ON THE COHOMOLOGY RING OF THE FREE LOOP SPACE OF A WEDGE OF SPHERES

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Abstract.

In this paper we consider $H^*(LX,k)$, the cohomology ring of the free loop space of $X$, when $X$ is a wedge of spheres of the same dimensions (both even and odd), i.e., $X = S^{2n+1} \vee S^{2n+1} \ldots \vee S^{2n+1}$ or $X = S^{2n} \vee S^{2n} \ldots \vee S^{2n}$. We prove that in the odd-dimensional case, this ring is an algebra with an infinite number of generators and an infinite number of relations, which however has a very nice algebraic structure. It is the trivial extension of the ring consisting of the elements of degrees $(2n)k, k = 1, 2, \ldots$, by the module consisting of the elements of degrees $(2n + 1)k, k = 1, 2, \ldots$. In the even-dimensional case, we prove that this ring is the trivial extension of the ring consisting of the elements of degrees $(2n - 1)k, k = 1, 2, \ldots$, by the module consisting of the elements of degrees $(2n)k, k = 1, 2, \ldots$. We prove that $H^*(LX,k)$ (in low dimensions) is a Koszul algebra when $X = \bigvee_{i=1}^d S^i (d \geq 3, \text{odd})$, but it is not a Koszul algebra when $X = S^4 \vee S^4$. However, we get strong indications that this algebra satisfies a condition $M_3$ that has been studied by Löffwall and Roos. We study the torsion of these cohomology rings with coefficients in $Z$ and prove that in odd-dimensional case there is no torsion at all, whereas in even-dimensional case we have torsion. We prove that only 2-torsion is present in this case and determine the number of generators for the 2-torsion part. A general tool that we use is the Eilenberg-Moore spectral sequence

$$E_2^{p,q} = Tor^H_p(H^*(X,k), H^*(X;k))^q \Rightarrow H^n(LX,k).$$

This spectral sequence degenerates if $X$ is a formal space and $k$ is a field of characteristic zero. It reduces our work to the calculation of the Hochschild homology $H_*(\Lambda, \Lambda) = Tor^\Lambda_*(\Lambda, \Lambda)$. We also study another Eilenberg-Moore spectral sequence and find that it degenerates for $LS^3$ but does not for $LS^1$. This gives a clear indication that the ring structure of $H^*(LX,k)$, should be more complicated when $X$ is a wedge of even-dimensional spheres.

0. Introduction.

In this paper $k$ is usually a field of characteristic 0, $X$ is a CW-complex with a basepoint, $LX$ is the free loop space of $X$, i.e., the space of continuous maps $S^1 \to X$ with the compact open topology and $\Omega X$ is the loop space of $X$, basepoint preserving maps, $S^1 \to X$. In the literature (cf. [20]) the Betti numbers of the free loop spaces have been studied.

If $X$ is a finite CW-complex, then the “fiber homotopy pull-back diagram” (cf. e.g. [20] page 182):

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\[
\begin{array}{ccc}
\mathcal{L}X & \rightarrow & X^I \\
\downarrow & & \downarrow \\
X & \xrightarrow{\Delta} & X \times X
\end{array}
\]

where \(\Delta\) is the diagonal map, \(X^I = \{\phi : I = [0,1] \rightarrow X\}\) and \(X^I \rightarrow X \times X\) is defined by \(\phi \rightarrow (\phi(0), \phi(1))\), gives rise to an Eilenberg–Moo spectral sequence in the second quadrant.

\[
E_2^{p,q} = \text{Tor}_p^{H_*(X \times X, k)}(H^*(X, k), H^*(X, k))^q \Rightarrow H^n(\mathcal{L}X, k)
\]

where

\[
H^*(X \times X, k) \cong H^*(X, k) \otimes H^*(X, k).
\]

If

\[
X = \bigvee_{i=1}^m S^{n_i}
\]

is a wedge of spheres, then this spectral sequence degenerates and, as a result of the following theorem due to D. Anick, we obtain

\[
\prod_{i \geq 0} \text{Tor}_i^{H_*(X, k) \otimes H_*(X, k)}(H^*(X, k), H^*(X, k))^{t+n} \simeq H^n(\mathcal{L}X, k).
\]

**Theorem 0.1.** Let \(k\) be a field of characteristic zero and let \(X\) be a formal space, then \(H^*(\mathcal{L}X, k)\) is naturally bigraded and

\[
\prod_{i \geq 0} \text{Tor}_i^{H_*(X, k) \otimes H_*(X, k)}(H^*(X, k), H^*(X, k))^{t+n} \simeq H^n(\mathcal{L}X, k).
\]

**Proof.** (cf. [1], page 489).

In fact (0.1) is an isomorphism of rings (on the left hand side we have the Hochschild homology and on the right hand side the cohomology ring) when \(X\) is a wedge of spheres. See [12] for more information. In [20] the groups

\[
\text{Tor}_i^{H_*(X, k) \otimes H_*(X, k)}(H^*(X, k), H^*(X, k)) =
\]

\[
\frac{(H^+(X, k))^\otimes t+1}{\text{Im } S_{t+1}} \prod \text{Ker}((H^+)\otimes S_i (H^+)\otimes t)
\]

is calculated when \(H^*(X, k) = k + \Lambda\) is concentrated in even degrees and where

(0.2)

\[
S_n : \Lambda^\otimes n \rightarrow \bar{\Lambda}^\otimes n
\]

is defined by \(S_n = 1 - s_n\) and \(s_n : \Lambda^\otimes n \rightarrow \bar{\Lambda}^\otimes n\) is defined by:
\[ \lambda_1 \otimes \lambda_2 \otimes \ldots \otimes \lambda_n \mapsto (-1)^{n-1} \lambda_n \otimes \lambda_1 \otimes \ldots \otimes \lambda_{n-1}. \]

In this paper we calculate the groups
\[
\text{Tor}^H_{(X,k) \otimes H^*(X,k)}(H^*(X,k), H^*(X,k)) = \frac{(H^*(X,k))^\otimes_{t+1}}{\text{Im } T_{t+1}} \bigoplus \ker((H^+)^\otimes t) \xrightarrow{T_t} (H^+)^\otimes t,
\]
when \( H^*(X,k) = k + \tilde{\Lambda} \) is concentrated in odd degrees and where
\[
(0.3) \quad T_n : \tilde{\Lambda}^\otimes n \to \tilde{\Lambda}^\otimes n
\]
is defined by \( T_n = 1 - t_n \) and \( t_n : \tilde{\Lambda}^\otimes n \to \tilde{\Lambda}^\otimes n \) is defined by
\[
\lambda_1 \otimes \lambda_2 \otimes \ldots \otimes \lambda_n \mapsto \lambda_n \otimes \lambda_1 \otimes \ldots \otimes \lambda_{n-1}.
\]

Using this, we give an explicit formula for \( H^*(\mathcal{L}X,k) \), when \( X = \bigvee_{i=1}^m S^d \) (\( d \geq 3 \), odd, and \( m \geq 2 \), see Table 2.1.1 below. We prove that the ring \( H^*(\mathcal{L}X,k) \) is the trivial extension of \( H^{2*}(\mathcal{L}X,k) \) by the module
\[
H^{2*}(\mathcal{L}X,k) = s^{-1}H^{2*}(\mathcal{L}X,k)
\]
in the case when \( X = S^{d_1} \bigvee S^{d_2} \bigvee \ldots \bigvee S^{d_s} \) (\( d_i \), odd) and that the ring \( H^*(\mathcal{L}(S^4 \bigvee S^4),k) \) is the trivial extension of \( H^{3*}(\mathcal{L}(S^4 \bigvee S^4),k) \) by the module
\[
H^{3*+1}(\mathcal{L}(S^4 \bigvee S^4),k) = s^{-1}H^{3*}(\mathcal{L}(S^4 \bigvee S^4),k).
\]

Moreover we prove that \( H^*(\mathcal{L}X,k) \) is isomorphic in low dimensions to a Koszul algebra, i.e., \( \text{Tor}^R_{p,q}(k,k) = 0 \) for \( p \neq q \) in the case \( X = \bigvee_{i=1}^2 S^d \) (\( d \geq 3 \), odd). Note that the even dimensional subalgebra, i.e., \( H^{2*}(\mathcal{L}X,k) \) is not free (see [12]).

In section 3, we show that the ring \( H^*(\mathcal{L}(S^4 \bigvee S^4),k) \) in low dimensions is not a Koszul algebra, but that
\[
\text{Ext}^H_{H^*(\mathcal{L}(S^4 \bigvee S^4),k)}(k,k)
\]
has a very nice form. We also study the torsion of these cohomology rings with coefficients in \( Z \) and prove that in the odd-dimensional case there is no torsion at all (section 2.1 below), whereas in the even-dimensional case we have only 2-torsion (section 3.1 below). We also determine the number of generators for the 2-torsion part.

In section four we consider the degeneration of the Eilenberg-Moore spectral sequence in some special cases as follows.

Let \( X \) be a finite simply connected CW-complex with a basepoint \( x_0, PX \)
the space of paths in \( X \) starting in \( x_0 \) and \( PX \xrightarrow{\pi} X \) the map that to each path associates its endpoint. We have a pull-back diagram

\[
\begin{array}{ccc}
\Omega X & \rightarrow & PX \\
\downarrow & & \downarrow \pi \\
\{x_0\} & \rightarrow & X
\end{array}
\]

This diagram gives rise to another Eilenberg-Moore spectral sequence:

\[
E^2_{p,q} = \text{Ext}^p_{H^*(X,k)}(k,k)_q \Rightarrow \text{gr}(H_*(\Omega X,k))
\]

**Theorem 0.2.** ([21] page 25). The Eilenberg-Moore spectral sequence (0.4) degenerates if \( X \) is a finite, simply connected CW-complex with \( \dim X \leq 4 \).

Whenever this spectral sequence degenerates,

\[
\dim_k(H_n(\Omega X,k)) = \sum_{p \geq 0} \dim_k \text{Ext}^p_{H^*(X,k)}(k,k)_{p+n}
\]

where the sum is finite.

In section four we replace \( X \) in (0.4) by \( \mathcal{L}X \) the free loop space of \( X \) and prove that it degenerates in the case \( X = S^5 \) but it does not if \( X = S^4 \).

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1. **Algebraic preliminaries.**

1.1. *The normalized standard free resolution.*

Let \( \Lambda = k \oplus \overline{\Lambda} \) be an associative \( k \)-algebra with unit (non-graded). Moreover assume that the product is zero in \( \overline{\Lambda} \). Then

\[
H_*(\Lambda, \Lambda) = \text{Tor}^*_{\Lambda'}(\Lambda, \Lambda)
\]

is the homology of the complex

\[
\begin{array}{cccc}
\cdots \rightarrow & \Lambda \otimes_k \overline{\Lambda} \otimes^{n+1} & \rightarrow & \Lambda \otimes_k \overline{\Lambda} \otimes^n \\
& d_n & \rightarrow & \Lambda \otimes_k \overline{\Lambda} \otimes^{n-1} \rightarrow \cdots
\end{array}
\]

where

\[
d_n(\lambda \otimes [\lambda_1 \otimes \ldots \otimes \lambda_n]) = \lambda \lambda_1 \otimes [\lambda_2 \otimes \ldots \otimes \lambda_n] \\
+ (-1)^n \lambda_n \lambda \otimes [\lambda_1 \otimes \ldots \otimes \lambda_{n-1}].
\]

(See [20] page 178).

Now assume that \( \Lambda \) is a graded algebra which is connected, that is, the
unit \( \eta : k \to \Lambda \) is an isomorphism in degree 0. Denote the cokernel of \( \eta \) by \( \tilde{\Lambda} \), since \( \Lambda \) is connected we have
\[
\tilde{\Lambda} = \Lambda^+ = \{ \gamma \in \Lambda \mid \deg \gamma > 0 \}.
\]
Recall (cf. [18] page 228) the bar construction.
\[
B^{-n}(\Lambda, N) = \Lambda \otimes_k \tilde{\Lambda} \otimes_k \cdots \otimes_k \tilde{\Lambda} \otimes_k N.
\]
Notice that \( B^{-n}(\Lambda, N) \) is a left \( \Lambda \)-module with the extended module action. It is customary to write an element of \( B^{-n}(\Lambda, N) \) as \( \gamma \otimes [\gamma_1 \otimes \gamma_2 \otimes \cdots \otimes \gamma_n] \otimes a \) and of \( B^0(\Lambda, N) \) as \( \gamma \otimes [\gamma_1 \cdots \cdots \gamma_n] \otimes a \). If we write \( \tilde{\alpha} = (-1)^{1+\deg \alpha} \alpha \) for a homogeneous element, then we can assemble the \( B^{-n}(\Lambda, N) \) into a resolution by introducing an external differential
\[
\delta : B^{-n}(\Lambda, N) \to B^{-n+1}(\Lambda, N)
\]
where
\[
\delta(\gamma[\gamma_1|\gamma_2|\cdots|\gamma_n]a) = (-1)^{\deg \gamma}(\gamma\gamma_1)[\gamma_2|\cdots|\gamma_n]a
\]
\[
+ \sum_{i=1}^{n-1} (-1)^{\deg \gamma}[\gamma_1|\gamma_2|\cdots|\gamma_{i-1}|\gamma_i|\gamma_{i+1}|\cdots|\gamma_n]a
\]
\[
+ (-1)^{\deg \gamma}[\gamma_1|\cdots|\gamma_{n-1}](\gamma_n a)
\]
Now in the case of product zero we obtain:
\[
(1.1.3) \quad d_n \lambda \otimes [\lambda_1 \otimes \cdots \otimes \lambda_n] = (-1)^{\deg \lambda} \lambda \lambda_1 \otimes [\lambda_2 \otimes \cdots \otimes \lambda_n]
\]
\[
+ (-1)^{\deg \lambda + \deg \lambda_1 + \cdots + \deg \lambda_{n-1} + n-1} (-1)^n \lambda_n \lambda \otimes [\lambda_1 \otimes \cdots \otimes \lambda_{n-1}].
\]
Notice that in (1.1.3) if \( \lambda_i \)'s are concentrated in even degrees, then we have the same formula as in the non-graded case. But in this paper we also need to consider the case when \( \lambda_i \)'s are concentrated in odd degrees. If this is so, then
\[
(-1)^{\deg \lambda + \deg \lambda_1 + \cdots + \deg \lambda_{n-1} + n-1} (-1)^n = -1, \quad \forall \quad n \geq 0.
\]
We have assumed \( \lambda \) to be in degree 0, i.e., \( \deg \lambda = 0 \). Hence (1.1.3) can be written as
\[
(1.1.4) \quad d_n[\lambda_1 \otimes \cdots \otimes \lambda_n] = \lambda_1 \otimes [\lambda_2 \otimes \cdots \otimes \lambda_n]
\]
\[
- \lambda_n \otimes [\lambda_1 \otimes \cdots \otimes \lambda_{n-1}].
\]
In order to calculate
\[
\text{Tor}_t^{H^\ast(X,k) \otimes H^\ast(X,k)}(H^\ast(X,k), H^\ast(X,k))
\]
when $H^*(X,k)/k$ is concentrated in odd degrees, let $H^*(X,k) = \Lambda = k \oplus \bar{\Lambda}$ and use the isomorphism

(1.1.5) \[ \Lambda \otimes_k \bar{\Lambda}^\otimes n \longleftrightarrow \bar{\Lambda}^\otimes n+1 \oplus 1 \otimes \bar{\Lambda}^\otimes n. \]

Moreover define

(1.1.6) \[ T_n : \bar{\Lambda}^\otimes n \rightarrow \bar{\Lambda}^\otimes n \]

by $T_n = 1 - t_n$, where $t_n : \bar{\Lambda}^\otimes n \rightarrow \bar{\Lambda}^\otimes n$ is defined by

\[ \lambda_1 \otimes \lambda_2 \otimes \ldots \otimes \lambda_n \longrightarrow \lambda_n \otimes \lambda_1 \otimes \ldots \otimes \lambda_{n-1}. \]

Now the resolution (1.1.1) can be written as:

(1.1.7) \[ \ldots \longrightarrow \bar{\Lambda}^\otimes n+2 + 1 \otimes \bar{\Lambda}^\otimes n+1 d_{n+1} \longrightarrow \bar{\Lambda}^\otimes n+1 \]

\[ + 1 \otimes \bar{\Lambda}^\otimes n d_n \longrightarrow \bar{\Lambda}^\otimes n + 1 \otimes \bar{\Lambda}^\otimes n-1 \longrightarrow \ldots \]

where the map $d_n$ has the simple form:

(1.1.8) \[ d_n[(v_0 \otimes v_1 \otimes \ldots \otimes v_n) + (1 \otimes w_1 \otimes \ldots \otimes w_n)] \]

\[ = (w_1 \otimes w_2 \otimes \ldots \otimes w_n) \]

\[ - (w_n \otimes w_1 \otimes \ldots \otimes w_{n-1}) \]

\[ = T_n(w_1 \otimes w_2 \otimes \ldots \otimes w_n). \]

Therefore

\[ \text{Ker} \ d_n = \bar{\Lambda}^\otimes n+1 + 1 \otimes \text{Ker} \ T_n \]

and

\[ \text{Im} \ d_{n+1} = \text{Im} \ T_{n+1} \]

and hence

(1.1.9) \[ \text{Tor}^{H^*(X,k) \otimes H^*(X,k)}_{t}(H^*(X,k), H^*(X,k)) \]

\[ = \frac{(H^*(X,k)^{\otimes t+1})}{\text{Im} \ T_{t+1}} \coprod \text{Ker}(H^+)^{\otimes t} \xrightarrow{T_t} (H^+)^{\otimes t}). \]

**Definition 1.1.1.** (The shuffle product in the graded case cf. [18]). Let $\gamma_{p,q}$ denote the group of permutations of the set $\{1, 2, \ldots, p + q\}$. If $\sigma \in \gamma_{p,q}$, then $\sigma$ is a $(p, q)$ shuffle if the following holds

\[ \sigma(1) < \sigma(2) < \ldots < \sigma(p) \quad \text{and} \]

\[ \sigma(p + 1) < \sigma(p + 2) < \ldots < \sigma(p + q). \]
Define the shuffle product (*) on the level of the standard free resolution (1.1.1) as follows:

Suppose

\[ \lambda \otimes [a_1 \otimes a_2 \otimes \ldots \otimes a_p] \in \Lambda \otimes \bar{\Lambda}^{\otimes p} \]

and

\[ \mu \otimes [b_1 \otimes b_2 \otimes \ldots \otimes b_q] \in \Lambda \otimes \bar{\Lambda}^{\otimes q}, \]

then

\[ \lambda \otimes [a_1 \otimes a_2 \otimes \ldots \otimes a_p] \ast \mu \otimes [b_1 \otimes b_2 \otimes \ldots \otimes b_q] = \sum_{(p,q)\text{-shuffles}(\sigma)} (-1)^{s(\sigma)} \lambda \mu \otimes [c_{\sigma^{-1}(1)} \otimes c_{\sigma^{-1}(2)} \otimes \ldots \otimes c_{\sigma^{-1}(p+q)}] \]

where \( c_{\sigma(i)} = a_{\sigma(i)} \) if \( 1 \leq \sigma(i) \leq p \), \( c_{\sigma(i)} = b_{\sigma(i)-p} \) if \( p + 1 \leq \sigma(i) \leq p + q \)

and

\[ s(\sigma) = \sum (\deg c_i + 1)(\deg c_{p+j} + 1) \]

summed over all pairs \((i, p + j)\) with \( \sigma(i) > \sigma(p + j) \). This sign reflects the convention that \( \pm 1 \) is introduced when elements are switched past each other according to their total degrees. Here \( \Lambda \) is a graded algebra and

\[ \bar{\Lambda} = \Lambda^+ = \{ \gamma \in \Lambda \mid \deg \gamma > 0 \}. \]

We study the shuffle product, since this gives the product structure on

\[ \text{Tor}_r^{H^*(X,k)}(H^*(X,k), H^*(X,k)), \]

and hence on \( H^*(\mathcal{L}X, k) \).

