_+ \), then there exists a reduced word \( h \in \Gamma_+ \) satisfying the following conditions:

(i) \( h_1 h, h_2 h, \ldots, h_n h \in \Gamma_+ \),

(ii) \( k_1 h, k_2 h, \ldots, k_m h \in \Gamma \setminus \Gamma_+ \),

(iii) \( h_1 h, h_2 h, \ldots, h_n h \) are mutually incomparable, and
(iv) all $h, h_1 h, h_2 h, \ldots, h_n h$ end with a generator of infinite order.

**Proof.** Case 1. $\Gamma = (\mathbb{Z}_{n_1} \ast \mathbb{Z}_{n_2} \ast \ldots) \ast \mathcal{F}_n$, or $(\mathbb{Z}_{n_1} \ast \mathbb{Z}_{n_2} \ast \ldots) \ast \mathcal{F}_\infty$.

Assume that $g_1$ is of infinite order, $g_2$ and $g_3$ are any two distinct generators of finite order. If $n_0 \in \mathbb{N}$ is chosen to be large enough, all $h_1(g_2g_3)^{n_0}g_1$, $h_2(g_2g_3)^{n_0}g_1$, $\ldots$, and $h_n(g_2g_3)^{n_0}g_1$ are in $\Gamma_+$ and end with $g_1$, and all $k_1(g_2g_3)^{n_0}g_1$, $k_2(g_2g_3)^{n_0}g_1$, $\ldots$, and $k_n(g_2g_3)^{n_0}g_1$ are still in $\Gamma \setminus \Gamma_+$, since each $k_i$ contains at least one factor of the form $g_j^{-1}$ for some generator $g_j$ of infinite order that cannot be canceled with $(g_2g_3)^{n_0}g_1$ for any $n_0$.

We will further find a reduced word $h'$ in $\Gamma_+$ such that $h := (g_2g_3)^{n_0}g_1 h'$ is an irreducible product satisfying all the conditions (i), (ii), (iii) and (iv). Here we notice that $h_1(g_2g_3)^{n_0}g_1 h'$, $h_2(g_2g_3)^{n_0}g_1 h'$, $\ldots$, $h_n(g_2g_3)^{n_0}g_1 h'$ remain in $\Gamma_+$ and $k_1(g_2g_3)^{n_0}g_1 h'$, $k_2(g_2g_3)^{n_0}g_1 h'$, $\ldots$, $k_n(g_2g_3)^{n_0}g_1 h'$ remain in $\Gamma \setminus \Gamma_+$ for any irreducible product of the form $(g_2g_3)^{n_0}g_1 h'$.

The lemma is trivial in case $n' = 1$, since $h := (g_2g_3)^{n_0}g_1$ is as wanted. Consider the case $n' = 2$ (we will need the arguments later for the general situation). If $h_1(g_2g_3)^{n_0}g_1$ and $h_2(g_2g_3)^{n_0}g_1$ are incomparable, set $h := (g_2g_3)^{n_0}g_1$, as wanted. If $h_1(g_2g_3)^{n_0}g_1$ is comparable with $h_2(g_2g_3)^{n_0}g_1$, then

either $h_1(g_2g_3)^{n_0}g_1 < h_2(g_2g_3)^{n_0}g_1$ or $h_2(g_2g_3)^{n_0}g_1 < h_1(g_2g_3)^{n_0}g_1$.

We need only to consider one case, say $h_1(g_2g_3)^{n_0}g_1 < h_2(g_2g_3)^{n_0}g_1$, since a symmetric argument applies to the other. Write the following irreducible product:

$$h_1(g_2g_3)^{n_0}g_1 = h_2(g_2g_3)^{n_0}g_1 g_{i_1}^{n_1} g_{i_2}^{n_2} \ldots g_{i_t}^{n_t}(g_2g_3)^{n_0}g_1,$$

where $g_{i_1}, \ldots, g_{i_t}$ are some generators of $\Gamma$. Let $g_i$ be a generator of $\Gamma$ such that $g_i \neq g_{i_1}$, then $h_1(g_2g_3)^{n_0}g_1 g_i g_1$ and $h_2(g_2g_3)^{n_0}g_1 g_i g_1$ are incomparable. Set $h := (g_2g_3)^{n_0}g_1 g_i g_1$, as desired. In any case, for $h_1$ and $h_2$ we can choose a reduced word $h \in \Gamma_+$ that ends with $g_1$ and satisfies all the conditions (i), (ii), (iii) and (iv).

