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DEFINING RELATIONS FOR CLASSICAL LIE ALGEBRAS OF POLYNOMIAL VECTOR FIELDS

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Abstract.

We explicitly describe the defining relations for simple Lie algebra $\mathfrak{vect}(n) = \mathfrak{der}\, \mathbf{C}[x]$, where $x = (x_1, \dots, x_n)$ of vector fields with polynomial coefficients and its subalgebras of divergence free, hamiltonian and contact vector fields, and for the Poisson algebra (realized on polynomials). We consider generators of $\mathfrak{vect}(n)$ and its subalgebras corresponding to the system of simple roots associated with the standard grading of these algebras. (These systems of simple roots are *distinguished* in the sense of Penkov-Serganova [PS].)

Introduction.

The class of simple $\mathbf{Z}$-graded Lie algebras of polynomial growth (SZGLAPGs, for short) over $\mathbf{C}$ often appears in various problems of mathematics and physics. V. Kac conjectured [K] (recently O. Mathieu proved this [M]) that this class consists of the following algebras:

1) finite dimensional Lie algebras (each determined by a Cartan matrix or Dynkin diagram);

2) (twisted) loop algebras (each determined by the Cartan matrix corresponding to an extended Dynkin diagram);

3) four series of Lie algebras of vector fields with polynomial coefficients or, briefly, "vectoral" algebras. (The name "vectoral" is influenced by physical terminology. Some physicists working with string theories call the Lie superalgebras pertaining to these theories "stringy". This term is very accurate and doubly suggestive: it reminds the relation of these algebras to string theories and reveals that the stringy algebras do look like a bunch of strings – modules over the Witt algebras. Stringy superalgebras are a particular case of what we propose to call "vectoral algebras", with Laurent polynomials as coefficients. We concede that "vectoral" might sound a bit reckless; still, in our terminology there is no chance to look at something like a "Cartan subalgebra of a Lie subalgebra of Cartan type".)

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$\mathfrak{vect}(n) = \mathfrak{der}\, \mathrm{C}[x]$, where $x = (x_1, \ldots, x_n)$, the *general vectoral algebra*;

$\mathfrak{svect}(n) = \{D \in \mathfrak{vect}(n) : \operatorname{div} D = 0\}$, the *special or divergence-free algebras*;

$\mathfrak{h}(2n) = \{D \in \mathfrak{vect}(2n) : D\omega = 0\}$ for $\omega = \sum_i dp_i \wedge dq_i$, here $x = (q, p)$, the *Hamiltonian algebra*;

$\mathfrak{k}(2n+1) = \{D \in \mathfrak{vect}(2n+1) : D\alpha = f\alpha\}$ for $\alpha = dt + \sum_i (p_i dq_i + q_i dp_i)$, here $x = (t, q, p)$, the *contact* Lie algebra;

4) this class consists of one "stringy" (i.e., pertaining to the string theory) Lie algebra; the *Witt algebra* $\mathfrak{witt} = \mathfrak{der}\, \mathrm{C}[x^{-1}, x]$.

A central extension. The Lie algebra $\mathfrak{po}(2n)$ whose space is $\mathrm{C}[q, p]$ and the bracket is the Poisson bracket is called the *Poisson algebra.* It is the nontrivial central extension of $\mathfrak{h}(2n)$, with the 1-dimensional center generated by constants. Its geometric interpreation: $\mathfrak{po}(2n)$ preserves the connection with the above form α in a line bundle over C^{2n} with a symplectic structure. Though $\mathfrak{po}(2n)$ is not simple, it is useful in applications (similarly, Kac-Moody algebras are more useful than the loop algebras). So we will consider it as well.

The end product of the deformation (the physicists call it *"quantization"*) of $\mathfrak{po}(2n)$ is "$\mathfrak{gl}$" $(\mathrm{C}[q])$. (We used quatation marks because for infinite dimensional spaces there are many distinct "$\mathfrak{gl}$"s, cf. [E].) Our results might help to understand how to quantize and *"q*-quantize" the Poisson algebra of functions on $2n$-dimensional torus. In particular, our results are at variance with expectations of [S].

Related results. The description of the above algebras in terms of defining relations is vital in questions where it is necessary to identify an algebra determined by its defining relations (Eastbrook-Wahlquist prolongations, Drinfield's quantum algebras, etc.). Presentations of the algebras from the first two classes are known. We will give a presentation for the remaining of the above algebras.

It was in [FF1] that Feigin and Fuchs first published a presentation of a nilpotent part ($\mathfrak{g}_+$ in our notations, see sect. 0.3) of $\mathfrak{g} = \mathfrak{vect}(n)$, later generalized (with a gap in the case of $\mathfrak{g} = \mathfrak{h}(2n)$) for all simple vectoral Lie algebrs in [FF2]. However, both the generators and relations considered in [FF1] and [FF2] are too numerous. Besides, they were implicit.