1.2. Invariant subspaces of Hochschild homology.

Let \( k \) be a commutative field, \( V \) a finite dimensional vector space over \( k \) and \( \Lambda = k \oplus V \) the trivial extension of \( k \) by \( V \). In other words \( \Lambda \) consist of the set of pairs \((\kappa, v), \kappa \in k, v \in V \) with pairwise addition and multiplication

\[ (k_1, v_1) \cdot (k_2, v_2) = (k_1 k_2, k_1 v_2 + k_2 v_1). \]

Let \( \Lambda^e = \Lambda \otimes_k \Lambda^o \), where \( \Lambda^o \) is the oposite ring (here \( \Lambda^o = \Lambda \)), and let \( \Lambda \) be considered as \( \Lambda^e \) module in the natural way. In [20], the Hochschild homology of trivial ring extension is calculated as follows:

\[ H_n(\Lambda, \Lambda) = \text{Tor}_n^\Lambda(\Lambda, \Lambda) = \frac{V^{\otimes n+1}}{\text{Im } S_{n+1}} \prod \text{Ker}(V^{\otimes n} \xrightarrow{S_n} V^{\otimes n}), \]

where
(1.2.2) \[ S_n = 1 - s_n \text{ and } s_n(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = (-1)^{n-1} v_n \otimes v_1 \otimes \cdots \otimes v_{n-1}. \]

Now \( \text{Ker } S_n \) in (1.2.1) is exactly the vector space \( (V^\otimes n)^{Z_n} \) of invariants in \( V^\otimes n \) for the group \( Z/nZ \) acting through its generator \( s_n = \bar{1} \) and hence

\[ \text{Ker } S_n = (V^\otimes n)^{Z_n} = \frac{1 + s_n + s_n^2 + \cdots + s_n^{n-1}}{n} V^\otimes n. \]

In a similar way, \( \text{Coker } S_{n+1} \) in (1.2.1) is the space of coinvariants for the analogous action of \( Z/(n + 1)Z \). But the exact sequence

(1.2.3) \[ 0 \rightarrow \text{Ker } S_n \rightarrow V^\otimes n s_n \rightarrow V^\otimes n \rightarrow \text{Coker } S_n \rightarrow 0 (\text{cf. [20]}) \]

shows that \( \text{Ker } S_n \) and \( \text{Coker } S_n \) always have the same dimension over \( k \). Therefore it follows from (1.2.1) and the preceding discussion that

\[ |\text{Tor}_*^N(\Lambda, \Lambda)| = |(V^\otimes n)^{Z_n}| + |(V^\otimes (n+1))^{Z/(n+1)}| \text{ for } n \geq 1. \]

Using the endomorphism

\[ N = \frac{1 + s_n + s_n^2 + \cdots + s_n^{n-1}}{n} \]

of \( V^\otimes n \) as a projection of \( V^\otimes n \) onto \( (V^\otimes n)^{Z_n} \), J.-E. Roos (cf. [20]) found the dimension of the invariant subspace by the following formula, over a field of characteristic 0:

\[ |(V^\otimes n)^{Z_n}| = \text{trace } N = \frac{1}{n} \sum_{i=0}^{n-1} \text{trace } (s_n)^i \]

where \((s_n)^0\) is the identity on \( V^\otimes n \).

In this part we try to find the dimension of the invariant and coinvariant subspaces over a field of arbitrary characteristic.

**Theorem 1.2.1.** Assume the group \( \frac{Z}{nZ} \) acts through its generator \( t_n = \bar{1} \) on \( V^\otimes n \) by

\[ t_n(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = v_n \otimes v_1 \otimes \cdots \otimes v_{n-1}. \]

Then the dimension of the invariant subspace is independent of the characteristic of the field, moreover

\[ |(V^\otimes n)^{Z_n}| = \frac{1}{n} \sum_{i=0}^{n} |V|^{(i,n)} \]
where \((V^\otimes n)^{Z_n}\) is the invariant subspace under the action of \(t_n\) and \(\|\) is the dimension as a vector space over \(k\).

**Proof.** Let \(|V| = m, |V^\otimes n| = m^n = l\) and let
\[
\{\alpha_1, \alpha_2, \ldots, \alpha_l\}
\]
be a basis for \(V^\otimes n\). If the characteristic of \(k\) is zero, then the distinct elements of
\[
(1.2.4) \quad \left\{ \frac{1 + t_n + t_n^2 + \cdots + t_n^{n-1}}{n} \alpha_i \mid \alpha_i \in V^\otimes n \right\}
\]
form a basis for
\[
\text{Ker } T_n = \text{Ker}(1 - t_n) = (V^\otimes n)^{Z_n}.
\]
We denote this basis by
\[
(1.2.5) \quad M = \left\{ \frac{1 + t_n + t_n^2 + \cdots + t_n^{n-1}}{n} \alpha_{i1}, \frac{1 + t_n + t_n^2 + \cdots + t_n^{n-1}}{n} \alpha_{i2}, \ldots, \frac{1 + t_n + t_n^2 + \cdots + t_n^{n-1}}{n} \alpha_{im} \right\}
\]
Note that
\[
(1.2.6) \quad \frac{1 + t_n + t_n^2 + \cdots + t_n^{n-1}}{n} \alpha_i = \frac{1 + t_n + t_n^2 + \cdots + t_n^{n-1}}{n} \alpha_j
\]
if and only if \(\alpha_i = t_n^k \alpha_j\) for some \(1 \leq k \leq n\). By eliminating the denominator \(n\) we get the basis
\[
(1.2.7) \quad M_1 = \{(1 + t_n + t_n^2 + \cdots + t_n^{n-1})(\alpha_{i1}), (1 + t_n + t_n^2 + \cdots + t_n^{n-1})(\alpha_{i2}), \ldots, (1 + t_n + t_n^2 + \cdots + t_n^{n-1})(\alpha_{im})\}.
\]
For any \(1 \leq j \leq m\) let \(r_j\) be the least integer \(1 \leq r_j \leq n\) such that \(t_n^{r_j} \alpha_{ij} = \alpha_{ij}\), then
\[
(1 + t_n + t_n^2 + \cdots + t_n^{r_j-1})(\alpha_{ij}) = \frac{n}{r_j} (1 + t_n + t_n^2 + \cdots + t_n^{r_j-1})(\alpha_{ij}).
\]
Now put
\[ M' = \{ \beta_{i1} = (1 + t_n + t_n^2 + \cdots + t_n^{r_n-1})(\alpha_{i1}), \]
\[
\beta_{i2} = (1 + t_n + t_n^2 + \cdots + t_n^{r_2-1})(\alpha_{i2}), \cdots
\]
\[
\cdots, \beta_{im} = (1 + t_n + t_n^2 + \cdots + t_n^{r_{m-1}})(\alpha_{im}) \}. \]

We prove that \( M' \) is a basis for the invariant subspace regardless of the characteristic of \( k \). Clearly \( M' \) is a linearly independent set, since the elements of \( M' \) are different with coefficients one. To prove that \( M' \) generates the invariant subspace, let \( \alpha \) be any invariant element in \( V^{\otimes n} \). Arrange the basis \( \{ \alpha_1, \alpha_2, \ldots, \alpha_l \} \) of \( V^{\otimes n} \) as
\[
M_2 = \{ \alpha_{i1}, t(\alpha_{i1}), t^2(\alpha_{i1}), \ldots, t^{r_1-1}(\alpha_{i1}), \alpha_{i2}, t(\alpha_{i2}), t^2(\alpha_{i2}), \ldots, \]
\[
t^{r_2-1}(\alpha_{i2}), \ldots, \alpha_{im}, t(\alpha_{im}), t^2(\alpha_{im}), \ldots, t^{r_m-1}(\alpha_{im}) \} \]

In \( M_2 \) we replaced \( t_n \) by \( t \) for simplicity. As a linear combination of elements of \( M_2 \), let
\[ \alpha = b_{i1}^0 \alpha_{i1} + b_{i1}^1 t(\alpha_{i1}) + b_{i1}^2 t^2(\alpha_{i1}) + \cdots + b_{i1}^{r_1-1} t^{r_1-1}(\alpha_{i1}) \]
\[ + b_{i2}^0 \alpha_{i2} + b_{i2}^1 t(\alpha_{i2}) + b_{i2}^2 t^2(\alpha_{i2}) + \cdots + b_{i2}^{r_2-1} t^{r_2-1}(\alpha_{i2}) + \cdots \]
\[ + b_{im}^0 \alpha_{im} + b_{im}^1 t(\alpha_{im}) + b_{im}^2 t^2(\alpha_{im}) + \cdots + b_{im}^{r_m-1} t^{r_m-1}(\alpha_{im}) \]

Now applying \( t \) to \( \alpha \), we obtain:
\[ (1.2.9) \]  
\[ t(\alpha) = b_{i1}^0 t(\alpha_{i1}) + b_{i1}^1 t^2(\alpha_{i1}) + \cdots + b_{i1}^{r_1-2} t^{r_1-1}(\alpha_{i1}) + b_{i1}^{r_1-1} t^{r_1-1}(\alpha_{i1}) \]
\[ + b_{i2}^0 t(\alpha_{i2}) + b_{i2}^1 t^2(\alpha_{i2}) + \cdots + b_{i2}^{r_2-2} t^{r_2-1}(\alpha_{i2}) + b_{i2}^{r_2-1} t^{r_2-1}(\alpha_{i2}) + \cdots \]
\[ + b_{im}^0 t(\alpha_{im}) + b_{im}^1 t^2(\alpha_{im}) + \cdots + b_{im}^{r_m-2} t^{r_m-1}(\alpha_{im}) + b_{im}^{r_m-1} t^{r_m-1}(\alpha_{im}) \]

An element \( \alpha \in V^{\otimes n} \) is invariant if and only if
\[ (1.2.11) \]
\[ t(\alpha) = \alpha. \]

It follows from substituting (1.2.9) and (1.2.10) into (1.2.11) that
\[
b_{i1}^0 = b_{i1}^1 = b_{i1}^2 = \cdots = b_{i1}^{r_1-2} = b_{i1}^{r_1-1} = b_{i1}^0 \]
\[ \cdots \cdots \cdots \]
\[
b_{ij}^0 = b_{ij}^1 = b_{ij}^2 = \cdots = b_{ij}^{r_j-2} = b_{ij}^{r_j-1} = b_{ij}^0 \]
\[ \cdots \cdots \cdots \]
\[
b_{im}^0 = b_{im}^1 = b_{im}^2 = \cdots = b_{im}^{r_m-2} = b_{im}^{r_m-1} = b_{im}^0 \]
and hence
\[ \alpha = b_1^{(0)} \beta_1 + b_2^{(0)} \beta_2 + \cdots + b_m^{(0)} \beta_m. \]
This shows that \( \alpha \) is a linear combination of elements of \( M' \). This proves the first part of the theorem.

For the second part, later in section 2 in Theorem 2.1.1, we prove that
\[
(1.2.12) \quad |(V^\otimes n)^{Z_{\mathbb{Z}}}| = \frac{1}{n} \sum_{i=0}^{n} |V|^{(i,n)}
\]
when the characteristic of \( k \) is zero. Now the result follows by the first part of the theorem.

**Theorem 1.2.2.** Assume the group \( \mathbb{Z}/n\mathbb{Z} \) acts through its generator \( s_n = \bar{1} \) on \( V^\otimes n \) by
\[
s_n(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = (-1)^{n-1} v_n \otimes v_1 \otimes \cdots \otimes v_{n-1}.
\]
Then the dimension of the invariant subspace is independent of the characteristic of the field if the characteristic is different from 2. If this is the case, then
\[
|(V^\otimes n)^{Z_{\mathbb{Z}}}| = \frac{1}{n} \sum_{i=0}^{n} (-1)^{i(n-1)} |V|^{(i,n)}
\]
where \((V^\otimes n)^{Z_{\mathbb{Z}}}\) is the invariant subspace under the action of \( s_n \). In characteristic 2, the dimension of the invariant subspace can be calculated as in Theorem 1.2.1, i.e.,
\[
|(V^\otimes n)^{Z_{\mathbb{Z}}}| = \frac{1}{n} \sum_{i=0}^{n} |V|^{(i,n)}.
\]

**Proof.** If characteristic of \( k \) is 2, then \(-1 = +1\) and hence:
\[
s_n(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = (-1)^{n-1} (v_n \otimes v_1 \otimes \cdots \otimes v_{n-1})
\]
\[= (v_n \otimes v_1 \otimes \cdots \otimes v_{n-1}) = t_n(v_1 \otimes v_2 \otimes \cdots \otimes v_n).
\]
This proves the second part of the theorem. The proof of the first part is almost the same as in Theorem 1.2.1. If \( n \) is odd then:
\[
s_n(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = t_n(v_1 \otimes v_2 \otimes \cdots \otimes v_n)
\]
and hence the dimension of the invariant subspace is independent of the characteristic of the field as we proved in Theorem 1.2.1. Consider the case that \( n \) is even, i.e.,
\[ s_n(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = -(v_n \otimes v_1 \otimes \cdots \otimes v_{n-1}) \]

As in Theorem 1.2.1, let \(|V| = m, |V^{\otimes n}| = m^n = l\) and let
\[ \{\alpha_1, \alpha_2, \ldots, \alpha_l\} \]
be a basis for \(V^{\otimes n}\) over a field \(k\) of characteristic zero. Consider the set
\[ (1.2.13) \quad G = \{(1 + s + s^2 + \cdots + s^{n-1})(\alpha_i) | \alpha_i \text{ is a basis element of } V^{\otimes n}\} \]
(we have replaced \(s_n\) by \(s\) for simplicity). For any \(1 \leq i \leq l\) let \(k_i\) be the least integer \(1 \leq k_i \leq n\) such that \(s^{k_i}(\alpha_i) = \pm \alpha_i\). If \(s^{k_i}(\alpha_i) = -\alpha_i\), i.e., if \(k_i\) is odd, then
\[ (1 + s + s^2 + \cdots + s^{n-1})(\alpha_i) = 0 \]
and if \(s^{k_i}(\alpha_i) = \alpha_i\), i.e., if \(k_i\) is even, then
\[ (1 + s + s^2 + \cdots + s^{n-1})(\alpha_i) = \frac{n}{k_i} (1 + s + s^2 + \cdots + s^{k_i-1})(\alpha_i). \]

Now let \(G'\) be the set of all distinct non-zero elements of \(G\). Assume
\[ (1.2.14) \quad G' = \{(1 + s + s^2 + \cdots + s^{n-1})(\alpha_{i1}), (1 + s + s^2 + \cdots + s^{n-1})(\alpha_{i2}), \ldots, (1 + s + s^2 + \cdots + s^{n-1})(\alpha_{iq})\}. \]
To prove that \(G'\) is a basis for the invariant subspace is easy and is left to the reader. For any \(1 \leq j \leq q\) let \(u_j\) be the least integer \(1 \leq u_j \leq n\) such that \(s^{u_j}(\alpha_{ij}) = \alpha_{ij}\), then
\[ (1 + s + s^2 + \cdots + s^{n-1})(\alpha_{ij}) = \frac{n}{u_j} (1 + s + s^2 + \cdots + s^{u_j-1})(\alpha_{ij}). \]

Put
\[ (1.2.15) \quad G'' = \{\beta_{i1} = (1 + s + s^2 + \cdots + s^{n-1})(\alpha_{i1}), \beta_{i2} = (1 + s + s^2 + \cdots + s^{u_j-1})(\alpha_{i2}), \ldots, \beta_{iq} = (1 + s + s^2 + \cdots + s^{u_j-1})(\alpha_{iq})\}. \]
We prove that \(G'\) is a basis for the invariant subspace over a field \(k_p\) of characteristic \(p(p \neq 2)\) and hence the dimension of the invariant subspace is independent of the characteristic if the characteristic is not 2. Clearly \(G''\) is linearly independent over \(k_p\), since it contains distinct elements with coefficient 1. It remains to prove that it generates the invariant subspace. First let us construct an special basis of \(V^{\otimes n}\). Consider the basis \(\{\alpha_1, \alpha_2, \ldots, \alpha_l\}\) of \(V^{\otimes n}\) and put
(1.2.16) \[ H = \{(1 + s + s^2 + \cdots + s^{k_i-1})(\alpha_i)|\alpha_i \text{ is a basis element of } V^\otimes n\} \]