We now consider the general situation by induction on $n'$ for each fixed $n''$. Applying the above arguments for the case $n' = 2$ to $h_n(g_2g_3)^{n_0}g_1$ and $h_1(g_2g_3)^{n_0}g_1$, one gets a reduced word $h_1'$ such that $h_1(g_2g_3)^{n_0}g_1 h_1'$ and $h_n(g_2g_3)^{n_0}g_1 h_1'$ are incomparable, and $h_1'$ ends with $g_1$. Applying the same argument to $h_n(g_2g_3)^{n_0}g_1 h_1'$ and $h_2(g_2g_3)^{n_0}g_1 h_1'$, one gets a reduced word $h_2'$ such that $h_2'$ ends with $g_1$, and $h_n(g_2g_3)^{n_0}g_1 h_1' h_2'$ and $h_2(g_2g_3)^{n_0}g_1 h_1' h_2'$ are incomparable. Furthermore,

$$h_n(g_2g_3)^{n_0}g_1 h_1' h_2' \text{ and } h_1(g_2g_3)^{n_0}g_1 h_1' h_2'$$
remain incomparable, since the product \((g_2 g_3)^{n_0} g_1 h'_1 h'_2\) is irreducible. Repeat this process \(n' - 1\) times, one gets an irreducible product \(h'_0 := h'_1 h'_2 \ldots h'_{n'-1}\) such that \(h'_0\) ends with \(g_1\), and \(h'_m (g_2 g_3)^{n_0} g_1 h'_0\) is incomparable with each of

\[ h_1 (g_2 g_3)^{n_0} g_1 h'_0, \ h_2 (g_2 g_3)^{n_0} g_1 h'_0, \ldots, \ h'_{n'-1} (g_2 g_3)^{n_0} g_1 h'_0. \]

By the inductive assumption, there exists a reduced word \(h_0\) such that \(h_0\) ends with \(g_1\), and

\[ h_1 (g_2 g_3)^{n_0} g_1 h'_0 h_0, \ h_2 (g_2 g_3)^{n_0} g_1 h'_0 h_0, \ldots, \ h'_{n'-1} (g_2 g_3)^{n_0} g_1 h'_0 h_0 \]

are mutually incomparable. Since \((g_2 g_3)^{n_0} g_1 h'_0 h_0\) is irreducible, \(h'_m (g_2 g_3)^{n_0} g_1 h'_0 h_0\) remains incomparable with each \(h_i (g_2 g_3)^{n_0} g_1 h'_0 h_0\) for \(1 \leq i \leq n' - 1\). Set \(h := (g_2 g_3)^{n_0} g_1 h'_0 h_0\). Then \(h\) satisfies all the conditions (i), (ii), (iii), and (iv).

Case 2. \(\Gamma = F_\infty\) or \(Z_{n_0} \ast F_\infty\).

In this case one can take a generator \(g\) of \(\Gamma\) with an infinite order such that \(g\) and \(g^{-1}\) is not a factor of any \(h_i\) and \(k_j\) for \(1 \leq i \leq n'\) and \(1 \leq j \leq n''\). Then \(h_0 g^0\) and \(k_j g^0\) are irreducible products for any \(n_0 \geq 1\), \(1 \leq i \leq n'\), and \(1 \leq j \leq n''\). Using the same arguments as in the above case 1 (just replace \(g_2 g_3\) by \(g\) everywhere), one can find a reduced word \(h' \in \Gamma_+\) such that \(h := g^{n_0} h'\) satisfies all the conditions (i), (ii), (iii), and (iv).

4.4. Lemma. Assume that \(\Gamma\) is as in the statement of Theorem 4.1. If \(X = \sum_{j=1}^{m} \alpha_j T_{k(j_1)} T_{k(j_2)} \ldots T_{k(j_m)} \in F_+\), where \(\{k(j) : 1 \leq j \leq m, \ 1 \leq l \leq m_j\}\) is a subset of \(\Gamma\) and \(T_{k(j_1)} T_{k(j_2)} \ldots T_{k(j_m)} \neq 0\) for \(1 \leq l \leq m\), then for any \(\epsilon > 0\) there exists a projection \(Q \in F_+\) satisfying that following conditions:

(i) \(\|XQ\| \geq \|X\| - \epsilon\).
(ii) \(XQX^*\) generates a finite dimensional \(*\)-subalgebra of \(X F_+ X^*\).

Proof. Since the proof is almost exactly the same as the one for [47,5,1], we only sketch the main ideas as follows and leave the details to the reader.