This was no wonder: in a similar situation for loop and affine Kac-Moody algebras $\mathfrak{g}^{(1)} = \mathfrak{g} \otimes \mathrm{C}[t^{-1}, t]$ with the grading $\deg g = 0$ for $g \in \mathfrak{g}$, $\deg t = 1$ it is more advisable to consider the generators of $\mathfrak{g}$ and the lowest and highest generators of $\mathfrak{g} \otimes t$ and $\mathfrak{g} \otimes t^{-1}$, respectively. So we suggest to enlarge the algebra considered by Feigin and Fuchs. In the same manner as is done for loop or Kac-Moody algebras, we add to $\mathfrak{g}_+$ the positive root vectors of $\mathfrak{g}_0$.

In this way we only have to slightly work on the results of [FF1] and [FF2] to get an explicit answer. Our results are based on the general theorems from [FF1], [FF2] related to the Hochschild-Serre's spectral sequence applied to locally nilpotent Lie algebras. (Similar statements for Kac-Moody algebras are well-known, cf. [K].) The exceptional cases of small numer of indeterminates, not covered by the general theorems of [FF1], [FF2], were studied in [HP1], [HP2] with the help of a computer.

This paper, preprinted in [L, n31], is the first in a series of papers devoted to presentation of simple Z-graded Lie *super*algebras of polynomial growth over C: the main ideas and some examples were first delivered by D. Leites at L. Faddeev's seminar in 1981, and later at the lectures at CWI, Amsterdam in 1986.

The publication of the text was delayed for 10–15 years for no reason; meanwhile there appeared several results with description of presentations of simple Lie algebras and Lie superalgebras, or their bases, connected with this paper: [GL1], [GL2], [HP1], [HP2], [LSe].

A. Dzhumadil'daev iinformed us that with a student he recently (1992) obtained a description identical to ours for simple vectoral Lie algebras over an algebraically closed field of large characteristics ($p > 7$).

§0. Preliminaries.

0.1. *Defining relations*. Presentations of Lie algebras of classes 1) and 2) of the above list of SZGLAPGs are known; the most popular is the one that can be neatly encoded in terms of a graph (Dynkin diagram). Still, we have to explain what does this common knowledge amount to since there is no "natural" set of generators for a simple Lie algebra.

The common presentation of the simple Lie algebras is performed in terms of Chevalley generators. A rival, less popular but also very useful in various problems, set of generators for a (say, finite dimensional) simple Lie algebra is to pick just two generators (as suggested first by N. Jacobson). These generators are related to the principal embedding of $\mathfrak{sl}(2)$; for proof see [BO], and for applications [F2], [FNZ], [LS]. What should be meant under *relations* for such a choice of generators and the degree of the ambiguity of these relations is, however, very far from clear, cf. [F2], [FNZ]; for the answer see [GL1].

Contrarywise, for a nilpotent Lie algebra $\mathfrak{n}$ the problem has a natural and unambiguous solution: a basis of the space $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}] = H_1(\mathfrak{n})$ is a set of *generators* of $\mathfrak{n}$. Suppose, as will be the case in our examples, that there is a set of outer derivations acting on $\mathfrak{n}$ so that $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$ splits into the direct sum of 1-dimensional eigenspaces. Then the choice of a basis is unique up to scalar factors.

To describe relations between the generators, consider the standard homology complex for $\mathfrak{n}$ with trivial coefficients ([Fu]):

$$0 \leftarrow \mathfrak{n} \overset{d_1}{\longleftarrow} \mathfrak{n} \wedge \mathfrak{n} \overset{d_2}{\longleftarrow} \mathfrak{n} \wedge \mathfrak{n} \wedge \mathfrak{n} \overset{d_3}{\longleftarrow} \cdots$$

By definition

$$d_1(x \wedge y) = [x, y], d_2(x \wedge y \wedge z) = [x, y] \wedge z + [y, z] \wedge x + [z, x] \wedge y.$$

The condition $d_1 d_2 = 0$ expresses the Jacobi identity. The elements of Ker d_1 are obviously the relations. The relations which are consequences of the Jacobi identity should be considered as trivial: they constitute Im d_2. Thus a basis of $H_1(\mathfrak{n})$ allows to construct the generators of $H_2(\mathfrak{n})$, and applying d_1 to them we obtain *defining relations* for a nilpotent Lie algebra $\mathfrak{n}$. Observe that the same arguments apply as well to *locally nilpotent* Lie algebras (as the algebras $\mathfrak{N}_\pm$ and $\mathfrak{g}_\pm$ considered in [FF1], [FF2] and below).

0.2. *Serre's relations.* It was J.-P. Serre who, perhaps, for the first time, wrote down relations between Chevalley generators of a simple finite dimensional Lie algebra $\mathfrak{g}$. They can be also written for the algebras of class 2) and more general Kac-Moody algebras, cf. [GKLP] and [K]; [GL2].