Recall that for any \(1 \leq i \leq l, k_i \) is the least integer, \(1 \leq k_i \leq n\) such that \(s^{k_i}(\alpha_i) = \pm \alpha_i\). If \(k_i\) is odd then

\[ (1 + s + s^2 + \cdots + s^{n-1})(\alpha_i) = 0 \]

and hence the element \((1 + s + s^2 + \cdots + s^{k_i-1})(\alpha_i)\) of \(H\) in (1.2.16) does not appear in \(G''\) in (1.2.15). But if \(k_i\) is even then the element \((1 + s + s^2 + \cdots + s^{k_i-1})(\alpha_i)\) lies in both \(H\) in (1.2.16) and \(G''\) in (1.2.15). So \(G''\) is a subset of \(H\). Rewrite the set \(H\) by arranging the elements as follows:

\[
H \{ \beta_{l_1} = (1 + s + s^2 + \cdots + s^{u_{l-1}})(\alpha_{l_1}), \\
\beta_{l_2} = (1 + s + s^2 + \cdots + s^{u_{l-2}})(\alpha_{l_2}), \ldots, \\
\beta_{lq} = (1 + s + s^2 + \cdots + s^{u_{l-1}})(\alpha_{lq}), \\
\beta_{l(q+1)} = (1 + s + s^2 + \cdots + s^{u_{l(q+1)-1}})(\alpha_{l(q+1)}), \\
\beta_{l(q+2)} = (1 + s + s^2 + \cdots + s^{u_{l(q+2)-1}})(\alpha_{l(q+2)}), \ldots, \\
\beta_{l(q+w)} = (1 + s + s^2 + \cdots + s^{u_{l(q+w)-1}})(\alpha_{l(q+w)}) \}
\]

where \(u_1, u_2, \ldots, u_q\) are even and \(u_{q+1}, u_{q+2}, \ldots, u_{q+w}\) are odd. Now an special basis for \(V^\otimes n\) is:

\[
G_1 = \{ \alpha_{l_1}, -s(\alpha_{l_1}), s^2(\alpha_{l_1}), \ldots, s^{u_{l-2}}(\alpha_{l_1}), -s^{u_{l-1}}(\alpha_{l_1}), \\
\alpha_{l_2}, -s(\alpha_{l_2}), s^2(\alpha_{l_2}), \ldots, s^{u_{l-2}}(\alpha_{l_2}), -s^{u_{l-1}}(\alpha_{l_2}), \ldots, \\
\alpha_{lq}, -s(\alpha_{lq}), s^2(\alpha_{lq}), \ldots, s^{u_{l-2}}(\alpha_{lq}), -s^{u_{l-1}}(\alpha_{lq}), \\
\alpha_{l(q+1)}, -s(\alpha_{l(q+1)}), s^2(\alpha_{l(q+1)}), \ldots, -s^{u_{l(q+1)-1}}(\alpha_{l(q+1)}), \\
s^{u_{l(q+1)-1}}(\alpha_{l(q+1)}), \\
\alpha_{l(q+2)}, -s(\alpha_{l(q+2)}), s^2(\alpha_{l(q+2)}), \ldots, -s^{u_{l(q+2)-1}}(\alpha_{l(q+2)}), \\
s^{u_{l(q+2)-1}}(\alpha_{l(q+2)}), \ldots, \\
\alpha_{l(q+w)}, -s(\alpha_{l(q+w)}), s^2(\alpha_{l(q+w)}), \ldots, -s^{u_{l(q+w)-1}}(\alpha_{l(q+w)}), \\
s^{u_{l(q+w)-1}}(\alpha_{l(q+w)}) \}
\]

Now let \(\alpha\) be an invariant element. As an element of \(V^\otimes n\), \(\alpha\) is a linear combination of elements of \(G_1\) in (1.2.18). Let
\[
\alpha = c^0_{i1} \alpha_{i1} - c^1_{i1} s(\alpha_{i1}) + c^2_{i1} s^2(\alpha_{i1}) + \cdots + c^{\mu_{i1}-2}_{i1} s^{\mu_{i1}-2}(\alpha_{i1}) \\
- c^{\mu_{i1}-1}_{i1} s^{\mu_{i1}-1}(\alpha_{i1}) \\
+ c^0_{i2} \alpha_{i2} - c^1_{i2} s(\alpha_{i2}) + c^2_{i2} s^2(\alpha_{i2}) - \cdots + c^{\mu_{i2}-2}_{i2} s^{\mu_{i2}-2}(\alpha_{i2}) \\
- c^{\mu_{i2}-1}_{i2} s^{\mu_{i2}-1}(\alpha_{i2}) + \cdots \\
+ c^0_{iq} \alpha_{iq} - c^1_{iq} s(\alpha_{iq}) + c^2_{iq} s^2(\alpha_{iq}) - \cdots + c^{\mu_{iq}-2}_{iq} s^{\mu_{iq}-2}(\alpha_{iq}) \\
- c^{\mu_{iq}-1}_{iq} s^{\mu_{iq}-1}(\alpha_{iq}) \\
+ c^0_{i(q+1)} \alpha_{i(q+1)} - c^1_{i(q+1)} s(\alpha_{i(q+1)}) + c^2_{i(q+1)} s^2(\alpha_{i(q+1)}) + \cdots \\
- c^{\mu_{i(q+1)}-2}_{i(q+1)} s^{\mu_{i(q+1)}-2}(\alpha_{i(q+1)}) + c^{\mu_{i(q+1)}-1}_{i(q+1)} s^{\mu_{i(q+1)}-1}(\alpha_{i(q+1)}) \\
+ c^0_{i(q+2)} \alpha_{i(q+2)} - c^1_{i(q+2)} s(\alpha_{i(q+2)}) + c^2_{i(q+2)} s^2(\alpha_{i(q+2)}) - \cdots \\
- c^{\mu_{i(q+2)}-2}_{i(q+2)} s^{\mu_{i(q+2)}-2}(\alpha_{i(q+2)}) + c^{\mu_{i(q+2)}-1}_{i(q+2)} s^{\mu_{i(q+2)}-1}(\alpha_{i(q+2)}) + \cdots \\
+ c^0_{i(q+w)} \alpha_{i(q+w)} - c^1_{i(q+w)} s(\alpha_{i(q+w)}) + c^2_{i(q+w)} s^2(\alpha_{i(q+w)}) - \cdots \\
- c^{\mu_{i(q+w)}-2}_{i(q+w)} s^{\mu_{i(q+w)}-2}(\alpha_{i(q+w)}) + c^{\mu_{i(q+w)}-1}_{i(q+w)} s^{\mu_{i(q+w)}-1}(\alpha_{i(q+w)}).
\]

Now our aim is to prove that \(\alpha\) is a linear combination of elements of \(G''\) to conclude that \(G''\) is really a basis for the invariant subspace. To do this we prove that all coefficients of \(\alpha_{ij}(j \geq q + 1)\) in (1.2.19) must vanish. As a sample we prove that the coefficients of \(\alpha_{i(q+1)}\) are zero. Rewrite \(\alpha\) as:

\[
\alpha = c^0_{i(q+1)} \alpha_{i(q+1)} - c^1_{i(q+1)} s(\alpha_{i(q+1)}) + c^2_{i(q+1)} s^2(\alpha_{i(q+1)}) + \cdots \\
- c^{\mu_{i(q+1)}-2}_{i(q+1)} s^{\mu_{i(q+1)}-2}(\alpha_{i(q+1)}) + c^{\mu_{i(q+1)}-1}_{i(q+1)} s^{\mu_{i(q+1)}-1}(\alpha_{i(q+1)}) + \beta
\]

Applying \(s\) to \(\alpha\), we obtain

\[
s(\alpha) = c^0_{i(q+1)} s(\alpha_{i(q+1)}) - c^1_{i(q+1)} s^2(\alpha_{i(q+1)}) + c^2_{i(q+1)} s^3(\alpha_{i(q+1)}) + \cdots \\
- c^{\mu_{i(q+1)}-2}_{i(q+1)} s^{\mu_{i(q+1)}-2}(\alpha_{i(q+1)}) + c^{\mu_{i(q+1)}-1}_{i(q+1)} s^{\mu_{i(q+1)}-1}(\alpha_{i(q+1)}) + s(\beta).
\]

Equating (1.2.20) and (1.2.21), since \(\alpha\) is an invariant element, we get

\[
c^0_{i(q+1)} = -c^1_{i(q+1)} = c^2_{i(q+1)} = c^3_{i(q+1)} = \cdots = -c^{\mu_{i(q+1)}-2}_{i(q+1)} = c^{\mu_{i(q+1)}-1}_{i(q+1)} = -c^0_{i(q+1)}
\]

and this implies

\[2c^0_{i(q+1)} = 0.
\]

Since the characteristic of \(k\) is not 2, \(c^0_{i(q+1)} = 0\) and hence

\[
c^0_{i(q+1)} = c^1_{i(q+1)} = c^2_{i(q+1)} = c^3_{i(q+1)} = \cdots = c^{\mu_{i(q+1)}-2}_{i(q+1)} = c^{\mu_{i(q+1)}-1}_{i(q+1)} = 0.
\]
2. The free loop space on a wedge of odd spheres.

2.1. Graded vector space structure and torsion.

**Theorem 2.1.1.** Let \( X = S^d \sqcup \cdots \sqcup S^d \) be the wedge of \( m, d \)-spheres \((d \geq 3, \text{odd})\). We have the following explicit formula for \( H^*(\mathcal{L}X, k) \), where \( \overline{\Lambda} = H^{>0}(\mathcal{L}X, k) \).

\[
\begin{array}{cccccc}
\text{deg} & 0 & d - 1 & d & 2(d - 1) & 2(d - 1) + 1 & \cdots \\
H^*(\mathcal{L}X, k) & k & 1 \otimes \overline{\Lambda} & \overline{\Lambda} & 1 \otimes \text{Ker } T_2 & \frac{\overline{\Lambda} \otimes 2}{\text{Im } T_2} & \cdots \\
\end{array}
\]

\( T_n \) is defined in (1.1.6), and moreover

\[
| H^{s(d-1)}(\mathcal{L}X, k) | = | 1 \otimes \text{Ker } T_s | = | \frac{\overline{\Lambda} \otimes s}{\text{Im } T_s} | = | H^{s(d-1)+1}(\mathcal{L}X, k) | = \frac{1}{s} \sum_{i=1}^{s} m^{(i,s)}
\]

In the above formula \((i, s)\) means the greatest common divisor of \(i\) and \(s\) and \(|\cdot|\) is the dimension as a vector space over \(k\).

**Remark 2.1.2.** The proof of this theorem is essentially identical to that in the even case given by Roos in [20].

**Proof.**

\[
H^n(\mathcal{L}X, k) = \prod_{n \geq 0} \text{Tor}^n_{H^*(X, k) \otimes H^*(X, k)}(H^*(X, k), H^*(X, k))^{N+n}
\]

(cf. the introduction). For given \(N\), the \(\text{Tor}_n\) in (2.1.2) can according to (1.1.9) only occur if \(n\) satisfies either \(n + N = dn\) or \(n + N = d(n+1)\), i.e., only if \(n = N/(d-1)\) or \(n = (N - d)/(d - 1)\), which requires \(N\) or \(N - 1\) to be divisible by \(d - 1\). In the first case the contribution to \(H^n(\mathcal{L}X, k)\) is \(1 \otimes \text{Ker } T_n\) and in the second case the contribution to \(H^N(\mathcal{L}X, k)\) is \(\overline{\Lambda} \otimes n^{+1}/\text{Im } T_{n+1}\) but the first case occurs when \(N = k(d - 1)\), for \(k = 1, 2, 3, \cdots \) and the second case occurs when \(N = k(d - 1) + 1\), for \(k = 1, 2, 3, \cdots \). This proves the first part of the theorem.

For the second part, note that it has been shown in [20] that

\[
| H^{s(d-1)}(\mathcal{L}X, k) | = | \text{Ker } T_s | = | \frac{\overline{\Lambda} \otimes s}{\text{Im } T_s} | = \frac{1}{s} \sum_{i=1}^{s} (-1)^{i(s-1)}m^{(i,s)}
\]

when \(T_s : \overline{\Lambda} \otimes s \to \overline{\Lambda} \otimes s\) is defined by \(T_s = 1 - t_s\), and where \(t_s : \overline{\Lambda} \otimes s \to \overline{\Lambda} \otimes s\) is defined by
\[ \lambda_1 \otimes \lambda_2 \otimes \ldots \ldots \otimes \lambda_s \longrightarrow (-1)^{s-1} \lambda_s \otimes \lambda_1 \otimes \ldots \ldots \otimes \lambda_{s-1}. \]

The only thing we need to do is to eliminate the sign coefficient \((-1)^{(s-1)}\) (see the proof in [20]) because now \(t_s : \Lambda^\otimes s \rightarrow \Lambda^\otimes s\) is defined by \(\lambda_1 \otimes \lambda_2 \otimes \ldots \ldots \otimes \lambda_s \longrightarrow \lambda_s \otimes \lambda_1 \otimes \ldots \ldots \otimes \lambda_{s-1}\) and has no sign.

Put \(N = \frac{1 + t + t^2 + \ldots + t^{n-1}}{n}\), where \(t\) is defined in (1.1.6), and define the map

\[ B : \Lambda^\otimes n \longrightarrow \Lambda^\otimes (n+1) \]

by

\[ B(v_1 \otimes v_2 \otimes \ldots \ldots \otimes v_n) = \sum_{i=1}^{n} 1 \otimes v_i \otimes v_{i+1} \otimes \ldots \ldots \otimes v_n \otimes v_1 \otimes \ldots \ldots \otimes v_{i-1}. \]

It is clear that

\[ 1 \otimes \ker T_n = 1 \otimes N(\Lambda^\otimes n) = B(\Lambda^\otimes n). \]

Moreover as the following lemma shows, \(B\) induces an isomorphism:

\[ H^{(d-1)n+1}(\mathcal{L}X, k) = \frac{\Lambda^\otimes n}{\text{Im } T_n} \subseteq \text{Tor}_{n-1}^\ast(X, k) \otimes H^\ast(X, k)(H^\ast(X, k), H^\ast(X, k)) \]

\[ \longrightarrow \frac{\Lambda^\otimes n}{\text{Im } T_n} \prod 1 \otimes \ker T_{n-1} \]

\[ H^{(d-1)n}(\mathcal{L}X, k) = 1 \otimes \ker T_n \subseteq \text{Tor}_{n}^\ast(X, k) \otimes H^\ast(X, k)(H^\ast(X, k), H^\ast(X, k)) \]

\[ = \frac{\Lambda^\otimes n + 1}{\text{Im } T_{n+1}} \prod 1 \otimes \ker T_n \]

where \(X = S^d \sqcup S^d\).

**Lemma 2.1.3.** Let \(\{B(\alpha_1), B(\alpha_2), \ldots, B(\alpha_l)\}\) be a basis for

\[ H^{(d-1)n}(\mathcal{L}X, k) = 1 \otimes \ker T_n \]

as a vector space over \(k\), then \(\{[\alpha_1], [\alpha_2], \ldots, [\alpha_l]\}\) can be chosen as a basis for

\[ H^{(d-1)n+1}(\mathcal{L}X, k) = \frac{\Lambda^\otimes n}{T_n} \]

as a vector space over \(k\).
Proof. It is enough to prove that $[\alpha_1], [\alpha_2], \ldots, [\alpha_l]$ are linearly independent. If

$$c_1[\alpha_1] + c_2[\alpha_2] + \cdots + c_l[\alpha_l] = 0,$$

then

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_l\alpha_l \in \text{Im } T_n$$

and hence

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_l\alpha_l = (1 - t)\alpha \quad \text{for some } \alpha.$$

Thus

$$c_1B\alpha_1 + c_2B\alpha_2 + \cdots + c_lB\alpha_l = B(1 - t)\alpha = 1 \otimes N((1 - t)\alpha) = 0$$

which implies that

$$c_1 = c_2 = \cdots = c_l = 0.$$

Lemma 2.1.4.

$$B(\alpha) = 0 \Rightarrow [\alpha] = 0 \text{in } \frac{\widetilde{\Lambda}^\otimes n}{T_n}.$$

Proof. We have

$$B(\alpha) = 1 \otimes N(\alpha) = 0 \Rightarrow N(\alpha) = 0$$

but

$$\widetilde{\Lambda}^\otimes n = N(\Lambda^\otimes n) \oplus (1 - t)(\Lambda^\otimes n),$$

hence

$$N(\alpha) = 0 \Rightarrow \alpha \in (1 - t)(\Lambda^\otimes n).$$

Lemma 2.1.5. The spectral sequence:

$$E_2^{p,q} = \text{Tor}_p^{H^*}(X \times X, k) \text{(H}^*(X, k), H^*(X; k))^q \Rightarrow H^n(LX, k)$$

degenerates in the case

$$X = S^d \vee S^d \cdots \vee S^d \quad (d, \text{odd})$$

over a field $k$ of arbitrary characteristic.

Proof. The elements of $H^*(X, k)$ are concentrated in degree $d$ and hence by (1.1.9), the elements of
are concentrated in degrees \( dp \) and \( dp + d \). This shows that only the terms
\[
E_2^{-p, dp} = \text{Tor}_p^{H^*(X, k) \otimes H^*(X, k)}(H^*(X, k), H^*(X, k))^{dp}
\]
and
\[
E_2^{-p, dp+d} = \text{Tor}_p^{H^*(X \times X, k)}(H^*(X, k), H^*(X, k))^{dp+d}
\]
are non zero. Note that the spectral sequence is \((E_r, d_r)\) where \( d_r \) has bidegree \((-r, r-1)\). Consider the two complexes
\[
(2.1.3) \quad 0 = E_r^{-p+r, dp-r+1} d_r \rightarrow E_r^{-p, dp} d_r \rightarrow E_r^{-p-r, dp+r-1} = 0
\]
and
\[
(2.1.4) \quad 0 = E_r^{-p+r, dp+d-r+1} d_r \rightarrow E_r^{-p, dp+d} d_r \rightarrow E_r^{-p-r, dp+d+r-1} = 0
\]
Since (2.1.3) and (2.1.4) hold for all \( r \geq 2 \), we have
\[
E_2^{-p, q} = E_\infty^{-p, q} \text{ for all } p, q
\]
and hence the spectral sequence degenerates.

Lemma 2.1.5, together with Theorem 1.2.1, which asserted that the dimension of the invariant subspace under the action of
\[
l(v_1 \otimes v_2 \otimes \ldots \otimes v_n) = v_n \otimes v_1 \otimes \ldots \otimes v_{n-1}
\]
is independent of the characteristic of the field, together with (1.1.9) imply that
\[
H^*(\mathcal{L}(S^d \bigvee S^d), Z)
\]
has no torsion part at all.

By the same argument as above it can easily be seen that
\[
H^*(\mathcal{L}(S^g \bigvee S^g \ldots \bigvee S^g), k) \quad (g, \text{odd})
\]
has no torsion part at all.