To get such a projection \(Q\), we start with a vector \(\xi = \sum_{j=1}^{n_0} \beta_j f_h\) where \(h_i \in \Gamma_+\) such that

\[ \|\xi\| \leq 1 \quad \text{and} \quad \|X(\xi)\| > \|X\| - \epsilon. \]

Observe that \(T_{k(j_1)} T_{k(j_2)} \ldots T_{k(j_m)} (f_h) \neq 0\) if and only if \(T_{k(j_1)} \ldots T_{k(j_m)} (f_h) \neq 0\) for \(1 \leq l \leq m_j\), again if and only if \(k(j_1)^{-1} \ldots k(j_m)^{-1} h_i \in \Gamma_+\) for \(1 \leq l \leq m_j\). If \(T_{k(j_1)} T_{k(j_2)} \ldots T_{k(j_m)} (f_h) \neq 0\), then \(T_{k(j_1)} T_{k(j_2)} \ldots T_{k(j_m)} (f_h) = f_{k_j h_i}\), where \(k_j\) denotes the reduced word obtained by simplifying the product \(k(j_m) \ldots k(j 2) k(j 1)\). Write
\[ X(\xi) = \sum_{k_j^{-1}h_i} \left( \sum_{l} \alpha_{jl} \beta_{li} \right) f_{k_j^{-1}h_i}. \]

Then
\[ \|X(\xi)\|^2 = \sum_{k_j^{-1}h_i} \left( \sum_{l} \alpha_{jl} \beta_{li} \right)^2, \]

where the sum \( \sum_{k_j^{-1}h_i} \) is indexed by all different resulting reduced words from the products \( k_j^{-1}h_i \) for which \( T_{k(j1)}T_{k(j2)}...T_{k(jm)}(f_{h_i}) \neq 0 \) (i.e., all those products \( k_j^{-1}h_i \) satisfying \( k_j^{-1}h_i = k_j^{-1}h_i \) give only one term which is indexed by \( k_j^{-1}h_i \)), and the sum \( \sum_{l} \alpha_{jl} \beta_{li} \) is indexed by pairs \( (i_l, j_l) \) such that \( k_j^{-1}h_i = k_j^{-1}h_i \).

Apply Lemma 4.3 to the following set
\[ \mathcal{W}_0 := \{ h_i : 1 \leq i \leq m_0 \} \cup \left( \bigcup_{l=1}^{m_1} \{ k(jl)^{-1}...k(jm_l)^{-1}h_i : 1 \leq l \leq m_l \} \right). \]

Some elements of \( \mathcal{W}_0 \) are in \( \Gamma_+ \) and some are not. We get a reduced word \( h \in \Gamma_+ \) satisfying the following conditions:

(a) \( \{ h_i h : 1 \leq i \leq m_0 \} \subset \Gamma_+ \) and any two elements in this set are incomparable,

(b) \( k(jl)^{-1}...k(jm_l)^{-1}h_i h \in \Gamma_+ \) if and only \( k(jl)^{-1}...k(jm_l)^{-1}h_i \in \Gamma_+ \),

(c) all elements in \( \{ h, h_1 h, h_2 h, ..., h_m h \} \) end with the same generator \( g_1 \) of infinite order, and

(d) all elements in \( \{ k_j^{-1}h_i h : 1 \leq i \leq m_0, 1 \leq j \leq m \} \cap \Gamma_+ \) are mutually incomparable.

Set \( \xi' = \sum_{i=1}^{m_0} \beta_i f_{h_i h} \); then \( \|\xi'\| = \|\xi\| \). Observe that \( k_j^{-1}h_i = k_j^{-1}h_i \) if and only if \( k_j^{-1}h_i h = k_j^{-1}h_i h \). Then the above condition (b) warrants
\[ \|X(\xi')\| = \|X(\xi)\|. \]

By the condition (a) above one sees that \( R_{h_1 h}, R_{h_2 h}, ..., R_{h_m h} \) are mutually orthogonal projections in \( \mathcal{F}_+ \), and furthermore, all these projections are equivalent to \( R_{g_1} \) by the condition (c) above, and in turn, equivalent to \( R_+ \). It is obvious that \( R_{h_i h} \) is a subprojection of the initial projection of \( T_{k(j1)}T_{k(j2)}...T_{k(jm_l)}(f_{h_i}) \neq 0 \). Set
\[ Q = R_{h_1 h} \oplus R_{h_2 h} \oplus ... \oplus R_{h_m h}. \]

Then \( Q \in \mathcal{F}_+ \) satisfies the two conditions (i) and (ii) required. In fact,
\[ \|XQ\| \geq \|X(\xi')\| \geq \|X\| - \epsilon, \]

and the condition (d) above implies that \( XQX^* \) is in a finite dimensional \(*\)-subalgebra of \( X\mathcal{F}_+X^* \). For more details the reader is referred to the proof of [47,5.1].
4.5. Proof of Theorem 4.1. (i) To show $\mathcal{I}_+ := R_+ C_\gamma^*(\Gamma, R_+) R_+$ is a purely infinite simple C*-algebra, we prove by definition [16] that the norm closure of $A \mathcal{I}_+ A$ contains a projection equivalent to $R_+$ for each nonzero positive element $A \in \mathcal{I}_+$. We use the same argument as in the proof of [47,5.2]; here is a sketch of the main ideas.