The relations are expressed in terms of Chevalley generators, so let us first define Chevalley generators. Let

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

be the root decomposition of $\mathfrak{g}$, where $\mathfrak{h}$ is a *maximal torus* or, more exactly, toral, i.e., diagonalizable subalgebra and $\mathfrak{n}_\pm$ are maximal nilpotent subalgebras generated by the root vectors corresponding to all positive (for +), respectively negative (for˘-), roots. (The notion of a Cartan subalgebra – a nilpotent subalgebra coinsiding with its normalizer˘- plays a modest role in the theory of Lie algebras over fields of prime characteristic, or for superalgebras, or in the case we are considering, that of infinite dimension: notice that for the simple finite dimensional Lie algebras over $\mathbf{C}$ Cartan subalgebra is both (maximal) commutative and diagonalizable. Sometimes, e.g., for $\mathfrak{po}(2n)$, $\mathfrak{h}(2n)$, it is neither.) Setting

$$\deg(X_{\pm\alpha}) = \pm 1$$

for a root vector corresponding to a simple root α (or its opposite), we endow $\mathfrak{n}_\pm$ with a $\mathbf{Z}$-grading, and it turns out that

$$[\mathfrak{n}_\pm, \mathfrak{n}_\pm] = \{X \in \mathfrak{n}_\pm;\ \deg X \geq 2 \quad (\text{for } +), \quad \deg X \leq -2 \ (\text{for } -)\}.$$

Hence, the elements $X_{\pm\alpha}$ themselves represent their homology classes in $H_1(\mathfrak{n}_\pm)$: $H_1(\mathfrak{n}_\pm) = \oplus_{\alpha_i \in S} \mathbf{C} X_{\pm\alpha_i}$, where S is a system of simple roots. Informally speaking, these generators are *pure*: they are not definied modulo anything.

Let us normalize these generators as follows. Set $X_i^{\pm} = X_{\pm\alpha_i}$ and introduce auxiliary generators $H_i = [X_i^+, X_i^-]$. Using rescaling $H_i \mapsto \lambda H_i$ it is possible to select the $X_i^{\pm}$ so that

$$[H_i, X_j^{\pm}] = \pm a_{ij} X_j^{\pm},$$

where the matrix (a_{ij}) is normed so that $a_{ii} = 2$ for all i. The matrix (a_{ij}) is called the *Cartan matrix* of $\mathfrak{g}$, the generators $X_i^{\pm}$ of $\mathfrak{n}_{\pm}$ the *Chevalley generators.*

The *relations* between Chevalley generators $X_i^{\pm}$ of $\mathfrak{n}_{\pm}$ are called *Serre's relations.* They are expressed in terms of the Cartan matrix as follows:

$$(SR_{\pm}) \qquad (\mathrm{ad}(X_i^{\pm}))^{1-a_{ij}}(X_j^{\pm}) = 0 \quad (i \neq j).$$

Serre's relations represent homology classes from $H_2(\mathfrak{n}_{\pm})$ which are also pure:

$$H_2(\mathfrak{n}_{\pm}) = \oplus_{\alpha_i \neq \alpha_j \in S} \mathbf{C}(X_{\pm\alpha_i} \wedge X_{\pm r_{\alpha_i}(\alpha_j)}),$$

where r_{α_i} is the reflection associated to the root $\alpha_i : r_{\alpha_i}(\alpha_j) = \alpha_j - a_{ij}\alpha_i$. The elements $X_{\pm\alpha_i} \wedge X_{\pm r_{\alpha_i}(\alpha_j)}$ correspond (via d_1) to the relations $(SR_{\pm})$.

Remark. The Serre's relations show that the lowest weight of the $\mathfrak{sl}(2)$-module generated by X_j^+ (or the highest weight of the module generated by X_j^-) is a_{ij} (resp. $-a_{ji}$) with the copy of $\mathfrak{sl}(2)$ we are talking about being generated by $X_i^{\pm}$.

There are also relations between $\mathfrak{n}_+$ and $\mathfrak{n}_-$ (and $\mathfrak{h}$):

$$(SR_0) \qquad [H_i, H_j] = 0, \quad [X_i^+, X_j^-] = \delta_{ij} H_i, \quad [H_i, X_j^{\pm}] = \pm a_{ij} X_j^{\pm}.$$

One of the reasons why the above relations (SR) won priority over any other presentation is that they can be encoded in a single nice-looking graph, the *Dynkin diagram* of $\mathfrak{n}_{\pm}$ (or, which is the same, of $\mathfrak{g}$), see [B], [OV]. There are, however, other possibilities, and recently it became clear that they are not so awful, cf. [GL1].

0.3. *Vectoral algebras.* About 1980 D. Leites conjectured that the Lie algebras of vector fields with polynomial coefficients are finitely presented. I. Kantor informed us then that much earlier he also arrived to the same conjecture; he believed that these algebras are determined by a pair of "Cartan matrices" or "Dynkin diagrams" and even produced diagrams hypothetically corresponding to $\mathfrak{vect}(n)$. As we will see, Kantor's conjecture was almost true. (It can be considered true if we change the rules of recovering relations from a graph; the new rules are unknown, cf. with the case of Lie superalgebras, cf. [GL2].)