2.2. The ring structure of \( H^*(\mathcal{L}(S^d \bigvee S^d), k)(d \geq 3, \text{odd}) \).

2.2.1. Global observations.

Lemma 2.2.1. Let \( X = S^d \bigvee \ldots \bigvee S^d \) be a wedge of \( m, d-\)spheres \((d \geq 3, \text{odd})\) and let \( \Lambda = H^*(X, k) \). Then the shuffle product on the level of the standard free resolution (1.1.1) is given by
\[ \lambda[a_1 \otimes a_2 \otimes \ldots \otimes a_p] \ast \mu[a_{p+1} \otimes a_{p+2} \otimes \ldots \otimes a_{p+q}] \]

\[ = \sum_{(p,q)-\text{shuffles}(\sigma)} \lambda\mu[a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \otimes \ldots \otimes a_{\sigma^{-1}(p+q)}] \]

**Proof.** All elements in \( \bar{\Lambda} = \text{coker}(k \to \Lambda) \) are concentrated in degree \( d \) and \( d \) is odd, so

\[ s(\sigma) = \sum (\deg c_i + 1)(\deg c_{p+j} + 1) = \sum (d+1)(d+1). \]

This implies \((-1)^{s(\sigma)} = 1.\)

**Theorem 2.2.2.** Let \( Q^* \) denote the indecomposables in \( H^*(\mathcal{L}(S^d \vee S^d), k), \)

\( d \geq 3, \) odd. The elements \( Y_1, Y_2, \ldots, Y_s \) are representatives of a basis of \( Q^{(d-1)n+1} \) if and only if the elements \( X_1 = B(Y_1), X_2 = B(Y_2), \ldots, X_s = B(Y_s) \)

are representatives of a basis of \( Q^{(d-1)n}. \)

**Proof.** We prove this theorem by induction. We have two generators \( X_1, X_2 \) in degree \((d-1)\) and two generators \( Y_1, Y_2 \) in degree \( d \) that satisfy

\[ X_1 = B(Y_1) \quad \text{and} \quad X_2 = B(Y_2). \]

Assume that the claim is true for \( n \leq p. \) We prove that

\[ X = B(Y) \text{ represents an indecomposable in } H^{(d-1)(p+1)}(\mathcal{L}(S^d \vee S^d), k) \]

if and only if

\[ Y \text{ represents an indecomposable in } H^{(d-1)(p+1)+1}(\mathcal{L}(S^d \vee S^d), k). \]

To prove this, assume that \( X = B(Y) \) represents an indecomposable in degree \((d-1)(p+1)\) but \( Y \) does not represent an indecomposable in degree \((d-1)(p+1) + 1, \) i.e.,

\[ Y = \sum a_{k_1, k_2, \ldots, k_t} X_1^{k_1} X_2^{k_2} \ldots X_t^{k_t} Y_j \]

where

\[ \deg X_i \leq (d-1)p, \deg Y_j \leq (d-1)p + 1 \quad \text{and} \]

\[ X_i = B(Y_i) \quad \text{for all} \quad i = 1, 2, 3, \ldots, t. \]

Since

\[ B(Y') \ast B(Y) = B(Y' \ast B(Y)) = B(B(Y') \ast Y) \text{ (cf. [13]),} \]
we obtain that
\[
X = B(Y) = \sum a_{k_1k_2\ldots k_t} B(X_1^{k_1}X_2^{k_2}\ldots X_t^{k_t}Y_j) = \sum a_{k_1k_2\ldots k_t} X_1^{k_1}X_2^{k_2}\ldots X_t^{k_t}B(Y_j)
\]
\[
= \sum a_{k_1k_2\ldots k_t} X_1^{k_1}X_2^{k_2}\ldots X_t^{k_t}X_j.
\]

But this is a contradiction to the fact that \(X\) was an indecomposable element. To prove the converse, assume that \(Y\) appears as an indecomposable in degree \((d-1)(p+1)+1\) but \(X = B(Y)\) does not appear as an indecomposable in degree \((d-1)(p+1)\), i.e.,
\[
X = \sum a_{k_1k_2\ldots k_t} X_1^{k_1}X_2^{k_2}\ldots X_t^{k_t},
\]
where
\[
\deg X_i \leq (d-1)p \quad \text{and} \quad X_i = B(Y_i) \text{ for all } i = 1, 2, 3, \ldots, t.
\]

This implies that
\[
B(Y) = \sum a_{k_1k_2\ldots k_t} B(Y_1)^{k_1}B(Y_2)^{k_2}\ldots B(Y_t)^{k_t}
\]
\[
= \sum a_{k_1k_2\ldots k_t} B[Y_1B(Y_1)^{k_1-1}B(Y_2)^{k_2}\ldots B(Y_t)^{k_t}]
\]
and hence
\[
Y = \sum a_{k_1k_2\ldots k_t} Y_1^{k_1-1}Y_2^{k_2}\ldots Y_t^{k_t},
\]
which is a contradiction.

We have used the following

**Lemma 2.2.3.** Let \(\alpha \in V^\otimes n\) and \(\gamma \in V^\otimes m\), then
\[
\alpha \ast B(\gamma) = B(\alpha) \ast \gamma
\]

**Proof.** (cf. [13]). This lemma has been proved in the non graded case, i.e., when \(t : V^\otimes n \to V^\otimes n\) is defined by
\[
t(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = (-1)^{n-1}v_n \otimes v_1 \otimes \cdots \otimes v_{n-1}
\]
and
\[
B : V^\otimes n \to V^\otimes (n+1)
\]
by
$B(v_1 \otimes v_2 \otimes \cdots \otimes v_n)$

$$= \sum_{i=1}^{n} (-1)^{i-1} \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \otimes v_1 \otimes \cdots \otimes v_{i-1}.$$ 

Here we have a similar proof with no sign coefficient.

Denote the ring structure of $H^2*(\mathcal{L}(S^d \vee S^d), k)$ by $R$ and denote $H^{odd}(\mathcal{L}(S^d \vee S^d), k)$ by $M$. Then $M = S^{-1}R$ (where $\bar{R} = R/k$ and $(S^{-1}R)^p = (\bar{R})^{p-1}$) is an $R$ module with shuffle product and we have

**Theorem 2.2.4.** Let $X = S^d \vee S^d$ be a wedge of 2, $d$--spheres ($d \geq 3$, odd), then the ring structure of $H^*(\mathcal{L}X, k)$ is $R \oplus s^{-1}\bar{R}$, the trivial extension of $R$ by $s^{-1}\bar{R}$ (see Definition 2.2.11 below).

**Proof.** Both $R = H^2*(\mathcal{L}(S^d \vee S^d), k)$ and $s^{-1}\bar{R} = H^{odd}(\mathcal{L}(S^d \vee S^d), k)$ are $R$ modules with respect to the shuffle product. The relation

$$B(B(Y) \ast Y') = B(Y) \ast B(Y')$$

for $B(Y) \in R$ and $Y' \in M$ shows that the map:

$$H^{odd}(\mathcal{L}(S^d \vee S^d), k) \xrightarrow{B} H^2*(\mathcal{L}(S^d \vee S^d), k)$$

is an $R$ module isomorphism.

**Remark 2.2.5.** Theorem 2.2.4 is true even for $X = S^{d_1} \vee S^{d_2} \vee \cdots \vee S^{d_n}$ ($d_i$, odd).

Although we are not able to compute the entire ring structure of $R \oplus s^{-1}\bar{R}$ we prove the following lemma.

**Lemma 2.2.6.** All relations involving elements of odd degree, except commutators of elements of odd degree, are induced by $B^{-1}$ from relations involving only elements of even degree.

**Proof.** Any relation not of the form $Y_iY_j$ in odd degrees must be of the form

$$(*) \quad \sum a_{k_1,k_2,\ldots,k_i} x_1^{k_1} x_2^{k_2} \cdots x_i^{k_i} Y_j = 0$$

If we set $X_j = B(Y_j)$, then the relation $(*)$ is induced by the relation

$$\sum a_{k_1,k_2,\ldots,k_i} x_1^{k_1} x_2^{k_2} \cdots x_i^{k_i} X_j = 0$$

**Theorem 2.2.7.** If $H^2*(\mathcal{L}X, k) \cong T(W)/I$, then $H^*(\mathcal{L}X, k)$ is isomorphic as a ring to
\[
\frac{T(W \oplus W')}{(I + I' + W'^2 + C)},
\]

where \( W' = B^{-1}(W), I' = 1 \otimes B^{-1}(I) \) and \( C = \{XY - YX | X \in W, Y \in W'\} \).

**Proof.** The map \( B : W' \to W \) induces the following diagram:

\[
\begin{array}{ccc}
I' & \to & T(W) \\
\downarrow & & \downarrow 1 \otimes B \\
I & \to & T(W)
\end{array}
\]

Here \( i \) and \( i' \) are inclusion maps and \( 1 \otimes B : W^\otimes n \otimes W' \to W^\otimes n \otimes W \). Now if the element

\[
\sum a_{k_1k_2\ldots k_r}X_1^{k_1}X_2^{k_2}\ldots X_i^{k_i}X_j
\]

lies in \( I \), then the above diagram shows that the corresponding element, i.e.,

\[
\sum a_{k_1k_2\ldots k_r}X_1^{k_1}X_2^{k_2}\ldots X_i^{k_i}Y_j
\]

lies in \( I' \).

2.2.2. Explicit low-dimensional calculations.

**Theorem 2.2.8.** Let \( X = S^d \vee S^d \) be a wedge of 2, \( d \)-spheres \((d \geq 3, \text{odd})\), then the ring structure of \( H^*(L'X, k) \) in low dimensions (up to degree \( 12(d - 1) - 1 \)) is of the form

\[
R = \frac{k[X_1, X_2, \ldots, X_{76}, Y_1, Y_2, \ldots, Y_{76}]}{(I)}
\]

where \( 2, 1, 4, 9, 8, 20, 32 \) of \( X_i \)'s are respectively in degrees

\((d - 1), 4(d - 1), 6(d - 1), 8(d - 1), 9(d - 1), 10(d - 1) \text{ and } 11(d - 1)\)

and \( 2, 1, 4, 9, 8, 20, 32 \) of \( Y_i \)'s are respectively in degrees

\((d - 1) + 1, 4(d - 1) + 1, 6(d - 1) + 1, 8(d - 1) + 1, 9(d - 1) + 1, 10(d - 1) + 1 \text{ and } 11(d - 1) + 1\)

and where

\( I = \{Y_iY_j(i \leq j), (X_iY_j - X_jY_i)(i < j)\} \).

In other words, \( I \) is generated by the relations of the form \( Y_iY_j \) and the relations that make the following matrix symmetric:
\[ A = \begin{pmatrix}
X_1 Y_1 & X_1 Y_2 & \cdots & X_1 Y_n \\
X_2 Y_1 & X_2 Y_2 & \cdots & X_2 Y_n \\
\vdots & \vdots & \ddots & \vdots \\
X_n Y_1 & X_n Y_2 & \cdots & X_n Y_n
\end{pmatrix} \]

**Remark 2.2.9.** In Theorem 2.2.8 we have chosen the algebra generators \(X_i\)'s in degree \((d - 1)n\) and the algebra generators \(Y_i\)'s in degree \((d - 1)n + 1\) such that \(X_i = B(Y_i)\) which is possible by Theorem 2.2.2.

**Proof of Theorem 2.2.8.** Put \(H(S^d \vee S^d, k) = k + \tilde{\Lambda}\), then \(\tilde{\Lambda} = \langle x, y \rangle\) is a vector space of dimension 2 and both \(x\) and \(y\) are in degree \(d\). By Theorem 2.1.1, we obtain

\[(2.2.2)\]

\[
\begin{array}{cccccccc}
H^*(\mathcal{L}X, k) & k & 1 \otimes \tilde{\Lambda} & \tilde{\Lambda} & 1 \otimes \ker T_2 & \frac{\tilde{\Lambda} \otimes 2}{\text{Im} T_2} & 1 \otimes \ker T_3 & \frac{\tilde{\Lambda} \otimes 3}{\text{Im} T_3} & \cdots \\
\hline
\text{deg} & 0 & d - 1 & d & 2(d - 1) & 2(d - 1) + 1 & 3(d - 1) & 3(d - 1) + 1 & \cdots \\
\text{dim} & 1 & 2 & 2 & 3 & 4 & 4 & 4 & \cdots \\
\end{array}
\]

Recall that \(\ker T_n\) in (2.2.2) is exactly the vector space \(\left(\frac{\tilde{\Lambda} \otimes n}{nZ}\right)^\mathbb{Z}\) of invariants in \(\tilde{\Lambda} \otimes n\) for the group \(\frac{Z}{nZ}\) acting through its generator \(t_n = 1\) by

\[ t_n(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = v_n \otimes v_1 \otimes \cdots \otimes v_{n-1}. \]

Hence

\[ \ker T_n = \left(\frac{\tilde{\Lambda} \otimes n}{nZ}\right)^\mathbb{Z} = \frac{1 + t_n + t_n^2 + \cdots + t_n^{n-1}}{n} \tilde{\Lambda} \otimes n. \]

Consider \(H^*(\mathcal{L}(S^d \vee S^d), k)\) in even degrees and denote it by \(H^{2*}(\mathcal{L}X, k)\). In low dimensions we have:

\[(2.2.3)\]

\[
\begin{array}{cccccccc}
H^{2*}(\mathcal{L}X, k) & k & 1 \otimes \tilde{\Lambda} & 1 \otimes \ker T_2 & 1 \otimes \ker T_3 & 1 \otimes \ker T_4 & 1 \otimes \ker T_5 & 1 \otimes \ker T_6 & \cdots \\
\hline
\text{deg} & 0 & d - 1 & 2(d - 1) & 3(d - 1) & 4(d - 1) & 5(d - 1) & 6(d - 1) & \cdots \\
\text{dim} & 1 & 2 & 3 & 4 & 6 & 8 & 14 & \cdots \\
\end{array}
\]

Start with \(R_{d-1} = k[X_1, X_2]\), where \(X_1, X_2\) are the two basis elements of \(H^{d-1}(\mathcal{L}(S^d \vee S^d), k)\) that commute in \(R_{d-1}\) and contain no other relations. We have chosen the notation \(R_{d-1}\) because this ring contains all generators up to degree \(d - 1\). Denote the Hilbert series of \(R_{d-1}\) by \(H_{R_{d-1}}(t)\). We have

\[(2.2.4)\]

\[ H_{R_{d-1}}(t) = \frac{1}{(1 - t^4)^2} = 1 + 2t^4 + 3t^8 + 4t^{12} + 5t^{16} + \cdots. \]
Comparing (2.2.3) and (2.2.4) shows that a generator of degree $4(d - 1)$ is needed. Call this new generator $X_3$ and add it to $R_{d-1}$. Denote the new ring by $R_{4(d-1)}$ then

$$R_{4(d-1)} = k[X_1, X_2, X_3],$$

and

$$(2.2.5) \quad H_{R_{4(d-1)}}(t) = \frac{1}{(1 - t^4)^2(1 - t^{16})} = 1 + 2t^4 + 3t^8 + 4t^{12} + 6t^{16} + 8t^{20} + 10t^{24} + \cdots.$$  

Comparing (2.2.3) and (2.2.5), we realize that four new generators are needed in degree $6(d - 1)$. Continuing this way, we can find the number of generators in each degree (even degrees). This ring, i.e., $H^{2*}(\mathcal{L}X, k)$, is a subring of a polynomial ring. If it were a polynomial ring it would have: $2, 1, 4, 9, 8, 20, 32, 68, \cdots$ generators respectively in degrees:

$$(d - 1), 4(d - 1), 6(d - 1), 8(d - 1), 9(d - 1),$$

$$10(d - 1), 11(d - 1), 12(d - 1), \cdots.$$

However this is only true up to degree $11(d - 1)$ since C. Löffwall and L. Lambe have proved the existence of four relations in degree $12(d - 1)$. Moreover they have proved the non-existence of any relations in previous degrees, i.e., degrees less than $12(d - 1)$. (see [12]). It remains an open problem to find out what this ring looks like in all dimensions.

By continuing, we find the ring

$$R_{11(d-1)} = k[X_1, X_2, X_3, \ldots, X_{76}]$$

isomorphic to $H^{2*}(\mathcal{L}X, k)$ up to degree $12(d - 1) - 1$.

In order to prove that

$$R = \frac{k[X_1, X_2, X_3, \ldots, X_{76}, Y_1, Y_2, Y_3, \ldots, Y_{76}]}{(Y_iY_j[i \leq j], (X_iY_j - X_jY_i)[i < j])}$$

and $H^*(\mathcal{L}X, k)$ are isomorphic up to degree $12(d - 1)$, first we prove that there is an onto ring map $R \rightarrow H^*(\mathcal{L}X, k)$ and then that $R$ and $H^*(\mathcal{L}X, k)$ have the same Hilbert series. It is clear that $H^*(\mathcal{L}X, k)$ contains all relations $Y_iY_j[i \leq j]$, by definition of the shuffle product in $H^*(\mathcal{L}X, k)$ and also contains the relations of the form $X_iY_j - X_jY_i$, (see Lemma 2.2.3).

To prove that $R$ and $H^*(\mathcal{L}X, k)$ have the same Hilbert series up to degree $12(d - 1)$, let
\[ R'_n = K[X_1, X_2, \ldots, X_n], \]
\[ R_n = \frac{k[X_1, X_2, X_3, \ldots, X_n, Y_1, Y_2, Y_3, \ldots, Y_n]}{(Y_i Y_j[i \leq j], (X_i Y_j - X_j Y_i)[i < j])}, \]
\[ A_n = \frac{k[X_1, X_2, X_3, \ldots, X_n, Y_1, Y_2, Y_3, \ldots, Y_n]}{(Y_i Y_j[i \leq j], X_i Y_j[i < j])}. \]

Considering all \( X_i \)'s and \( Y_j \)'s as of degree 1 we get
\[ \text{Hilb}_{R_n}(t) = \sum h_{k,n} t^k = \frac{1}{(1-t)^n}, \quad h_{k,n} = \binom{n+k-1}{n-1}. \]

It can easily be seen that \( (Y_i Y_j[i \leq j], (X_i Y_j - X_j Y_i)[i < j]) \) is a Groebner basis, so
\[ \text{Hilb}_{R_n}(t) = \text{Hilb}_{A_n}(t) = \sum h_k t^k, \quad h_0 = 1. \]

The monomials of degree \( k \) in \( A_n \) are of the following types:

\[ M_0 = \text{monomials of degree } k \text{ in } \{X_1, X_2, \ldots, X_n\} \quad \text{dim } M_0 = h_{k,n} \]
\[ M_1 = Y_1, \text{ monomials of degree } k-1 \text{ in } \{X_1, X_2, \ldots, X_n\} \quad \text{dim } M_1 = h_{k-1,n} \]
\[ M_2 = Y_2, \text{ monomials of degree } k-1 \text{ in } \{X_2, X_3, \ldots, X_n\} \quad \text{dim } M_2 = h_{k-1,n-1} \]
\[ M_3 = Y_3, \text{ monomials of degree } k-1 \text{ in } \{X_3, X_4, \ldots, X_n\} \quad \text{dim } M_3 = h_{k-1,n-2} \]
\[ \ldots \]
\[ M_n = Y_n, \text{ monomials of degree } k-1 \text{ in } \{X_n\} \quad \text{dim } M_n = h_{k-1,1} = 1 \]

and hence
\[ h_k = h_{k,n} + h_{k-1,1} + h_{k-1,2} + \cdots + h_{k-1,n} \]

(2.2.6)

It is also easy to see that
\[ h_{k-1,1} + h_{k-1,2} + \cdots + h_{k-1,n} = h_{k,n} \]

(2.2.7)

By (2.2.6) and (2.2.7) we get \( h_k = 2h_{k,n} \) for \( k \geq 1 \) and hence
\[ \text{Hilb}_{R_n}(t) = \text{Hilb}_{A_n}(t) = \sum h_k t^k = \frac{2}{(1-t)^n} - 1. \]

This shows that \( R_n \) and \( H^*(\mathcal{L}X, k) \) have the same Hilbert series, up to some degree. The above argument works, however, for any \( R_n(n \geq 1) \), so
\[ R = \frac{k[X_1, X_2, X_3, \ldots, X_{76}, Y_1, Y_2, Y_3, \ldots, Y_{76}]}{(Y_i Y_j[i \leq j], (X_i Y_j - X_j Y_i)[i < j])} \quad \text{and } \quad H^*(\mathcal{L}X, k) \]

have the same Hilbert series up to degree 12(d - 1) - 1.