Without loss of generality, we assume that $\|A\| = 1$. Let $\epsilon \in (0, \frac{1}{2})$. By the construction of $\mathcal{I}_+$ there exists an element $X = \sum_{j=1}^m \alpha_i T_{k(j_1)} T_{k(j_2)} \cdots T_{k(j_m)}$ such that

$$\|A - X\| < \frac{\epsilon}{3}.$$ 

By Lemma 4.4 there exists projection $Q \in \mathcal{I}_+$ satisfying:

(i) $\|XQ\| \geq \|X\| - \epsilon/3$; and

(ii) $XQX^*$ generates a finite dimensional C*-subalgebra of the hereditary C*-subalgebra $(X \mathcal{I}_+ X^*)^-$. Now the following estimates are in order:

$$\|XQX^* - AQA^*\| \leq \|(X - A)QX^*\| + \|AQ(X^* - A^*)\| < \epsilon.$$ 

Take the largest eigenvalue $\mu$ of $XQX^*$ with the corresponding spectral projection $P' \in \mathcal{I}_+$. Then

$$\mu = \|XQX^*\| > \left(\|X\| - \frac{\epsilon}{3}\right)^2 \geq \frac{25}{36}, \quad \text{and}$$

$$\|\mu P' - P'AQA^* P'\| = \|P'XQX^* P' - P'AQA^* P'\| < \epsilon.$$ 

It follows that

$$\|P' - \frac{1}{\mu} P'AQA^* P'\| < \frac{\epsilon}{\mu} < \frac{36\epsilon}{25} < 1.$$ 

Then $P'AQA^* P'$ is an invertible element in $P' \mathcal{I}_+ P'$. Set

$$W := (P'AQA^* P')^{-\frac{1}{2}}(AQA^*)^\frac{1}{2}.$$ 

Then $W \in \mathcal{I}_+$, and $WW^* = P'$. It follows that $W^* W$ is a projection in the norm closure of $A \mathcal{I}_+ A^*$ which equivalent to $P'$. By the construction of $Q$ one sees that $R_+$ is equivalent to a subprojection of $P'$.

(ii) We now show that $\mathcal{I}_{R_+}$ is *-isomorphic to $\mathcal{I}_+ \otimes \mathcal{K}$. Clearly, $\mathcal{I}_{R_+}$ is generated by $\mathcal{I}_+$, and thus $\mathcal{I}_{R_+} \otimes \mathcal{K} \cong \mathcal{I}_+ \otimes \mathcal{K}$ by [6,2.8]. Since $\mathcal{I}_+$ is a purely infinite, simple C*-algebra, the stabilization $\mathcal{I}_+ \otimes \mathcal{K}$ is also a purely infinite, simple C*-algebra. Thus, $\mathcal{I}_{R_+}$ is a purely infinite simple C*-algebra. Observe that $\mathcal{I}_{R_+}$ is a non-unital separable C*-algebra. Then $\mathcal{I}_{R_+}$ must be stable by our result in [44], asserting that a σ-unital, purely infinite, simple C*-algebra is either unital or stable. Therefore, $\mathcal{I}_{R_+} \cong \mathcal{I}_+ \otimes \mathcal{K}$.

Now $C_\gamma^*(\Gamma, R_+) / \mathcal{I}_{R_+} \cong C_\gamma^* \Gamma$ which is a simple C*-algebra by a result in
[33]. We conclude that $\mathcal{A}_{\mathcal{I}}^r$ is the only nontrivial closed ideal of $C^r(\Gamma, R_+)$.
We conclude this note with the following problem:

4.6. **Problem.** Calculate $K_0$, $K_1$, and Ext of $\mathcal{F}_+$ in case

$$\Gamma = \mathcal{F}_\infty, \, \mathbb{Z}_{n_0} \ast \mathcal{F}_\infty, \, (\mathbb{Z}_{n_1} \ast \mathbb{Z}_{n_2} \ast \ldots) \ast \mathcal{F}_n, \, (\mathbb{Z}_{n_1} \ast \mathbb{Z}_{n_2} \ast \ldots) \ast \mathcal{F}_\infty,$$

where $2 \leq m \leq +\infty$ and $1 \leq n < +\infty$.

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