A weaker conjecture, on finite determination, had been independently proved by Bondal and Ufnarovsky in [BU] as a by-product in their answer to another problem. They did not appreciate, however, the importance of this by-product, and no explicit formulas were written neither then nor later, cf. [U].

By the time [BU] was out of print it became manifest that the simplest examples shatter any hope for neat relations. Indeed, consider that $\mathfrak{n}_\pm$-part of the very first examples, $\mathfrak{witt}$ and $\mathfrak{vect}(1)$. Recall that a natural basis in $\mathfrak{witt}$ is

$$e_i = x^{i+1}\frac{d}{dx} \quad \text{for } i \in \mathbb{Z}$$

with relations

$$[e_i, e_j] = (j-i)e_{i+j}.$$

Clearly, $\mathrm{Span}(e_0)$ is the maximal torus, $\mathfrak{n}_\pm$ is generated by $e_{\pm 1}$ and $e_{\pm 2}$. (Note that $\mathfrak{n}_-$ for $\mathfrak{vect}(1)$ is just $\mathrm{Span}(e_{-1})$.)

The relations in $\mathfrak{n}_-$ and $\mathfrak{n}_+$ and between them, not as popular as Serre's relations, are, nevertheless, known, thanks to B. Feigin [F1], since 1979, cf. [FNZ]:

$$(DR_0) \qquad [e_{-1}, e_1] = 2e_0,\ [e_{-2}, e_2] = 4e_0,\ [e_0, e_i] = ie_i\ (i = \pm 1, \pm 2),$$
$$[e_{-1}, e_2] = 3e_1,\ [e_1, e_{-2}] = -3e_{-1};$$

$$(DR_\pm) \qquad (ade_{\pm 1})^3 e_{\pm 2} + 6(ade_{\pm 2})^2 e_{\pm 1} = 0,\ (ade_{\pm 1})^5 e_{\pm 2} + 40(ade_{\pm 2})^3 e_{\pm 1} = 0.$$

The relations that do not involve e_{-2} are the defining relations for $\mathfrak{vect}(1)$.

The relations of degree ± 7 are "dirty". This means that (say, for the plus sign) in the 2-dimensional space of cycles

$$\mathrm{Span}(3e_1 \wedge e_6 - 5e_2 \wedge e_5,\ e_2 \wedge e_5 - 3e_3 \wedge e_4)$$

the relations span any line transversal to the line of boundaries:

$$d_2(e_1 \wedge e_2 \wedge e_4) = e_3 \wedge e_4 - 3e_5 \wedge e_2 + 2e_6 \wedge e_1.$$

By that time (1980) D. Fuchs himself got interested in the problem and made a major step forward. He studied the relations for $\mathfrak{g}_+ = \oplus_{i>0}\, \mathfrak{g}_i$, where $\mathfrak{g}$ is a Lie algebra of vector fields (class 3) above) in the standard $\mathbb{Z}$-grading ($\deg x_i = 1$ for all i, except for $\deg t = 2$). For $\mathfrak{svect}(2) = \mathfrak{h}(2)$ and for $\mathfrak{k}(3)$ there are also some "dirty" relations. Still, further investigation showed that for vectoral Lie algebras depending on sufficiently large number of indeterminates there are no "dirty" relations and this "large" number is actually pretty small: it is equal to 3 except for the series $\mathfrak{k}$ when it is equal to 5.

Fuchs conjectured (Fuchs proved this conjecture for $\mathfrak{g} = \mathfrak{vect}(n)$, the statement and its proof ([FF1]) is also true for the other series except for the Hamiltonian algebra. Fuchs' statement might have been corrected already in 1984 when Kochetkov discoverd that the similar relations for the Lie superalgebra $\mathfrak{h}(0\,|\,n)$ always contain a component of degree 3 if we would have put more faith in Kochetkov's result, cf. [GKLP], [HP1], [HP2]. We describe the relations here.), see [FF2] and references therein, that for any simple vectoral $\mathfrak{g}$ in the standard grading all the relations between the generators of $\mathfrak{g}_+$ are of degree 2:

$$H_2(\mathfrak{g}_+) = [H_2(\mathfrak{g}_+)]_2 = \wedge^2(\mathfrak{g}_1)/\mathfrak{g}_2.$$

These relations, however, are implicit and inconvenient: if $\dim \mathfrak{g}_{-1} = n$, then there are $\dim \mathfrak{g}_1 \approx n^3$ of generators of $\mathfrak{g}_+$ and there are $\approx n^6$ relations between these generators.

In what follows we will proceed precisely as for the (twisted) loop or Kac-Moody algebras, namely, replace $\mathfrak{g}_+$ by the maximal locally nilpotent subalgebra $\mathfrak{N}_+$ of $\mathfrak{g}$: this sleeply diminishes the number of generators and relations and makes them graphic.