In this part we are going to show that the ring
\[ R = \frac{k[X_1, X_2, X_3, \ldots, X_{76}, Y_1, Y_2, Y_3, \ldots, Y_{76}]}{(Y_i Y_j[i \leq j], (X_i Y_j - X_j Y_i)[i < j])}, \]

i.e., \( H^*(\mathcal{L}X, k) \) up to degree \( 12(d - 1) - 1 \) is a homogeneous Koszul algebra (see Definition 2.2.10 below). In order to do this we need the following.

**Definition 2.2.10.** Let \( R \) be an algebra over \( k \). We call \( R \) a homogeneous Koszul algebra, if the ring \( \text{Ext}^*_R(k, k) \) is generated by \( \text{Ext}^1_R(k, k) \).

**Definition 2.2.11.** Let \( R \) be a local ring and let \( M \) be an \( R \)-module. The ring \( R \oplus M \) whose elements consist of pairs \((r, m)\) with addition componentwise and multiplication

\[(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)\]

is called the trivial extension of \( R \) by \( M \).

**Definition 2.2.12.** We say that the bigraded ring \( R \) is a Koszul algebra (with respect to the first degree) up to degree \( \leq n \) (with respect to the second degree), if there is a ring \( K \) (\( K \) a Koszul algebra) (with respect to the first degree) and a ring homomorphism \( K \to R \) that is isomorphism up to degree \( \leq n \) (with respect to the second degree).

**Theorem 2.2.13.** If the finitely generated ring \( R \) is a homogeneous Koszul-algebra then the trivial extension ring \( R \oplus \bar{R} \) is also a homogeneous Koszul-algebra.

**Remark 2.2.14.** We have considered the generators \( X_i s \) and \( Y_i s \) as bigraded elements with bidegree \((1, \deg X_i s)\) and \((1, \deg Y_i s)\) respectively. Now if we only consider this new degree \( 1 \) (the first degree), then \( H^*(\mathcal{L}(S^d \vee S^d), k) \) is nothing but \( R \oplus \bar{R} \). This is why we consider the ring extension \( R \oplus \bar{R} \) in Theorem 2.2.13.

**Proof of Theorem 2.2.13.** According to the Fröberg formula, [8], It is enough to show that

\[ P_R(x, y) \cdot H_{R \oplus \bar{R}}(-xy) = 1 \]

where \( P \) denotes the Poincaré-betti series and where \( H \) denotes the Hilbert series. It is clear that

\[ H_{R \oplus \bar{R}}(t) = H_R(t) + (H_R(t) - 1). \]  

Next we prove that

\[ P_{R \oplus \bar{R}}(x, y) = P_R(x, y)/(1 - (P_R(x, y) - 1)) \]
In [11] Herzog has found

\begin{equation}
(2.2.10) \quad P_{R \oplus \bar{R}}(z) = P_R(z)/(1 - zP_{\bar{R}}(z))
\end{equation}

but the exact sequence

\[ 0 \longrightarrow \bar{R} \longrightarrow R \longrightarrow k \longrightarrow 0 \]

gives

\[ \text{Ext}_R^*(\bar{R}, k) = \text{Ext}_R^{*+1}(k, k), \]

i.e.,

\[ P_{\bar{R}}^R(z) = 1/z(P_R(z) - 1). \]

This gives the proof of (2.2.9). Now using (2.2.8) and (2.2.9) we get

\begin{equation}
(2.2.11) \quad H_{R \oplus \bar{R}}(-xy) \cdot P_{R \oplus \bar{R}}(x, y) = (H_R(-xy) + H_R(-xy) - 1) \\
\quad \cdot P_R(x, y)/(1 - (P_R(x, y) - 1)) \\
\quad = (2 - P_R(x, y))/(1 - (P_R(x, y) - 1)) = 1.
\end{equation}

**Theorem 2.2.15.** The ring:

\[ k[X_1, X_2, X_3, \ldots, X_{76}, Y_1, Y_2, Y_3, \ldots, Y_{76}] \\
\quad (Y_i Y_j [i \leq j], (X_i Y_j - X_j Y_i)[i < j]) \]

which is isomorphic to \( H^*(\mathcal{L}(S^d \vee S^d), k) \) in degrees less than \( 12(d - 1) - 1 \) is a Koszul-algebra, where all generators \( X_i \) and \( Y_i \) are considered to be in degree 1.

To prove Theorem 2.2.15 we need the following:

**Definition 2.2.16.** Let \( M = \oplus_{i,j \geq 0} M^{ij} \) be bigraded vector space. We define the bigraded vector space \( s^p q M \) by putting

\[ (s^p q M)^{ij} = M^{i+p_j+q}. \]

Now if we consider the generators \( X_i \)'s of \( H^{2*}(\mathcal{L}(S^d \vee S^d), k) \) and the generators \( Y_i \)'s of \( H^{2*+1}(\mathcal{L}(S^d \vee S^d), k) \) as bigraded elements with bidegrees \((1, \deg X_i)\) and \((1, \deg Y_i)\) respectively, then using the notation of Definition 2.2.16, we have

\[ H^*(\mathcal{L}(S^d \vee S^d), k) = H^{2*}(\mathcal{L}(S^d \vee S^d), k) \oplus s^{0,1} H^{2*}(\mathcal{L}(S^d \vee S^d), k). \]

In another words the ring structure of \( H^*(\mathcal{L}(S^d \vee S^d), k) \) is the trivial extension of \( H^{2*}(\mathcal{L}(S^d \vee S^d), k) \) by \( s^{0,1} H^{2*}(\mathcal{L}(S^d \vee S^d), k) \).
PROOF OF THEOREM 2.2.15. The ring $R = H^{2*}(\mathcal{L}(S^d \vee S^d), k)$ is a polynomial ring in degrees less than $12(d - 1) - 1$ and hence a Koszul-algebra, if we consider the generators to be in degree 1. Now the result follows from Theorem 2.2.13.

We devote the last part of this section to finding an explicit formula for $\text{Ext}_R(k, k)$, when

\begin{equation}
R = \frac{k[X_1, X_2, X_3, \ldots, X_n, \ldots, Y_1, Y_2, Y_3, \ldots, Y_n, \ldots]}{(Y_i Y_j[i \leq j], (X_i Y_j - X_j Y_i)[i < j])}
\end{equation}

DEFINITION 2.2.17. If $a$ and $b$ are bigraded elements with bigrade $(s_1, t_1)$ and $(s_2, t_2)$, then the \textit{graded commutator} $[a, b]$ is defined by $ab - (-1)^{s_1 s_2 + t_1 t_2}ba$ for $a \neq b$ and $[a, a] = a^2$ or zero according to whether $s_1 + t_1$ is odd or even.

The following lemma is due to Löfwall ([14]).

LEMMA 2.2.18. Let $k[X_1, X_2, X_3, \ldots, X_n]$ denote the free graded strictly commutative algebra on the variables $X_1, X_2, X_3, \ldots, X_n$ of nonnegative degree. Let

$$f_i = \sum_{j \leq k} b_{ijk} X_j X_k,$$

be homogeneous elements. Put

$$R = \frac{k[X_1, X_2, X_3, \ldots, X_n]}{(f_1, f_2, \ldots, f_r)},$$

then

$$A = [\text{Ext}_R^1(k, k)] = \frac{k < T_1, T_2, \ldots, T_n >}{(\phi_1, \phi_2, \ldots, \phi_s)}.$$

As bigraded algebras, $T_i$ has bidegree $(1, \deg X_i)$, and

$$\phi_i = \sum_{j \leq k} c_{ijk} [T_j, T_k],$$

$c_{ijk} \in k, i = 1, 2, 3, \ldots, s.$

where $[T_j, T_k]$ is graded commutator, and $(c_{ijk})_{jk}, i = 1, 2, 3, \ldots, s$ is a basis to the solutions of the linear system

$$\sum_{j \leq k} b_{ijk} X_{jk} = 0,$$

$i = 1, 2, 3, \ldots, r$
\[
\left(\text{hence } s = \frac{n(n + 1)}{2} - r\right)
\]

**Proof.** (cf. [19] and [14]).

Using lemma 2.2.18, we easily compute \([\text{Ext}^1_R(k, k)]\), where \(R\) is the ring in (2.2.12) as the following

\[
\text{Ext}_R(k, k) = [\text{Ext}^1_R(k, k)] = \left\langle k < T_1, T_2, T_3, \ldots, T_n, \ldots, S_1, S_2, S_3, \ldots, S_n, \ldots \right\rangle
\]

\((F)\),

where \(F\) is generated by all graded commutators \([T_i, T_j]\) \(i \leq j\) and those relations that make the following matrix skew symmetric, i.e., \(A = -A^T\).

\[
A = \begin{pmatrix}
T_1 S_1 & T_1 S_2 & \cdots & T_1 S_n \\
T_2 S_1 & T_2 S_2 & \cdots & T_2 S_n \\
\vdots & \vdots & \ddots & \vdots \\
T_n S_1 & T_n S_2 & \cdots & T_n S_n \\
\vdots & \vdots & \ddots & \vdots
\end{pmatrix}
\]

In other words, \(F\) is generated by all relations of the form \([T_i, S_i]\) \(i = 1, 2, 3, \ldots, n\) and \([T_i, S_j] + [T_j, S_i]\) \(i < j\)

3. **The free loop space on a wedge of even spheres.**

3.1. **Graded vector space structure and torsion.**

The following theorem is due to J. E. Roos.

**Theorem 3.1.1.** Let \(X = S^4 \vee S^4\) and put \(H(S^4 \vee S^4, k) = k + V\), where \(V = \langle x, y \rangle\) is a vector space of dimension 2 and both \(x\) and \(y\) are in degree 4, then we have the following explicit formula for \(H^*(\mathcal{L}X, k)\):

\[
\begin{array}{ccccccccc}
\text{deg} & 0 & 4 - 1 & 4 & 2(4 - 1) & 2(4 - 1) + 1 & 3(4 - 1) & 3(4 - 1) + 1 & \cdots & \cdots \\
H^*(\mathcal{L}X, k) & k & 1 \otimes V & V & 1 \otimes \text{Ker } S_2 & \frac{V \otimes^2}{\text{Im } S_2} & 1 \otimes \text{Ker } S_3 & \frac{V \otimes^3}{\text{Im } S_3} & \cdots & \cdots \\
\end{array}
\]

where \(S_n\) is defined in (1.2.2). Moreover we have
\[(3.1.2) \quad |H^{(4-1)}(\mathcal{L}X, k)| = |1 \otimes \text{Ker } S_x| = \frac{|V^{\otimes s}|}{\text{Im } S_x} \]

\[= |H^{i(4-1)+1}(\mathcal{L}X, k)| = \frac{1}{s} \sum_{i=1}^{s} (-1)^{(s-1)i} 2^{(i, s)} \]

In the above formula \((i, s)\) means the greatest common divisor of \(i\) and \(s\) and \(||\) is the dimension as a vector space over \(k\).

**Proof.** (cf. [20]).

Definition 1.1.1 implies that the shuffle product in

\[\text{Tor}_t^{H^*(X, k) \otimes H^*(X, k)}(H^*(X, k), H^*(X, k))\]

is given by

\[\lambda[a_1 \otimes a_2 \otimes \ldots \otimes a_p] \ast \mu[a_{p+1} \otimes a_{p+2} \otimes \ldots \otimes a_{p+q}] = \sum_{(p, q) \text{-shuffles} (\sigma)} \text{sign}(\sigma) \lambda \mu[a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \otimes \ldots \otimes a_{\sigma^{-1}(p+q)}],\]

when \(X = H^*(\mathcal{L}(S^4 \vee S^4), k)\), since all elements of \(H^*(S^4 \vee S^4)\) are concentrated in even degrees.

If we put:

\[N = \frac{1 + s + s^2 + \cdots + s^{n-1}}{n}\]

and define the map:

\[B : V^{\otimes n} \longrightarrow V^{\otimes n+1}\]

by

\[B(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n} (-1)^{im} 1 \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \otimes v_1 \otimes \cdots \otimes v_{i-1}.\]

Then it is clear that

\[1 \otimes \text{Ker } S_n = 1 \otimes N(V^{\otimes n}) = B(V^{\otimes n}).\]

Moreover we have
\[ H^{3n+1}(\mathcal{L}(X, k)) = \frac{V^\otimes n}{\text{Im}S_n} \subseteq \text{Tor}^{H^*(X, k) \otimes H^*(X, k)}_{n-1}(H^*(X, k), H^*(X, k)) \]
\[ = \frac{V^\otimes n}{\text{Im}S_n} \bigoplus 1 \otimes \text{Ker}(S_{n-1}) \]
\[ \downarrow \text{B} \]
\[ H^{3n}(\mathcal{L}(X, k)) = 1 \otimes \text{Ker}(S_n) \subseteq \text{Tor}_{n}^{H^*(X, k) \otimes H^*(X, k)}(H^*(X, k), H^*(X, k)) \]
\[ = \frac{V^\otimes n+1}{\text{Im}S_{n+1}} \bigoplus 1 \otimes \text{Ker}S_n \]

where \( X = S^4 \vee S^4 \).

The proofs of the following two lemmas are essentially identical to those of Lemmas 2.1.3 and 2.1.4.

**Lemma 3.1.2.** If
\[ \{B(\alpha_1), B(\alpha_2), \ldots, B(\alpha_s)\} \]
is a \( k \)-basis of
\[ H^{3n}(\mathcal{L}(S^4 \vee S^4), k) = 1 \otimes \text{Ker}S_n, \]
then
\[ \{[\alpha_1], [\alpha_2], \ldots, [\alpha_s]\} \]
is a \( k \)-basis of
\[ H^{3n+1}(\mathcal{L}(S^4 \vee S^4), k) = \frac{V^\otimes n}{S_n}. \]

**Lemma 3.1.3.**
\[ B(\alpha) = 0 \Rightarrow [\alpha] = 0 \text{in } \frac{V^\otimes n}{S_n}. \]

In the last part of this section we try to compute the torsion part in
\[ H^*(\mathcal{L}(S^4 \vee S^4), Z) \]

**Lemma 3.1.4.** The spectral sequence
\[ E_2^{p,q} = \text{Tor}^{H^*(X \times X, k)}_{p}(H^*(X, k), H^*(X; k))^q \Rightarrow H^n(\mathcal{L}X, k) \]
degenerates in the case...
\[ X = S^d \bigvee S^d \cdots \bigvee S^d \quad (d, \text{ even}) \]

over a field \( k \) of arbitrary characteristic.

**Proof.** See Lemma 2.1.5.

Theorem 1.2.2, which asserted that the dimension of invariant under the action of

\[ s(v_1 \otimes v_2 \otimes \ldots \otimes v_n) = (-1)^{n-1} v_n \otimes v_1 \otimes \ldots \otimes v_{n-1} \]

is independent of the characteristic of the field if the characteristic is not 2, together with Lemma 3.1.4 imply that

\[ H^* (\mathcal{L}(S^4 \bigvee S^4), Z) \]

has only 2-torsion part. Moreover the series of this 2-torsion can be calculated by considering the equality of formal power series, provided by the universal coefficient theorem:

\[ \sum_{i \geq 0} v_2 (H_i(\mathcal{L}X, Z)) t^i = (1 + t)^{-1} (H^{Z/2Z}_{\mathcal{L}X}(t) - H^O_{\mathcal{L}X}(t)), \]

where \( v_p(\text{Ab}) \) denotes the minimal number of generators of the \( p \)-torsion subgroup of an abelian group \( \text{Ab} \).

Theorem 3.1.1 implies that over a field of characteristic zero the Hilbert series of

\[ H^* (\mathcal{L}(S^4 \bigvee S^4), Z) \]

is

\[ H^O_{\mathcal{L}X}(t) = 1 + (1 + t) \sum_{n \geq 1} |V^\otimes n / S_n| t^{3n} \]

where

\[ |V^\otimes n / S_n| = \frac{1}{n} \sum_{i=1}^n (-1)^{i(n-1)} 2^{(i,n)} \]

and over a field of characteristic 2

\[ H^* (\mathcal{L}(S^4 \bigvee S^4), Z) \]

has the Hilbert series:

\[ H^{Z/2Z}_{\mathcal{L}X}(t) = 1 + (1 + t) \sum_{n \geq 1} |V^\otimes n / T_n| t^{3n} \]
where
\[ \left| \frac{V \otimes n}{T_n} \right| = \frac{1}{n} \sum_{i=1}^{n} 2^{(1,n)}. \]

By replacing (3.1.4) and (3.1.5) in (3.1.3) we obtain:
\[ (3.1.6) \quad \sum_{i \geq 0} v_2(H_i(\mathcal{L} X, Z))t^i = \sum_{n \geq 1} \left( \left| \frac{V \otimes n}{T_n} \right| - \left| \frac{V \otimes n}{S_n} \right| \right) t^{3n} \]

and this can be written as
\[ \sum_{i \geq 0} v_2(H_i(\mathcal{L} X, Z))t^i = \sum_{n \geq 2, \text{even}} \frac{2}{n} (2 + 2^{(3,n)} + \cdots + 2^{(n-1,n)}) t^{3n}. \]

By the same argument as above it can easily be seen that
\[ H^*(\mathcal{L}(S^g \vee S^g \cdots \vee S^g), k) \quad (g, \text{ even}) \]

has only 2-torsion.

3.2. The ring structure of \( H^*(\mathcal{L}(S^4 \vee S^4), k) \).

3.2.1. Global observations.

**Theorem 3.2.1.** Let \( Q^* \) denote the indecomposables in \( H^*(\mathcal{L} S^4 \vee S^4), k \). The elements \( Y_1, Y_2, \ldots, Y_s \) are representatives of a basis of \( Q^{3n+1} \) if and only if the elements \( X_1 = B(Y_1), X_2 = B(Y_2), \ldots, X_s = B(Y_s) \) are representatives of a basis of \( Q^{3n} \).

**Proof.** See the proof of Theorem 2.2.2.

Denote the ring \( H^{3*}(\mathcal{L}(S^d \vee S^d), k) \) by \( R \) and denote \( H^{3*+1}(\mathcal{L}(S^d \vee S^d), k) \) by \( M \). Then \( M = s^{-1} \bar{R} \) (where \( \bar{R} = R/k \) and \( (s^{-1} \bar{R})^p = (\bar{R})^{p-1} \)) is an \( R \) module (using shuffle product) and we have

**Theorem 3.2.2.** Let \( X = S^4 \vee S^4 \), then the ring structure of \( H^*(\mathcal{L} X, k) \) is \( R \oplus s^{-1} \bar{R} \), i.e., the trivial extension of \( R \) by \( s^{-1} \bar{R} \), where \( R = H^{3*}(\mathcal{L}(S^4 \vee S^4), k) \).

**Proof.** See the proof of Theorem 2.2.4.

**Remark 3.2.3.** Theorem 3.2.2 is true even for \( X = S^{d_1} \vee S^{d_2} \vee \cdots \vee S^{d_a}(d_i, \text{ even}) \).

Although we are not able to compute the entire ring structure of \( R \oplus s^{-1} \bar{R} \) we have the following result, the proof of which is essentially identical to that of Lemma 2.2.6.
Lemma 3.2.4. All relations involving elements of degree 3* + 1 except commutators of elements of degree 3* + 1, are induced by $B^{-1}$ from relations involving only elements of degree 3*.

Theorem 3.2.5. If

$$H^{3*}(\mathcal{L}(S^4 \vee S^4), k) \cong T(W)/I.$$ 

Then $H^*(\mathcal{L}(S^4 \vee S^4), k)$ is isomorphic as a ring to

$$\frac{T(W \oplus W')}{(I + I' + W'^2 + C)}$$

where $W' = B^{-1}(W), I' = 1 \otimes B^{-1}(I)$ and $C = \{XY - YX | X \in W, Y \in W'\}$.

Proof. See the proof of Theorem 2.2.7.

Remark 3.2.6. As a result of this section we show that

$$(3.2.1) \quad \overline{HC}_*(A) \text{ and } H^{3*}(\mathcal{L}(S^4 \vee S^4), k)$$

have the same ring structure. Here $A = k + V$ is the trivial extension of $k$ by $V$, where $V$ is a vector space of dimension 2 and $\overline{HC}_*(A)$ is the reduced cyclic homology of $A$.

In [13], Loday and Quillen have defined a product:

$$HC_n(A) \otimes HC_p(A) \longrightarrow HC_{n+p+1}(A)$$

for a commutative $k$–algebra $A$ as follows:

Let

$$x \in (\beta(A)_{\text{norm}})_{lm} = A \otimes \overline{A}^{m-l} \quad \text{and} \quad y \in (\beta(A)_{\text{norm}})_{rs} = A \otimes \overline{A}^{s-r}$$

where $\overline{A} = A/k$ and where $\beta(A)_{\text{norm}}$ is a double complex with:

$$HC_n(A) = H_n(\text{Tot} \beta(A)_{\text{norm}})$$

(c.f. [13], page 571). Define the product as:

$$(3.2.2) \quad x \cdot y = \begin{cases} x \ast B(y) & \text{if } r = 0 \\ 0 & \text{otherwise.} \end{cases}$$

(Note that $x \cdot y \in (\beta(A)_{\text{norm}})_{l+r,m+s+1}$) Then this formula is extended to Tot $\beta(A)_{\text{norm}} \otimes \text{Tot} \beta(A)_{\text{norm}}$ by linearity. In (3.2.2) $\ast$ is the shuffle product and $B : A \otimes A^\otimes n \longrightarrow A \otimes A^\otimes n+1$ is defined by:
\[(3.2.3) \quad B(a_0 \otimes a_1 \otimes \cdots \otimes a_n) \]
\[= \sum_{i=1}^{n} (-1)^{in_1} a_i \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}.\]

**Lemma 3.2.7.**

\[B(x \ast B(y)) = B(x) \ast B(y).\]

**Proof.** ([13]. page 575).

This Lemma shows that the product structure on cyclic homology is compatible with the shuffle product on Hochschild homology on the sense that the following diagram commutes.

\[(3.2.4) \quad \begin{array}{ccc}
HC_n(A) \otimes HC_m(A) & \xrightarrow{B \otimes B} & HH_{n+1}(A) \otimes HH_{m+1}(A) \\
\downarrow \ast & & \downarrow \ast \\
HC_{n+m+1}(A) & \xrightarrow{B} & HH_{n+m+2}(A)
\end{array}\]

In the above diagram \(\ast\) is the product defined in (3.2.2), \(\ast\) is the shuffle product and the map \(B\) is the same map in the long exact sequence:

\[(3.2.5) \quad \cdots \cdots \rightarrow HH_{n}(A) \xrightarrow{1} HC_{n}(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \rightarrow \cdots \]

The cyclic homology \(HC_\ast(A)\) and the reduced cyclic homology \(\overline{HC}_\ast(A)\) of \(A = k + V\), the trivial extension of \(k\) by \(V\) has been computed by many authors e.g. [13] as follows:

\[HC_n(A) = \begin{cases} 
  k + V^{\otimes n+1} / \text{Im} S_{n+1} & \text{if } n = 2t \\
  V^{\otimes n+1} / \text{Im} S_{n+1} & \text{otherwise}
\end{cases}\]

and

\[\overline{HC}_\ast(A) = \frac{V^{\otimes n+1}}{\text{Im} S_{n+1}}\]

where \(S_n : V^{\otimes n} \rightarrow V^{\otimes n}\) is defined in (1). Note that by definition of the product \(\ast\) in (3.2.2) we have:

\[(3.2.6) \quad x \in k \subseteq HC_2(A) t \geq 1 \Rightarrow x \ast y = 0 \quad \forall \ y \in HC_\ast(A).\]

Consider the following two tables:
\[(3.2.7)\]

<table>
<thead>
<tr>
<th>$V$</th>
<th>$V^{\otimes 2}$</th>
<th>$V^{\otimes 3}$</th>
<th>$V^{\otimes 4}$</th>
<th>$V^{\otimes 5}$</th>
<th>$V^{\otimes 6}$</th>
<th>$V^{\otimes 7}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Im } S_2$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>$\text{Im } S_3$</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td>8</td>
<td>10</td>
</tr>
</tbody>
</table>

\[(3.2.8)\]

| $H^3\text{( }\mathcal{L}(S^4 \vee S^4), k)\text{ )}$ | $1 \otimes V$ | $1 \otimes \text{Ker } S_2$ | $1 \otimes \text{Ker } S_3$ | $1 \otimes \text{Ker } S_4$ | $1 \otimes \text{Ker } S_5$ | $1 \otimes \text{Ker } S_6$ | $1 \otimes \text{Ker } S_7$ |
| deg | 3 | 6 | 9 | 12 | 15 | 18 | 21 | $\ldots \ldots$ |
| dim | 2 | 1 | 4 | 4 | 8 | 10 | 20 | $\ldots \ldots$ |

and the product:

\[(3.2.9)\]

$$
\frac{V^{\otimes n+1}}{\text{Im } S_{n+1}} \otimes \frac{V^{\otimes m+1}}{\text{Im } S_{m+1}} \bullet \rightarrow \frac{V^{\otimes n+m+2}}{\text{Im } S_{n+m+2}}
$$

Let $z$ be an element in the image of this product ($\bullet$ was defined in (3.2.2)), i.e., $z = x \bullet y$ for some $x \in \frac{V^{\otimes n+1}}{\text{Im } S_{n+1}}$ and $y \in \frac{V^{\otimes m+1}}{\text{Im } S_{m+1}}$

$$
z = x \bullet y = x * B(y).
$$

By Lemma 3.2.7:

$$
B(z) = B(x \bullet y) = B(x * B(y)) = B(x) * B(y).
$$

This shows that $B(z)$ is in the image of the product:

\[(3.2.10)\]

$$
(1 \otimes \text{Ker } S_{n+1}) \otimes (1 \otimes \text{Ker } S_{m+1}) * \rightarrow (1 \otimes \text{Ker } S_{n+m+2}),
$$

where $*$ is the shuffle product. Conversely if $B(z)$ is an element in the image of the product in (3.2.10), then $z$ is in the image of the product in (3.2.9). This shows that

$$
\overline{HC}_*(A) \text{ and } H^3\text{( }\mathcal{L}(S^4 \vee S^4), k)\text{ )}
$$

have the same ring structure.

3.2.2. Explicit low-dimensional calculations.

**Theorem 3.2.8.** The ring $R = H^3\text{( }\mathcal{L}(S^4 \vee S^4), k)$ is isomorphic in degrees less than 14 to the ring

\[(3.2.11)\]

$$
R = \frac{k[a, b, c, d, e, f, g]}{(I)}
$$

where $\deg a = \deg b = 4, \deg c = \deg d = \deg e = \deg f = 9 \text{ and } \deg g = 12,$
and where \( I \) is generated by the following 11 elements:

\[ a^2, b^2, c^2, d^2, e^2, f^2, ac, bf, (ad - bc), (ae - bd), (af - be). \]

**Proof.** Recall that \( H^*(\mathcal{L}(S^4 \vee S^4), k) \) has the following table

<table>
<thead>
<tr>
<th>( H^*(\mathcal{L}X, k) )</th>
<th>( k )</th>
<th>( 1 \otimes V )</th>
<th>( V )</th>
<th>( 1 \otimes \text{Ker} S_2 )</th>
<th>( V^{\otimes 2} / \text{Im} S_2 )</th>
<th>( 1 \otimes \text{Ker} S_3 )</th>
<th>( V^{\otimes 3} / \text{Im} S_3 )</th>
<th>( 1 \otimes \text{Ker} S_4 )</th>
<th>( V^{\otimes 4} / \text{Im} S_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>deg</strong></td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>7</td>
<td>9</td>
<td>10</td>
<td>12</td>
<td>13</td>
</tr>
<tr>
<td><strong>dim</strong></td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

In the above table \( H(S^4 \vee S^4, k) = k + V, \) where \( V = \langle x, y \rangle \) is a vector space of dimension 2 and both \( x \) and \( y \) are in degree 4. We consider \( H(S^4 \vee S^4, k) \) in dimensions 3, 6, 9, 12.

\[ H^3 \ast H^3 \rightarrow H^6 \]

\[ H^3 = \langle a = 1 \otimes x, b = 1 \otimes y \rangle, \quad H^6 = \left\langle \frac{1 \otimes x \otimes y - 1 \otimes y \otimes x}{2} \right\rangle \]

(3.2.12) \( (1 \otimes x) \ast (1 \otimes x) = 0 \) \( (1 \otimes y) \ast (1 \otimes y) = 0 \)

\( (1 \otimes x) \ast (1 \otimes y) = 1 \otimes x \otimes y - 1 \otimes y \otimes x \)

In (3.2.12) we have used the shuffle product

\[ (1 \otimes a_1) \ast (1 \otimes a_2) = 1 \otimes a_1 \otimes a_2 - 1 \otimes a_2 \otimes a_1. \]

The product is onto and no generator in degree six is needed.

\[ H^9 = \langle c = 1 \otimes x \otimes x \otimes x, \quad d = \frac{1 \otimes x \otimes x \otimes y + 1 \otimes y \otimes x \otimes x + 1 \otimes x \otimes y \otimes x}{3}, \]

\[ e = \frac{1 \otimes x \otimes y \otimes y + 1 \otimes y \otimes x \otimes y + 1 \otimes y \otimes y \otimes x}{3}, \quad f = 1 \otimes y \otimes y \otimes y \rangle. \]

The fact that \( a^2 = (1 \otimes x)^2 = 0 \) and \( b^2 = (1 \otimes y)^2 = 0 \), implies that

\[ H^3 \ast H^6 = H^3 \ast H^3 \ast H^3 = 0 \]

and hence no element of degree 9 is obtained by the elements of previous degrees. So we need four generators \( a, b, c, \) and \( d \) in degree 9. By an easy calculation the shuffle product.
\[(1 \otimes a_1 \otimes a_2 \otimes a_3) \ast (1 \otimes a_4 \otimes a_5 \otimes a_6)\]
\[= \sum_{(3,3)\text{-shuffles}(\sigma)} \text{sign}(\sigma) 1 \otimes a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \otimes \cdots \cdot \otimes a_{\sigma^{-1}(12)}\]

implies that:
\[(3.2.13) \quad c^2 = d^2 = e^2 = f^2 = 0\]
and the shuffle product:
\[(3.2.14) \quad (1 \otimes a_1) \ast (1 \otimes a_2 \otimes a_3 \otimes a_4) = 1 \otimes a_1 \otimes a_2 \otimes a_3 \otimes a_4\]
\[- 1 \otimes a_2 \otimes a_1 \otimes a_3 \otimes a_4 + 1 \otimes a_2 \otimes a_3 \otimes a_1 \otimes a_4\]
\[- 1 \otimes a_2 \otimes a_3 \otimes a_4 \otimes a_1\]

implies that
\[(3.2.15) \quad ac = bf = 0, \quad ad = bc, \quad ae = bd, \quad \text{and} \quad af = be.\]

Put
\[H^{12} = \langle g_1 = 1 \otimes x \otimes x \otimes x \otimes y - 1 \otimes y \otimes x \otimes x \otimes x + 1 \otimes x \otimes y \otimes x \otimes x - 1 \otimes x \otimes x \otimes y \otimes x \rangle \frac{1}{4}\]
\[g_2 = 1 \otimes x \otimes x \otimes y \otimes y - 1 \otimes y \otimes x \otimes x \otimes y + 1 \otimes y \otimes y \otimes x \otimes x - 1 \otimes x \otimes y \otimes y \otimes x \rangle \frac{1}{4}\]
\[g_3 = 1 \otimes x \otimes y \otimes x \otimes y - 1 \otimes y \otimes x \otimes y \otimes x \rangle \frac{1}{2}\]
\[g_4 = 1 \otimes x \otimes y \otimes y \otimes y - 1 \otimes y \otimes x \otimes y \otimes y + 1 \otimes y \otimes y \otimes x \otimes y - 1 \otimes y \otimes y \otimes y \otimes x \rangle \frac{1}{4}\]

By using the shuffle product (3.2.14), it can easily be proved that the image of the product
\[H^3 \ast H^9 \rightarrow H^{12}\]
has dimension 3 and hence one generator in degree 12 is needed. We call this generator \(g\).

**Remark 3.2.9.** The above calculations can be also carried out by using the model of Sullivan and Vigué. See [26].
Definition 3.2.10. We say that a local ring \((R, m)\) satisfies the condition \(M_3\) if

\[
(3.2.16) \quad P_R(z)^{-1} = (1 + 1/z)/A(z) - H_R(-z)/z,
\]

where \(k = R/m, A\) is the subalgebra of \(\text{Ext}_R^*(k, k)\) generated by \(\text{Ext}_R^1(k, k), P_R(z)\), is the Poincaré-Betti series, \(A(z)\) is the Hilbert series of \(A\) and \(H_R(z)\) is the Hilbert series of \(R\).

In the graded case, i.e., when \(R\) is a quotient of a polynomial ring

\[
k[X_1, X_2, \ldots, X_n]
\]

(where the generators \(X_i\) have degree 1) by an ideal generated by homogeneous elements, it is clear that the vector space \(\text{Ext}_R^*(k, k)\) has an extra grading so that we can introduce a Poincaré-Betti series in two variables

\[
(3.2.17) \quad P_R(x, y) = \sum_{i \geq 0, j \geq i} \dim_k \text{Ext}_R^i(k, k)^j x^j y^i.
\]

It follows easily that if \((R, m)\) satisfies \(M_3\) then we have the even more precise two-variable version of (3.2.16):

\[
(3.2.18) \quad P_R(x, y)^{-1} = (1 + 1/x)/A(xy) - H_R(-xy)/x.
\]

We refer the reader to [22] for more details about the condition \(M_3\).

Theorem 3.2.11. If the finitely generated ring \(R\) satisfies the condition \(M_3\), then the trivial extension ring \(R \oplus R\) also satisfies the condition \(M_3\).

Proof. In the proof of Theorem 2.2.13 we proved that

\[
(3.2.19) \quad H_{R \oplus R}(t) = H_R(t) + (H_R(t) - 1) \quad \text{and} \quad P_{R \oplus R}(x, y) = P_R(x, y)/(1 - (P_R(x, y) - 1)).
\]

Let \(A\) be the subalgebra of \(\text{Ext}_R^*(k, k)\) generated by \(\text{Ext}_R^1(k, k)\) and \(A'\) be the subalgebra of \(\text{Ext}_{R \oplus R}^*(k, k)\) generated by \(\text{Ext}_{R \oplus R}^1(k, k)\). Theorem 2.2.13 implies that

\[
(3.2.20) \quad A'(t) = A(t)/(1 - (A(t) - 1))
\]

and hence

\[
(3.2.21) \quad A(t) = 2A'(t)/(1 + A'(t)).
\]

Moreover since \(R\) satisfies the condition \(M_3\) we have

\[
(3.2.22) \quad P_R(x, y)^{-1} = (1 + 1/x)/A(xy) - H_R(-xy)/x.
\]
Now
\[ P_{R \oplus \bar{R}}(x, y)^{-1} = \left( P_R(x, y)/(1 - (P_R(x, y) - 1)) \right)^{-1} \]
\[ = 2 P_R(x, y)^{-1} - 1 \]
\[ = 2((1 + 1/x)/A(xy) - H_R(-xy)/x) - 1 \]
\[ = 2(1 + 1/x)((1 + A'(xy))/(2A'(xy))) - 2(H_{R \oplus \bar{R}}(-xy) + 1)/(2x) - 1 \]
\[ = (1 + 1/x)/(A'(xy)) + 1 + 1/x - H_{R \oplus \bar{R}}(-xy)/x - 1/x - 1 \]
\[ = (1 + 1/x)/(A'(xy)) - H_{R \oplus \bar{R}}(-xy)/x. \]

This shows that \( R \oplus \bar{R} \) satisfies the condition \( M_3 \).