Thus, our proof is: to look at the 2nd term of Hochschild-Serre's spectral sequence constructed for the pair $\mathfrak{g}_\pm \subset \mathfrak{N}_\pm$ (see p2) and unite the bases of three spaces obtained $(H_i(\mathfrak{n}_\pm; H_j(\mathfrak{g}_\pm))$ for $i+j=2$, where $\mathfrak{g}_+ = \oplus_{i>0}\mathfrak{g}_i$ and $\mathfrak{g}_- = \oplus_{i<0}\mathfrak{g}_i)$; two of them are calculated with the help of the Borel–Weil-Bott–L theorem, to the third we apply (amended) Fuchs' result.

We started this work with expectations to get simple, Serre-type looking relations. To an extent our relations *are* simple if we

a) do NOT express them in terms of generators;

b) simplify them by taking not necessarily the one which corresponds to the lowest weight of the corresponding $\mathfrak{g}_0$-module but a linear combination of it and other relations of the same weight obtained from other modules of relations in order to get a simpler, "more factorizable", cycle.

0.4. *The main result.* We give an explicit expression for the generic case of defining relations in the maximal locally nilpotent subalgebras $\mathfrak{N}_\pm = \mathfrak{n}_\pm \oplus \mathfrak{g}_\pm$ generated by positive and negative root vectors of $\mathfrak{g}$ in the standard $\mathbb{Z}$-grading of $\mathfrak{g}$, where $\mathfrak{g}_\pm$ are described above and $\mathfrak{n}_\pm$ are the maximal nilpotent subalgebras of $\mathfrak{g}_0$ described in textbooks (e.g., [B], [OV]).

The exceptional cases of small dimension are investigated by N. Hijligenberg and G. Post, for explicit formulas see [HP1], [HP2].

0.5. *A problem: what is an analog of the Weyl group for a vectoral algebra.*

Unlike the case of simple finite-dimensional Lie algebras with Cartan matrix over C, there are several non-isomorphic types of maximal nilpotent subalgebras $\mathfrak{g}_+$ of a vectoral Lie algebra. (The choice we made is distinguished by the fact that the subalgebra of elements of non-negative degree is the unique maximal subalgebra of finite codimension; it also happens that it is easier to deal with.)

We encounter similar phenomenon with (say, simple finite dimensional) Lie superalgebras, even if they have Cartan matrices, or in prime characteristic. The experience with Lie superalgebras ([LSe], [FLV]) suggests to consider all maximal nilpotent subalgebras (or, equivalently, bases, i.e., systems of simple roots) simultaneously and consider the group that transitively acts on the bases – an analog of the Weyl group. A solution to this problem will be discussed elsewhere.

§1. Generators in vectoral Lie algebras.

Set $\partial_i = \frac{\partial}{\partial x_i}$.

1.1. *Generators of* $\mathfrak{vect}(n)$. Some of the generators of $\mathfrak{vect}(n)$ generate its following subalgebras:

	$\mathfrak{sl}(n+1)$	
$\mathfrak{N}_+$	$x_1\partial_2,\ \dots,\ x_{n-1}\partial_n,\ \boldsymbol{x_n \sum x_i\partial_i}$	$x_n^2\partial_1$
$\mathfrak{N}_-$	$x_2\partial_1,\ \dots,\ x_n\partial_{n-1},\ \partial_n$	
notations	$X_1^\pm,\ \dots,\ X_{n-1}^\pm,\ X_n^\pm$	Y

The generators of $\mathfrak{svect}(n)$ are the same as of $\mathfrak{vect}(n)$ but without the boldfaced element X_n^+.

1.2. *Generators of* $\mathfrak{k}(2n+1)$. First of all it is convenient to express the elements of $\mathfrak{k}(2n+1)$ and its subalgebras $\mathfrak{po}(2n)$ and $\mathfrak{h}(2n)$ in terms of *generating functions* (for the series $\mathfrak{h}(2n)$ these generating functions are called *Hamiltonians*) as follows.

To every $f \in \mathbb{C}[q,p]$ assign the Hamiltonian field H_f corresponding to the Hamiltonian f:

$$H_f = \sum\left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}\right).$$

To every $f \in \mathbb{C}[t,q,p]$ assign the contact field

$$K_f = \Delta(f)\frac{\partial}{\partial t} + \frac{\partial f}{\partial t}E + H_f,$$

where $\Delta(f) = 2f - E(f)$ for $E = \sum_i y_i \frac{\partial}{\partial y_i}$; here the y are all the coordinates except t.

In particular, the functions that not depend on t generate the Poisson algebra realized by vector fields K_f.

To the commutator of hamiltonian vector fields $[H_f, H_g]$ there corresponds the *Poisson bracket* of generating functions, and to the commutator of contact vector fields $[K_f, K_g]$ there corresponds the *Lagrange* or, as it is more often called, *contact* bracket:

$$\{f, g\}_{c.b.} = \Delta(f)\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}\Delta(g) - \{f, g\}_{P.b.},$$

and the Poisson bracket is given by the formula

$$\{f, g\}_{P.b.} = \sum\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right).$$

It is well-known that

$$\mathfrak{h}(2n) = \mathrm{Span}(H_f : f \in \mathbb{C}[q,p]), \quad \mathfrak{k}(2n+1) = \mathrm{Span}(K_f : f \in \mathbb{C}[t,q,p]).$$

In what follows we will by abuse of language write just f instead of H_f or K_f; in so doing we must remember that the degree of the vector field K_f or H_f generated by a monomial f of degree k is equal to $k-2$.

The important for us subalgebras of $\mathfrak{k}(2n+1)$ and their generators in our notations are:

	$\mathfrak{sp}(2n+2)$	
$\mathfrak{N}_-$	$p_1 \quad q_1p_2 \quad \dots \quad q_{n-1}p_n \quad q_n^2$	
$\mathfrak{N}_+$	$\mathbf{tq_1} \quad p_1q_2 \quad \dots \quad q_np_{n-1} \quad p_n^2$	q_1^3
notations	$X_0^\pm, \quad X_1^\pm, \quad \dots \quad X_{n-1}^\pm, \quad X_n^\pm$	Y

The generators of $\mathfrak{h}(2n)$ and $\mathfrak{po}(2n)$ are those above without the boldfaced element $X_0^+ = tq_1$.

§2. How to calculate relations.

Clearly, $\mathfrak{N}_-$ for $\mathfrak{vect}(n)$ and $\mathfrak{svect}(n)$ coinsides with $\mathfrak{n}_-$ for $\mathfrak{sl}(n+1)$; $\mathfrak{N}_-$ for $\mathfrak{k}(2n+1)$ and $\mathfrak{po}(2n)$ coincides with $\mathfrak{n}_-$ for $\mathfrak{sp}(2n+2)$. Therefore, the relations for these Lie algebras are known.

The remaining defining relations are found with the help of the Hochschild-Serre's spectral sequence [Fu] for the pair $\mathfrak{g}_\pm \subset \mathfrak{N}_\pm$ and the results of [FF2] on $H_2(\mathfrak{g}_+)$. We estimate the relations from above: the space of relations is contained in the direct sum of the following spaces:

(1) $H_2(\mathfrak{n}_\pm)$;

(2) $H_1(\mathfrak{n}_\pm; H_1(\mathfrak{g}_\pm))$

(3$^+$) $H_0(\mathfrak{n}_+; H_2(\mathfrak{g}_+)) = \text{Span}(\text{the } \mathfrak{n}_- - \text{lowest vectors of } H_2(\mathfrak{g}_+))$;

(3$^-$) $H_0(\mathfrak{n}_-; H_2(\mathfrak{g}_-)) = \text{Span}(\text{the } \mathfrak{n}_+ - \text{highest vectors of } H_2(\mathfrak{g}_-))$.

Homology (1) is known, cf. [OV]; homology (2) is calculable with the help of the Borel–Weil–Bott theorem, cf. [Go]. When homologies (3) are known, it remains to compute the diferential in the spectral sequence. Usually it is zero.

Note that the first two types of these relations are of the form

$$(\text{ad } X_i^\pm)^{k_{ij}}(Y_j^\pm) = 0,$$

where the X_i^+ are the generators of $\mathfrak{n}_+$ and the Y_j^+ are the lowest weight vectors of the $\mathfrak{g}_0$-module $\mathfrak{g}_1$; the X_i^- are the generators of $\mathfrak{n}_-$ and the Y_j^- are the highest weight vectors of the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$.

REMARK. Similar calculations by induction on the rank for simple finite dimensional and (twisted) loop agebras give the shortest known to us proof of completeness of the Serre's defining relations.

§3. Relations in $\mathfrak{k}(2n+1)$, $\mathfrak{po}(2n)$ and $\mathfrak{h}(2n)$.

3.1. *Relations in* $\mathfrak{N}_-$ *for* $\mathfrak{k}(2n+1)$ *and* $\mathfrak{po}(2n)$. These relations are the same as for $\mathfrak{n}_-$ of $\mathfrak{sp}(2n+2)$.

3.2. *Relations in* $\mathfrak{N}_-$ *for* $\mathfrak{h}(2n)$. The Lie algebra $\mathfrak{h}(2n)$ is generated by the same elements as $\mathfrak{k}(2n+1)$ and $\mathfrak{po}(2n)$ but the relations are different: for $\mathfrak{h}(2n)$ there is an additional relation of weight (0,-L, 0) with respect to $\mathfrak{sp}(2n)$ because

$$H_2(\mathfrak{g}_{-1}) = \begin{cases} \Lambda^2(\mathfrak{g}_{-1}) = R(\pi_2) \oplus R(0) & \text{if } n > 1 \\ R(0) & \text{if } n = 1, \end{cases}$$

where $R(\pi_2)$ and $R(0)$ denote the irreducible $\mathfrak{sp}(2n)$-modules with highest weights π_2 and 0, respectively; π_i is the i-th fundamental weight (see [OV]).