We devote the last part of this section to prove that the ring \( H^*(\mathcal{L}(S^4 \vee S^4), k) \) in low dimensions satisfies the condition \( M_3 \). In Theorem 3.2.2 we proved that the ring \( H^*(\mathcal{L}(S^4 \vee S^4), k) \) is the trivial extension of \( R = H^3*(\mathcal{L}(S^4 \vee S^4), k) \) and in Theorem 3.2.11 we proved that the trivial extension ring \( R \oplus \bar{R} \) satisfies the condition \( M_3 \). If the finitely generated ring \( R \) satisfies the condition \( M_3 \). So in order to show that the whole ring \( H^*(\mathcal{L}(S^4 \vee S^4), k) \) in low dimensions satisfies the condition \( M_3 \) it is enough to show that the ring \( R = H^3*(\mathcal{L}(S^4 \vee S^4), k) \) in low dimensions satisfies the condition \( M_3 \).

**Theorem 3.2.12.** The ring \( H^3*(\mathcal{L}(S^4 \vee S^4), k) \) in low dimensions satisfies the condition \( M_3 \) if all generators are considered to be in degree 1.

In order to do prove Theorem 3.2.12, we take \( R_1 = H^3*(\mathcal{L}(S^4 \vee S^4), k) \) in low dimensions as
\[ R_1 = \frac{k[a, b, c, d]}{(a^2, b^2, c^2, d^2, ac, (ad - bc))} \]

The Hilbert series of \( R_1 \) is
\[ H_{R_1}(t) = 1 + 4t + 4t^2 \]

This shows that \( M^3 = 0 \) (\( M \) is the maximal ideal of \( R_1 \)) and hence the ring \( R_1 \) satisfies automatically the condition \( M_3 \).

In this case the Poincaré-Betti series of \( R_1 \) can be computed as follows. We first use MACAULAY to make a prediction. Introduce \( R_1 \) to MACAULAY (more details in [22]), issue the command:
\texttt{nres R1 t 8}
and then break the computations after a while and issue the new command:
\texttt{betti t}. 

MACAULAY produces the following table of graded Betti numbers for $R_1$:

<table>
<thead>
<tr>
<th>total :</th>
<th>1</th>
<th>4</th>
<th>12</th>
<th>33</th>
<th>89</th>
<th>240</th>
<th>649</th>
<th>1758</th>
<th>4765</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>4</td>
<td>12</td>
<td>32</td>
<td>81</td>
<td>200</td>
<td>488</td>
<td>1184</td>
<td>2865</td>
</tr>
<tr>
<td>1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>1</td>
<td>8</td>
<td>40</td>
<td>160</td>
<td>562</td>
<td>1816</td>
</tr>
<tr>
<td>2</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>1</td>
<td>12</td>
<td>84</td>
<td></td>
</tr>
</tbody>
</table>

Moreover MACAULAY determines the Hilbert series of $R_1$ as

$$H_{R_1}(t) = 1 + 4t + 4t^2.$$ 

Now let

$$f = 1 + 4t + 12t^2 + 32t^3 + 81t^4 + 200t^5 + 488t^6 + 1184t^7 + 2865t^8.$$ 

By using two successive commands:

convert ('', series); and convert ('', ratpoly);

we convert $f$ into

$$A(t) = -\frac{1}{(t-1)^2(t^2 + 2t - 1)}$$

(3.2.24)

where $A$ is the subalgebra of $\text{Ext}_{R_1}^*(k, k)$ generated by $\text{Ext}_{R_1}^1(k, k)$. Substitution of $A(t)$ and $H_{R_1}(t)$ in (3.2.18) ($t = xy$), we obtain

$$P_{R_1}(x, y) = -\frac{1}{(-1 + 4xy - 4x^2y^2 + x^3y^4 + x^4y^4)}$$

$$= 1 + 4yx + 12y^2x^2 + (32y^3 + y^4)x^3 + (81y^4 + 8y^5)x^4$$

$$+ (200y^5 + 40y^6)x^5 + (488y^6 + 160y^7 + y^8)x^6$$

$$+ (1184y^7 + 562y^8 + 12y^9)x^7$$

$$+ (2865y^8 + 1816y^9 + 84y^{10})x^8 + O(x^9)$$

(3.2.25)

and this determines the table (3.2.23) completely. This was a prediction by Macaulay. The only thing that is needed to be proved is why the subalgebra $A$ generated by $\text{Ext}_{R_1}^1(k, k)$ has the Hilbert series as asserted in (3.2.24). We prove this as follows:

The subalgebra $A = U(\eta)(\eta)$ is the Lie subalgebra of the homotopy Lie algebra of $R_1$) has according to the recipe of Lofwall [15] the presentation:

$$k\langle T_1, T_2, T_3, T_4 \rangle$$

$$\frac{1}{([T_1, T_2], [T_2, T_4], [T_3, T_4], [T_1, T_4] + [T_2, T_3])}$$

where $k < T_1, T_2, T_3, T_4 >$ is the free associative $k$-algebra in variables
$T_1, T_2, T_3, T_4$ of degree 1 and where $[T_i, T_j] = T_iT_j + T_jT_i$ is the graded commutator.

**Lemma 3.2.13.** The Lie algebra $\zeta$ generated by $T_1, T_3, [T_1, T_4]$ is an ideal in $\eta$.

Here $\eta$ is

$$L(T_1, T_2, T_3, T_4)$$

$$\frac{([T_1, T_2], [T_2, T_4], [T_2, T_4], [T_1, T_4] + [T_2, T_3])}{([T_1, T_2], [T_2, T_4], [T_2, T_4], [T_1, T_4] + [T_2, T_3])}$$

where $L < T_1, T_2, T_3, T_4 >$ is the free lie-algebra.

**Proof.** We prove the Lemma by induction. Using the Jacobi identity, i.e.,

$$(-1)^{|a||c|}[a, [b, c]] + (-1)^{|b||a|}[b, [c, a]] + (-1)^{|c||b|}[c, [a, b]] = 0$$

we have

$$[T_2, T_1] = 0, \quad [T_2, T_3] = -[T_1, T_4], \quad [T_2, T_1, T_4] = 0,$$

$$[T_4, T_1] = [T_1, T_4], \quad [T_4, T_3] = 0, \quad [T_4, [T_1, T_4]] = -[T_4, [T_2, T_3]] = 0.$$

Now assume for $\omega$ in $\zeta$ we have $[T_2, \omega]$ and $[T_4, \omega]$ is in $\zeta$, then

$$[T_2, [T_3, \omega]] = [[T_2, T_3], \omega] + [T_3, [T_2, \omega]] = -[T_1, T_4], \omega + [T_3, [T_2, \omega]] \in \zeta$$

$$[T_2, [T_1, \omega]] = [T_1, [T_2, \omega]] + [T_2, [T_1, \omega]] = [T_1, [T_2, \omega]] \in \zeta$$

$$[T_2, [[T_1, T_4], \omega]] = [[T_1, T_4], [T_2, \omega]] \in \zeta$$

$$[T_4, [T_3, \omega]] = [T_3, [T_4, \omega]] \in \zeta$$

$$[T_4, [T_1, \omega]] = [T_1, [T_4, \omega]] + [[T_4, T_1], \omega] \in \zeta$$

$$[T_4, [[T_1, T_4], \omega]] = [[T_1, T_4], [T_4, \omega]] \in \zeta.$$

Now the exact sequence

$$0 \longrightarrow \zeta \longrightarrow \eta \longrightarrow L(T_2, T_4)/[T_2, T_4] \longrightarrow 0$$

shows that

$$A(t) \leq \frac{1}{(1 - 2t - t^2)} \cdot \frac{(1 + t)^2}{(1 - t^2)^2}.\tag{3.2.27}$$
Because
\[ U(L(T_2, T_4)/[T_2, T_4])(t) = \frac{(1 + t)^2}{(1 - t^2)^2} \text{ and } U(\zeta)(t) \leq \frac{1}{(1 - 2t - t^2)}. \]

Equality in (3.2.27) holds if the ideal \( \zeta \) in (3.2.26) is free. To prove that this ideal is really free let \( L(T_2, T_4)/[T_2, T_4] \) acts on \( \zeta = L(T_1, T_3, [T_1, T_4]) \) by
\[
[T_2, T_1] = 0, \quad [T_2, T_3] = -[T_1, T_4], \quad [T_2, [T_1, T_4]] = 0,
\]
\[
[T_4, T_1] = [T_1, T_4], \quad [T_4, T_3] = 0, \quad [T_4, [T_1, T_4]] = 0.
\]

Take the semi-direct product. This semi-direct product is a quotient of
\[
\eta = \frac{L(T_1, T_2, T_3, T_4)}{([T_1, T_2], [T_2, T_4], [T_3, T_4], [T_1, T_4] + [T_2, T_3])}
\]
and has the series
\[
\frac{1}{(1 - 2t - t^2)} \cdot \frac{(1 + t)^2}{(1 - t^2)^2}.
\]

Hence
\[(3.2.28) \quad A(t) \geq \frac{1}{(1 - 2t - t^2)} \cdot \frac{(1 + t)^2}{(1 - t^2)^2}.
\]

The relations (3.2.27) and (3.2.28) show that \( A(t) \) has the desired Hilbert series.

Now we take another sample namely
\[
R_2 = \frac{k[a, b, c, d, e, f]}{(a^2, b^2, c^2, d^2, e^2, f^2, ac, bf, (ad - bc), (ae - bd), (af - be))}.
\]

MACAULAY produces the following table of graded Betti numbers for \( R_2 \):
(3.2.29)

<table>
<thead>
<tr>
<th>total</th>
<th>1</th>
<th>6</th>
<th>26</th>
<th>101</th>
<th>376</th>
<th>1376</th>
<th>5001</th>
<th>18126</th>
<th>65626</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>6</td>
<td>26</td>
<td>100</td>
<td>364</td>
<td>1288</td>
<td>4488</td>
<td>15504</td>
<td>53296</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td>12</td>
<td>88</td>
<td>512</td>
<td>2604</td>
<td>12144</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td>18</td>
<td>1186</td>
<td></td>
</tr>
</tbody>
</table>

Moreover MACAULAY determines the Hilbert series of \( R_2 \) as
\[
H_{R_2}(t) = 1 + 6t + 10t^2 + 4t^3 + t^4.
\]
Now let
\[ f = 1 + 6t + 26t^2 + 100t^3 + 364t^4 + 1288t^5 + 4488t^6 + 15504t^7 + 53296t^8. \]

By using two successive commands:
\[
\text{convert(',series'); and convert(',ratpoly);} \\
\]
we convert \( f \) into
\[
(3.2.30) \quad A(t) = -1/(2t-1)(2t^2-4t+1)
\]
where \( A \) be the subalgebra of \( \text{Ext}_{R_2}^*(k, k) \) generated by \( \text{Ext}_{R_2}^1(k, k) \). Substitution of \( A(t) \) and \( H_{R_2}(-t) \) in (3.2.18) \((t = xy)\), we obtain
\[
(3.2.31) \quad P(x,y) = -\frac{1}{(-1+6xy-10x^2y^2+4x^3y^3+x^3y^4)} \\
= 1 + 6xy + 26y^2x^2 + (100y^3 + y^4)x^3 + (364y^4 + 12y^5)x^4 \\
+ (1288y^5 + 88y^6)x^5 + (4488y^6 + 512y^7 + y^8)x^6 \\
+ (15504y^7 + 2604y^8 + 18y^9)x^7 \\
+ (53296y^8 + 12144y^9 + 186y^{10})x^8 + O(x^9)
\]
and this determines the table (3.2.29) completely. This was a prediction by Macaulay. We prove that the subalgebra \( A \) generated by \( \text{Ext}_{R_2}^1(k, k) \) has the Hilbert series as asserted in (3.2.30). We prove this as follows:

The subalgebra \( A = U(\eta)(\eta \) is the Lie sub algebra of the homotopy Lie algebra of \( R_2 \)) has according to the recipe of Löfwall [15] the presentation:
\[
A = \frac{k\langle T_1, T_2, T_3, T_4, T_5, T_6 \rangle}{I}
\]
\[
I = ([T_1, T_2], [T_3, T_4], [T_3, T_5], [T_3, T_6], [T_4, T_5], [T_4, T_6], [T_5, T_6], \\
[T_1, T_4] + [T_2, T_3], [T_1, T_5] + [T_2, T_4], [T_1, T_6] + [T_2, T_5]),
\]
where \( k\langle T_1, T_2, T_3, T_4, T_5, T_6 \rangle \) is the free associative \( k \)-algebra in variables \( T_1, T_2, T_3, T_4, T_5, T_6 \) of degree 1 and where \( [T_i, T_j] = T_iT_j + T_jT_i \) is the graded commutator. By changing the orders we rewrite \( A \) as
\[
(3.2.32) \quad A = \frac{k\langle T_1, T_2, T_3, T_4, T_5, T_6 \rangle}{I}
\]
where
(3.2.33) \[ I = ([T_1, T_3], [T_1, T_4], [T_3, T_4], [T_2, T_3], [T_1, T_6], [T_3, T_6], [T_4, T_6],
               [T_1, T_5] + [T_2, T_3], [T_3, T_5] + [T_2, T_4], [T_2, T_6] + [T_4, T_3]). \]

In order to compute the Hilbert series of \( A \), we use the program BERGMAN of Jörgen Backelin to compute the associated monomial ring of \( A \), i.e., \( \text{gr}(A) \) because \( A \) and \( \text{gr}(A) \) have the same Hilbert series. To use BERGMAN first create with a text editor the following file:

```lisp
(setq embdim 6)
(LISPFORMINPUT)
((1 1 3) (1 3 1))
((1 1 4) (1 4 1))
((1 1 6) (1 6 1))
((1 2 5) (1 5 2))
((1 3 4) (1 4 3))
((1 3 6) (1 6 3))
((1 4 6) (1 6 4))
((1 2 3) (1 3 2) (1 1 5) (1 5 1))
((1 2 4) (1 4 2) (1 3 5) (1 5 3))
((1 4 5) (1 5 4) (1 2 6) (1 6 2))
(LISPFORMINPUTEND)
```

Give a name to this file (e.g. TEXT). Then start BERGMAN and write:

```lisp
(ncpbbhgroebner 'TEXT' 't1' 't2' 't3').
```

The file t1 gives the Gröbner basis for the ideal \( I \) in (3.2.33) as follows:

(3.2.34) \[ G = ([T_3, T_1], [T_4, T_1], [T_4, T_3], [T_5, T_1] + [T_3, T_2], [T_5, T_2], [T_5, T_3] + [T_4, T_2], [T_6, T_1], [T_6, T_2] + [T_5, T_4], [T_6, T_3], [T_6, T_4], [[T_4, T_2], T_1]) + [T_3^2, T_2]) \]

and hence the associated monomial ring of \( A \) has the form:

\[
\text{gr}(A) = \frac{k(T_1, T_2, T_3, T_4, T_5, T_6)}{\text{In}(I)}
\]

where

(3.2.35) \[
\text{In}(I) = (T_3 T_1, T_4 T_1, T_4 T_3, T_5 T_1, T_5 T_2, T_5 T_3, T_6 T_1, T_6 T_2, T_6 T_3, T_6 T_4, T_4 T_2 T_1)
\]
The file $r2$ gives the Poincaré-Betti series of $\text{gr}(A)$ (up to degree 4) as
\[(3.2.36) \quad P_{\text{gr}(A)}(x,y) = 1 + 6xy + 10x^2y^2 + x^2y^3 + 5x^3y^3 + x^3y^4 + x^4y^4.\]

But by [4] page 843 THÉORÈME 1 the following is an $\text{gr}(A)$—free resolution of $k$.

\[
0 \longrightarrow \text{gr}(A)s_{6431} \longrightarrow \text{gr}(A)s_{431} \oplus \text{gr}(A)s_{531} \oplus \text{gr}(A)s_{631} \oplus \text{gr}(A)s_{641} \\
\quad \oplus \text{gr}(A)s_{643} \oplus \text{gr}(A)s_{6421} \longrightarrow \text{gr}(A)s_{31} \oplus \text{gr}(A)s_{41} \\
\quad \oplus \text{gr}(A)s_{43} \oplus \text{gr}(A)s_{51} \oplus \text{gr}(A)s_{52} \oplus \text{gr}(A)s_{53} \\
\quad \oplus \text{gr}(A)s_{61} \oplus \text{gr}(A)s_{62} \oplus \text{gr}(A)s_{63} \oplus \text{gr}(A)s_{64} \\
\quad \oplus \text{gr}(A)s_{421} \longrightarrow \text{gr}(A)s_{1} \oplus \text{gr}(A)s_{2} \oplus \text{gr}(A)s_{3} \\
\quad \oplus \text{gr}(A)s_{4} \oplus \text{gr}(A)s_{5} \oplus \text{gr}(A)s_{6} \longrightarrow R \longrightarrow k,
\]

where $\text{gr}(A)s_{i_1,i_2,\ldots,i_t}$ is the free $\text{gr}(A)$—module generated by $s_{i_1,i_2,\ldots,i_t}$, and where $\text{gr}(A)s_{i_1,i_2,\ldots,i_t}$ is sent to $\text{gr}(A)T_{i_1,s_{i_2,\ldots,i_t}}$. This shows that $\text{gr}(A)$ has Global-dimension 4 and hence (3.2.36) is the whole Poincaré-Betti series of $\text{gr}(A)$. This gives

\[
A(t) = \text{gr}(A)(t) = \frac{1}{P_{\text{gr}(A)}(-1,t)} = \frac{1}{1 - 6t + 10t^2 - 4t^3},
\]

since $\text{gr}(A)$ has monomial relations.