For $n = 1$ the corresponding relation is of the form

$$(\text{ad } X_0)^2 X_1 = 0.$$

For $n > 1$ the cycle of weight 0 is

$$(*) \qquad p_1 \wedge q_1 + \cdots + p_n \wedge q_n$$

and the relation expressed in terms of generators looks very ugly. It can be beautified as follows. In the space of relations corresponding to the other irreducible component, $R(\pi_2)$, the subspace of relations of weight 0 is of dimension $n-1$. Therefore, each summand in $(*)$ vanishes; select the simplest one of them, say, the following one:

$$\{p_1, q_1\} = 0.$$

In terms of generators this relation is:

$$\{p_1, \{\dots\{p_1, q_1p_2\}, q_2p_3\}, \dots, q_{n-1}p_n\}, q_n^2\}, q_{n-1}p_n\}, \dots, \}, q_1p_2\} = 0.$$

3.3. *Relations in* $\mathfrak{N}_+$ *for* $\mathfrak{h}(2n)$, $n > 1$. The space $H_1(\mathfrak{n}_+; \mathfrak{g}_1)$ is responsible for the following relations:

$$(adp_1q_2)^4(q_1^3) = 0,\ (adp_2q_3)(q_1^3) = 0, \dots,\ (adp_{n-1}q_n)(q_1^3) = 0,\ adp_n^2(q_1^3) = 0.$$

The space $H_2(\mathfrak{g}_+)$ is the direct sum of the irreducible $\mathfrak{g}_0$-modules with the following lowest weights:

N	the lowest weight	the corresponding cycle
1	$-5\epsilon_1 - \epsilon_2$	$q_1^3 \wedge q_1^2q_2$
2	$-3\epsilon_1 - 3\epsilon_2$	$3q_1q_2^2 \wedge q_1^2q_2 - q_2^3 \wedge q_1^3$
3	$-2\epsilon_1 - 2\epsilon_2$	$\sum_i [-q_1^2q_i \wedge q_2^2p_i + q_1^2p_i \wedge q_2^2q_i + 2q_1q_2q_i \wedge q_1q_2p_i]$
4	$-\epsilon_1 - \epsilon_2$	$\sum_{i,j} [2q_1q_ip_i \wedge q_2q_ip_i - q_1q_iq_j \wedge q_2p_ip_j + q_2q_iq_j \wedge q_1p_ip_j]$
5	0	$\sum_{i,j,k} [3q_iq_jp_k \wedge q_kp_ip_j - q_iq_jq_k \wedge p_ip_jp_k]$
6	$-\epsilon_1$	$(q_1 \wedge 1) \sum_i q_i^3 \wedge p_i^3$

The last relation is the slippery relation of degree 3.

3.4. *Relations in* $\mathfrak{N}_+$ *for* $\mathfrak{k}(2n+1)$, $n > 1$. For the X_i^+ the relations are the same as for $\mathfrak{n}_+$ of $\mathfrak{sp}(2n+2)$.

The relations between X_i^+, $1 \le i \le n$ and Y are the same as for $\mathfrak{h}(2n)$.

New relations involving X_0^+ and Y are:

N	the lowest weight	the corresponding cycle	a simplified relation
1	$-4\epsilon_1$	$\sum_i -q_1^2q_i \wedge q_1^2p_i + (n+2)tq_1 \wedge q_1^3$	
2	$-3\epsilon_1 - \epsilon_2$	$q_1^3 \wedge tq_2 + q_1^2q_2 \wedge tq_1$	$[X_1^+, [Y, X_0^+]] = 0$

3.5. *Relations between* $\mathfrak{N}_+$ *and* $\mathfrak{N}_-$ *for* $\mathfrak{h}(2n)$ *and* $\mathfrak{k}(2n+1)$, $n > 1$. These relations are as for $\mathfrak{sp}(2n+2)$ unless they involve Y; and the new, extra, ones are (the right hand side is just $-3q_1^2$):

$$[Y, X_0^+] = -\frac{3}{2^{n-1}}(\text{ad } X_1^+)^2 \dots (\text{ad } X_{n-1}^+)^2 X_n^+; \quad [Y, X_i^+] = 0 \text{ for } i > 0.$$

§4. Relations for $\mathfrak{vect}(n)$ and $\mathfrak{svect}(m), n, m > 2$.

4.1. *Relations in $\mathfrak{N}_-$ for $\mathfrak{vect}(n)$ and $\mathfrak{svect}(m)$, where $n, m > 2$.* These relations are the same as for $\mathfrak{n}_-$ of $\mathfrak{sl}(n+1)$.