To prove that $R_2$ has really the Poincaré-Betti series as asserted in (3.2.31), it is enough to prove that this ring satisfies the condition $M_3$. In order to do this we rewrite $R_2$ and $A$ here and replace $R_2$ by $R$ and the generators $T_1, T_2, T_3, T_4, T_5, T_6$ of $A$ by $X, Y, Z, U, V, W$ for simplicity. So:

\[
R = \frac{k[a, b, c, d, e, f]}{(a^2, b^2, c^2, d^2, e^2, f^2, ac, bf, (ad - bc), (ae - bd), (af - be))}
\]

and

\[
A = \frac{k(X, Y, Z, U, V, W)}{I}
\]

where

\[
I = ([X, Z], [X, U], [Z, U], [Y, V], [X, W], [Z, W], [U, W], [X, V] \\
+ [Y, Z], [Z, V] + [Y, U], [Y, W] + [U, V]).
\]

The complex $R^* \otimes A$ (for more details look at [22] page 306) is the following $(X, Y, Z, U, V, W$ is the basis for $A^1$ dual to $a, b, c, d, e, f$):
Let $\alpha$ be in the Ker of $d_4$, i.e., $d_4(\alpha) = 0$. We have:

$W\alpha = 0$

$U\alpha = 0$

$Z\alpha = 0$

$X\alpha = 0$.

Then Lemma B.7 ([22] page 310) gives $\alpha = 0$ and this shows that the homology is zero in degree 4. Now let $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be in the Ker of $d_3$, i.e., $d_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 0$. We have:

$U\alpha_1 + W\alpha_2 = 0$

$Z\alpha_1 + W\alpha_3 = 0$

$Z\alpha_2 + U\alpha_3 = 0$

$X\alpha_1 + W\alpha_4 = 0$

$X\alpha_2 + U\alpha_4 = 0$

$X\alpha_3 + Z\alpha_4 = 0$.

Lemma B.8 ([22] page 310) gives
\[(\alpha_1, \alpha_2) = (W, U)t_1\]
\[(\alpha_1, \alpha_3) = (W, Z)t_2\]
\[(\alpha_2, \alpha_3) = (U, Z)t_3\]
\[(\alpha_1, \alpha_4) = (W, X)t_4\]
\[(\alpha_2, \alpha_4) = (U, X)t_5\]
\[(\alpha_3, \alpha_4) = (Z, X)t_6.\]

Using once more Lemma B.7 ([22] page 310) we get \(t_1 = t_2 = t_3 = t_4 = t_5 = t_6\) and this implies that

\[(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (Wt_1, Ut_1, Zt_1, Xt_1)\]

and hence the homology is zero in degree 3 too. Next we prove that the homology is not zero in degree 2. In order to do this, we compute the Poincaré-Betti series of \(\mathcal{A}\). In (3.2.36) we computed the Poincaré-Betti series of \(\text{gr}(\mathcal{A})\) as

\[P_{\text{gr}(\mathcal{A})}(x, y) = 1 + 6xy + 10x^2y^2 + x^2y^3 + 5x^3y^3 + x^3y^4 + x^4y^4.\]

The Hilbert series of \(\mathcal{R}\) determines the \(|\text{Tor}_{1,1}^A(k, k)|\) as

\[|\text{Tor}_{1,1}^A(k, k)| = 6, \quad |\text{Tor}_{2,2}^A(k, k)| = 10, \quad |\text{Tor}_{3,3}^A(k, k)| = 4, \quad |\text{Tor}_{4,4}^A(k, k)| = 1.\]

To determine the Poincaré-Betti series of \(\mathcal{A}\) completely, we use the spectral sequence starting with \(\text{Tor}_{*,*}^\text{gr}(\mathcal{A})(k, k)\) and converging to \(\text{Tor}_{*,*}^A(k, k)\) (cf. [2]). Now by (3.2.36) we have only two nonzero element off the diagonal namely

\[\text{Tor}_{2,3}^{\text{gr}(\mathcal{A})}(k, k), \quad \text{and} \quad \text{Tor}_{3,4}^{\text{gr}(\mathcal{A})}(k, k).\]

The complex

\[0 = \text{Tor}_{1,3}^{\text{gr}(\mathcal{A})}(k, k) \longrightarrow \text{Tor}_{2,3}^{\text{gr}(\mathcal{A})}(k, k) \longrightarrow \text{Tor}_{3,3}^{\text{gr}(\mathcal{A})}(k, k) \longrightarrow \text{Tor}_{4,3}^{\text{gr}(\mathcal{A})}(k, k) = 0,\]

where \(d\) is the differential in the spectral sequence, shows that the map \(d\) is the inclusion map, because we know that \(\text{Tor}_{3,3}^{\text{gr}(\mathcal{A})}(k, k)\) has dimension 5 and \(\text{Tor}_{2,3}^{\text{gr}(\mathcal{A})}(k, k)\) has dimension 4, and hence \(\text{Tor}_{2,3}^{A}(k, k) = 0\).

To compute \(\text{Tor}_{3,4}^{A}(k, k)\) we use the complex

\[0 = \text{Tor}_{2,4}^{\text{gr}(\mathcal{A})}(k, k) \longrightarrow \text{Tor}_{3,4}^{\text{gr}(\mathcal{A})}(k, k) \longrightarrow \text{Tor}_{4,4}^{\text{gr}(\mathcal{A})}(k, k) \longrightarrow \text{Tor}_{5,4}^{\text{gr}(\mathcal{A})}(k, k) = 0.\]

Here the map \(f\) is zero because both \(\text{Tor}_{4,4}^{\text{gr}(\mathcal{A})}(k, k)\) and \(\text{Tor}_{3,3}^{A}(k, k)\) has dimension 1. This shows that \(|\text{Tor}_{3,4}^{A}(k, k)| = 1\) and hence the Poincaré-Betti series of \(\mathcal{A}\) is
\[ P_A(x, y) = 1 + 6xy + 10x^2y^2 + 4x^3y^3 + x^4y^4. \]

Now we go back to the complex \( R^* \otimes A \). The existence of a nonzero element, i.e., \( \text{Tor}_{3,4}^A(k, k) \) off the diagonal shows that \( H_2(R^* \otimes A) \) is different from zero. Theorem B.3 (\cite{22} page 306) now implies that \( R \) satisfies \( M_3 \). As a consequence, we obtain the following result.

**Theorem 3.2.14.** The whole ring \( H^*(\mathcal{L}(S^4 \vee S^4), k) \) in low dimensions satisfies the condition \( M_3 \) if all generators are considered to be in degree 1.

**Proof.** Consider the generators \( X_i \)'s of \( H^{3*}(\mathcal{L}(S^4 \vee S^4), k) \) and the generators \( Y_i \)'s of \( H^{3*+1}(\mathcal{L}(S^4 \vee S^4), k) \) as bigraded elements with bidegree \( (1, \deg X_i \text{'s}) \) and \( (1, \deg Y_i \text{'s}) \) respectively. Then using the notation of definition 2.2.13, we have (by Theorem 3.2.2)

\[
H^*(\mathcal{L}(S^4 \vee S^4), k) = H^{3*}(\mathcal{L}(S^4 \vee S^4), k) \oplus s^{0,-1}H^{3*}(\mathcal{L}(S^4 \vee S^4), k)
\]

In other words the ring structure of \( H^*(\mathcal{L}(S^4 \vee S^4), k) \) is the trivial extension of the ring \( H^{3*}(\mathcal{L}(S^4 \vee S^4), k) \) by the modules \( s^{0,-1}H^{3*}(\mathcal{L}(S^4 \vee S^4), k) \).

Now if we only consider this new degree 1 (the first degree), then \( H^*(\mathcal{L}(S^4 \vee S^4), k) \) is nothing but \( R \oplus \tilde{R} \) where \( R = H^{3*}(\mathcal{L}(S^4 \vee S^4), k) \). Now the proof follows of Theorem 3.2.11.

4. \( S^4, S^5 \), and the EMSS of the path fibration.

Recall (cf. introduction) that

\[
E_{-p,q}^2 = \text{Ext}^p_{H^*(\mathcal{L}X, k)}(k, k) \Longrightarrow (H_*(\Omega \mathcal{L}X, k))
\]

and if this spectral sequence degenerates, then

\[
\dim_k(H_n(\Omega \mathcal{L}X, k)) = \sum_{p \geq 0} \dim_k \text{Ext}^p_{H^*(\mathcal{L}X, k)}(k, k)_{p+n},
\]

where the sum is finite.

**Theorem 4.1.** The Eilenberg-Moore spectral sequence (4.1) does not degenerate when \( X = S^4 \).

**Proof.** (cf. \cite{28}) The series \( U(L) \) (universal enveloping algebra of \( L \)), where \( L \) is a graded lie algebra

\[
L = L_1 + L_2 + L_3 + \cdots \quad \text{with} \quad |L_i| = l_i \quad i = 1, 2, 3, \cdots
\]

can be written as
(4.3) \[ U(L)(z) = \frac{(1 + z)^{l_1} (1 + z^3)^{l_3}}{(1 - z^2)^{l_2} (1 - z^4)^{l_4}} \ldots \ldots. \]

Now set \( H_*(\Omega X, k) = U(L) \), then
\[
H_*(\Omega \mathcal{L}X, k) = U(L \oplus SL).
\]

where \((SL)_n = L_{n+1}\). In other words if \( g = L \oplus SL = g_1 + g_2 + g_3 + \ldots \ldots \), then
\[
|g_1| = l_1 + l_2, \quad |g_2| = l_2 + l_3, \quad |g_3| = l_3 + l_4, \ldots
\]

This last equality can be easily seen in cohomology case (cf. [28] page 183 and [26]). We have replaced cohomology with homology because here \( k \) is a field of characteristic zero. This implies

(4.4) \[ H_*(\Omega \mathcal{L}X, k)(z) = U(L \oplus SL)(z) = \frac{(1 + z)^{l_1 + l_2} (1 + z^3)^{l_3 + l_4}}{(1 - z^2)^{l_2 + l_3} (1 - z^4)^{l_4 + l_5}} \ldots \ldots. \]

Now let \( X = S^4 \), then

(4.5) \[ H_*(\Omega X, k)(z) = \frac{1}{1 - z^3} = \frac{1 + z^3}{1 - z^6} \]

and hence

(4.6) \[ H_*(\Omega \mathcal{L}X, k)(z) = \frac{(1 + z^3)(1 + z^5)}{(1 - z^2)(1 - z^6)} \]
\[ = 1 + z^2 + z^3 + z^4 + 2z^5 + 2z^6 + 2z^7 + 3z^8 + \ldots. \]

We have the following table for \( H^*(\mathcal{L}S^4, k) \) as a particular case of the table (3.1.1).

| \( H^*(\mathcal{L}S^4, k) \) | \( k \) | \( 1 \otimes x \) | \( x \) | \( 1 \otimes x^{\otimes 3} \) | \( x^{\otimes 3} \) | \( 1 \otimes x^{\otimes 5} \) | \( x^{\otimes 5} \) | \ldots |
|---|---|---|---|---|---|---|---|
| deg | 0 | 3 | 4 | 9 | 10 | 15 | 16 | \ldots |
| dim | 1 | 1 | 1 | 1 | 1 | 1 | 1 | \ldots |

To complete the proof of Theorem 4.1, we need

**Lemma 4.2.** The product in \( H^*(\mathcal{L}S^4, k) \) is 0.

**Proof.** Easy. Recall that \( x^{\otimes n} \ast x^{\otimes m} = 0 \) (where \( \ast \) is the shuffle product) because the product in \( H^*(X, k) \) is 0.

**Theorem 4.3.** \( H^*(\mathcal{L}S^4, k) \) is isomorphic as a ring to

\[
R = \frac{k[X_1, X_2, X_3, \ldots \ldots]}{(X_i X_j, i \leq j)}.
\]
**Proof.** Follows easily by lemma 4.2.

**Theorem 4.4.** \( \text{Ext}_{H^*(\mathcal{L}S^4, k)}(k, k) \) is generated by elements of degree one, i.e., \( \text{Ext}^1_{H^*(\mathcal{L}S^4, k)}(k, k) \). It has the following explicit form

\[
\text{Ext}_{H^*(\mathcal{L}S^4, k)}(k, k) = [\text{Ext}^1_{H^*(\mathcal{L}S^4, k)}(k, k)] = k(T_1, T_2, T_3, \ldots).
\]

where the bidegree of \( T_i \) is \( (1, \deg X_i) \).

**Proof.** Theorem 4.3 shows that the ring \( H^*(\mathcal{L}S^4, k) \) is a Koszul algebra and hence \( \text{Ext}_{H^*(\mathcal{L}S^4, k)}(k, k) \) is generated by elements of degree one. The second part follows by easy calculations (using Lemma 2.2.18).

Now by Theorem 4.4 we have

\[
\sum_{i \geq 0} \left( \sum_{p \geq 0} \dim_k \text{Ext}^p_{H^*(\mathcal{L}X, k)}(k, k)_{p+i} \right) z^i = \frac{1}{1 - \sum_{i \geq 1} (| H^i_c(\mathcal{L}X, k) | - 1) z^i} = 1 + z^2 + z^3 + z^4 + 2z^5 + 2z^6 + 3z^7 + 5z^8 + 6z^9 + \ldots.
\]

Comparing \( \dim_k(\mathcal{H}_n(\Omega \mathcal{L}X, k)) \) and \( \sum_{p \geq 0} \dim_k \text{Ext}^p_{H^*(\mathcal{L}X, k)}(k, k)_{p+n} \) for \( n = 7 \), we obtain

\[
\dim_k(\mathcal{H}_7(\Omega \mathcal{L}X, k)) = 2 \quad \text{(by (4.6))}
\]

and

\[
\sum_{p \geq 0} \dim_k \text{Ext}^p_{H^*(\mathcal{L}X, k)}(k, k)_{p+7} = \dim_k \text{Ext}^3_{H^*(\mathcal{L}X, k)}(k, k)_7 = 3 \quad \text{(by (4.7)).}
\]

Now (4.1) implies

\[
E^2_{-3,10} = \text{Ext}^3_{H^*(\mathcal{L}X, k)}(k, k)_7 \Rightarrow H^*_7(\Omega \mathcal{L}X, k)
\]

and hence

\[
\dim E^\infty_{-3,10} = \dim_k(\mathcal{H}_7(\Omega \mathcal{L}X, k)) = 2 \quad \text{and} \quad \dim E^2_{-3,10} = 3.
\]

This implies that we have a non zero differential in the spectral sequence \( (E', d') \). Notice that \( d' \) has bidegree \( (-r, r-1) \).

Comparing two complexes

\[
E^2_{-1,9} \xrightarrow{d^2_{-1,9}} E^2_{-3,10} \xrightarrow{d^2_{-3,10}} 0 = E^2_{-5,11}
\]

and
0 = E'_{r-3,-r+11} \Rightarrow E'_{-3,10} \Rightarrow E'_{-3-r,r+9} = 0 \quad \forall \quad r \geq 3,

we see that $d^2_{-1,9} \neq 0$ and hence the Eilenberg–Moore spectral sequence does not degenerate at $E^2$ though it seems to converges very quickly.

**Theorem 4.5.** The Eilenberg–Moore spectral sequence (4.1) degenerates when $X = S^5$.

**Proof.** We have $H_*(\Omega S^5, k)(z) = \frac{1}{1 - z^4}$ and hence by (4.4)

\[ H_*(\Omega S^5, k)(z) = \frac{(1 + z^3)}{(1 - z^4)} \]

\[ = 1 + z^3 + z^4 + z^7 + z^8 + z^{11} + z^{12} + z^{15} + z^{16} + \ldots \]

Now as a particular case of Theorem 2.1.1, we have the following table

<table>
<thead>
<tr>
<th>$H^*(\mathcal{L}S^5, k)$</th>
<th>$k$</th>
<th>$\otimes x$</th>
<th>$x$</th>
<th>$1 \otimes x \otimes^2$</th>
<th>$x \otimes^2$</th>
<th>$1 \otimes x \otimes^3$</th>
<th>$x \otimes^3$</th>
<th>$\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>deg</strong></td>
<td>0</td>
<td>4</td>
<td>5</td>
<td>8</td>
<td>9</td>
<td>12</td>
<td>13</td>
<td>$\ldots$</td>
</tr>
<tr>
<td><strong>dim</strong></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\ldots$</td>
</tr>
</tbody>
</table>

To complete the proof of Theorem 4.5, we need:

**Lemma 4.6.** The algebra $H^*(\mathcal{L}S^5, k)$ is generated by $1 \otimes x$ and $x$.

**Proof.** Lemma 2.2.1 implies

\[ 1 \otimes x \otimes^n \ast 1 \otimes x \otimes^m = \frac{(n + m)!}{n!m!} 1 \otimes x \otimes^{n+m} \]

and

\[ 1 \otimes x \otimes^n \ast x \otimes^m = \frac{(n + m - 1)!}{(n-1)!m!} x \otimes^{n+m} \]

where $\ast$ means shuffle product.

By Lemma 4.6 above we get:

\[ H^*(\mathcal{L}S^5, k) = R = \frac{k[X_1, X_2]}{X_2^2} \quad (\deg X_1 = 4, \quad \deg X_2 = 5). \]

This ring is a complete intersection and

\[ \text{Ext}_{H^*(\mathcal{L}S^5, k)}(k, k) = [\text{Ext}_{H^*(\mathcal{L}S^5, k)}^1(k, k)] = \frac{k\langle T_1, T_2 \rangle}{(T_1^2, T_1T_2 + T_2T_1)}, \]

where the monomial $T_1$ generates
ON THE COHOMOLOGY RING OF THE FREE LOOP...

\[ \text{Ext}^1_{H^*(\mathcal{S}^1, \mathbb{Z})}(k, k) \]

and the monomial \( T_2 \) generates

\[ \text{Ext}^1_{H^*(\mathcal{S}^1, \mathbb{Z})}(k, k) \]

Counting monomials in \( \text{Ext}^1_{H^*(\mathcal{S}^1, \mathbb{Z})}(k, k) \) we obtain

<table>
<thead>
<tr>
<th>deg</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>……</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monomials</td>
<td>( T_1, T_2 )</td>
<td>( T_1 T_2, T_2^2 )</td>
<td>( T_1 T_2^2, T_2^3 )</td>
<td>( T_1 T_2^3, T_2^4 )</td>
<td>( T_1 T_2^4, T_2^5 )</td>
<td>( T_1 T_2^5, T_2^6 )</td>
<td>……</td>
</tr>
</tbody>
</table>

The fact that the monomials \( T_1 T_2^n \) generate \( \text{Ext}^{n+1}_{H^*(\mathcal{S}^1, \mathbb{Z})}(k, k) \) together with (4.8) imply that

\[ \dim_k(H_n(\Omega \mathcal{S}^1 X, k)) = \sum_{p \geq 0} \dim_k \text{Ext}^p_{H^*(\mathcal{S}^1, \mathbb{Z})}(k, k)_{p+n} \]

and hence the Eilenberg–Moore spectral sequence (4.1) degenerates in this case.

REFERENCES

15. C. Löfwall, *The Yoneda Ext-algebra for an equi-characteristic local ring \((R, m)\) with \(m^2 = 0\), unpublished manuscript.

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