4.2. *Relations for $\mathfrak{N}_+$ of $\mathfrak{vect}(n), n \geq 4$.* The relations that constitute $H_1(\mathfrak{n}_+; \mathfrak{g}_1)$ are of two kinds:

1) those between $X_1^+, \dots, X_n^+$ which are the same as for $\mathfrak{sl}(n+1)$, namely,

$$(\text{ad } X_i^+)(X_j^+) = 0 \quad \text{for } |i-j| > 1, \ (\text{ad } X_{i\pm 1}^+)^2 X_i^+ = 0.$$

2) those that involve Y but not X_n^+:

$$(\text{ad } X_i^+)^2 Y = 0 \quad \text{for } i = 1,\ n-2,\ n-1;$$

$$(\text{ad } X_i^+) Y = 0 \quad \text{for } i \neq 1,\ n-2,\ n-1.$$

The space $H_2(\mathfrak{g}_+)$ is the direct sum of the irreducible $\mathfrak{g}_0$-modules with the following lowest weights:

N	the lowest weight	the corresponding cycle
1	$-\epsilon_1 - \epsilon_2 + 4\epsilon_n$	$x_n^2\partial_1 \wedge x_n^2\partial_2$
2	$-2\epsilon_1 + \epsilon_{n-1} + 3\epsilon_n$	$x_n^2\partial_1 \wedge x_n x_{n-1}\partial_1$
3	$-\epsilon_1 - 3\epsilon_n$	$n\sum_i (x_n^2\partial_1 \wedge x_n x_i\partial_i) + 2\sum_i (x_n^2\partial_i \wedge x_n x_i\partial_1)$
4	$-\epsilon_1 - \epsilon_{n-1} - 2\epsilon_n$	$\sum_i (x_n x_{n-1}\partial_1 \wedge x_n x_i\partial_i) - \sum_i (x_n^2\partial_1 \wedge x_{n-1}x_i\partial_i)$
5	$-\epsilon_1 - \epsilon_{n-1} - 2\epsilon_n$	$\sum_i (x_n x_{n-1}\partial_i \wedge x_n x_i\partial_1) - \sum_i (x_n^2\partial_i \wedge x_{n-1}x_i\partial_1)$
6	$-\epsilon_{n-1} - \epsilon_n$	$\sum_{i,j} x_n x_i\partial_i \wedge x_{n-1}x_j\partial_j$
7	$-\epsilon_{n-1} - \epsilon_n$	$\sum_{i,j} x_n x_i\partial_j \wedge x_{n-1}x_j\partial_i$
8	$-\epsilon_1 - \epsilon_2 + 2\epsilon_{n-1} + 2\epsilon_n$	$2x_n x_{n-1}\partial_1 \wedge x_n x_{n-1}\partial_2 - x_n^2\partial_1 \wedge x_{n-1}^2\partial_2 - x_{n-1}^2\partial_1 \wedge x_n^2\partial_2$

4.3. *Relations for $\mathfrak{N}_+$ of $\mathfrak{vect}(3)$.* The relations that constitute $H_1(\mathfrak{n}_+; \mathfrak{g}_1)$ are the same as for $n > 3$ except that some of the relations distinct for large n merge into one for $n = 3$.

The space $H_2(\mathfrak{g}_+)$ is the direct sum of the irreducible $\mathfrak{g}_0$-modules with the following lowest weights:

N	the lowest weight	the corresponding cycle
1	$-\epsilon_1 - \epsilon_2 + 4\epsilon_3$	$x_3^2\partial_1 \wedge x_3^2\partial_2$
2	$-2\epsilon_1 + \epsilon_2 + 3\epsilon_3$	$x_3^2\partial_1 \wedge x_3x_2\partial_1$
3	$-\epsilon_1 - 3\epsilon_3$	$3\sum_i x_3^2\partial_1 \wedge x_3x_i\partial_i + 2\sum_i x_3^2\partial_i \wedge x_3x_i\partial_1$
4	$-\epsilon_1 - \epsilon_2 - 2\epsilon_3$	$\sum_i x_3x_2\partial_1 \wedge x_3x_i\partial_i - \sum_i x_3^2\partial_1 \wedge x_2x_i\partial_i$
5	$-\epsilon_1 - \epsilon_2 - 2\epsilon_3$	$\sum_i x_3x_2\partial_i \wedge x_3x_i\partial_1 - \sum_i x_3^2\partial_i \wedge x_2x_i\partial_1$
6	$-\epsilon_2 - \epsilon_3$	$\sum_{i,j} x_3x_i\partial_i \wedge x_2x_j\partial_j$
7	$-\epsilon_2 - \epsilon_3$	$.\sum_{i,j} x_3x_i\partial_j \wedge x_2x_j\partial_i$

4.4. *The relations for* $\mathfrak{svect}(n)$. These relations are those of the relations for $\mathfrak{vect}(n)$ that do not involve $X_n^+ = x_n \sum x_i\partial_i$.

4.5. *Relations between* $\mathfrak{N}_+$ *and* $\mathfrak{N}_-$ *for* $\mathfrak{vect}(n)$ *and* $\mathfrak{svect}(m), n, m > 2$. These relations are as for $\mathfrak{sl}(n+1)$ unless they involve Y; the extra relations are:

$$[Y, X_n^+] = -2[X_{n-1}^+, [\dots, [X_2^+, X_1^+]\dots]](= -2x_n\partial_1), [Y, X_i^+] = 0 \quad \text{for } i < n.$$

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