MATH. SCAND. 80 (1997), 56–100

ON INDUCTIVE LIMITS OF MATRIX ALGEBRAS OVER HIGHER DIMENSIONAL SPACES, PART II

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Abstract.

In this paper, we will prove the following result. Suppose that $A = \lim_{n\to\infty} \bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i}))$ is of real rank zero, where the spaces $X_{n,i}$ are finite CW complexes with uniformly bounded dimension (or with slow dimension growth in a generalized sense for non simple C*-algebras). Then A can be written as an inductive limit of finite direct sums of matrix algebras over 3-dimensional finite CW complexes. (Hence it can be classified by its graded ordered K-group, if one supposes further that A is simple.)

1. Introduction.

In [EG2], George A. Elliott and the author proved the following theorem. If a simple C*-algebra A of real rank zero can be expressed as an inductive limit of matrix algebras over 3-dimensional finite CW complexes, then its isomorphism type is completely determined by its graded ordered K-group (with dimension range). Also we proved that the above classification theorem still holds if one replaces the condition of simplicity by the condition that $K_*(A)$ is torsion free (see also [G1]). In [G2], this result for the case that $K_*(A)$ is torsion free was generalized to include inductive limits of matrix-algebras over arbitrary finite CW complexes with uniformly bounded dimension (rather than dimension ≤ 3). In this paper, we will prove that if a simple C*-algebra A of real rank zero can be expressed as an inductive limit form $\lim_{n\to\infty} \bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i}))$ with

$$\lim_{n\to\infty} \frac{\dim(X_{n,i})}{[n,i]} = 0$$

(this condition is called slow dimension growth condition; see [BDR]), then A can be written as an inductive limit of matrix algebras over 3-dimensional

Received March 24, 1995.

finite CW complexes. (Hence it can be classified by its graded ordered K-group.)

The theorem also holds without the simplicity condition. But we need to use a slow dimension growth condition in a generalized sense (see §2), or simply suppose that the spaces $X_{n,i}$ have uniformly bounded dimension. Recall that if we drop both the condition that A be simple and the condition that $K_*(A)$ be torsion free, then the isomorphism type of A is not completely determined by their K-theory, even though they can be written as inductive limits of matrix algebras over 3-dimension spaces (see [G1]). But one can prove that the isomorphism type of such a C*-algebra is completely determined by its unsuspended E-equivalence type (or asymptotic isomorphism type). This is a generalization of a result in [G1] in which we suppose that the base spaces have dimension ≤ 2.

This is a revision of the author's paper of the same title.The results on dimension drop C*-algebras contained in the original version will be included in a joint paper with George A. Elliott and Hongbing Su. (Also see [G3].) The results in [G2] and this paper have been announced in [G3].

The material is organized as follows. In §2, we will review some known results and give some preliminary results. In §3, we will prove our theorem for a special inductive limit which involves only one fixed space X (X can be an arbitrary fixed finite CW complex). In §4, we will prove the general theorem. In §5, we will present some remarks. In our main result, we have actually proved that the inductive limit algebra of any system with slow dimension growth can be rewritten as an inductive limit of matrix algebras over certain very special 3-dimensional CW complexes, namely, S^1, S^2, and the spaces $T_{II,k}$ and $T_{III,k}$ (defined in [EG2]). It should be pointed out that to produce all real rank zero inductive limit C^*-algebras, we need to use S^2 or a space with similar properties (see §5). (One does not need S^2 to produce all ordered K-groups.) In §5, we give a theorem which says that the isomorphism type of such an inductive limit C*-algebra (within the class of such algebras) is completely determined by its unsuspended E-equivalence type (or asymptotic isomorphism type of Dadarlat).

ACKNOWLEDGMENT. This work was done while the author was visiting the University of Toronto during 1993–1994. The author would like to thank Professor G. Elliott for many helpful conversations. Shortly after the proof of the theorem was completed, the author was informed that Professor M. Dadarlat had completed a paper with results similar to our main result, Theorem 4.4, (and probably also Theorem 5.4 concerning classification by unsuspended E-equivalence type).

2. Preliminaries.

2.1. Let X and Y be path connected finite CW complexes with base points x_0 and y_0. (All the CW complexes in this paper are assumed to be connected.) Let $C_0(X)$ denote the collection of continuous functions on X vanishing at the base point. Write $F(X) = \mathrm{Hom}(C_0(X), \mathcal{K})$, where $\mathcal{K}$ is the C*-algebra of all compact operators on an infinite dimensional separable complex Hilbert space. Also write

$$\mathrm{kk}(Y, X) \;=\; \left[C_0(X), C_0(Y) \otimes \mathcal{K}\right],$$

which is the collection of homotopy classes of homomorphisms between $C_0(X)$ and $C_0(Y) \otimes \mathcal{K}$. Let us use $[A, B]_1$ to denote the collection of homotopy classes of unital homomorphisms. As a consequence of 4.2.11 of [DN], we have,

$$\mathrm{kk}(Y, X) \;=\; \left[C_0(X), PM_n C_0(Y) P\right] \;=\; \left[C(X), PM_n(C(Y))P\right]_1\,,$$

where $P \in M_n(C(Y))$ is a projection with $\mathrm{rank}(P) \geq 3\dim Y + 1$. (In [EGLP2], Theorem 8, we gave a direct proof of the above fact from 6.4.4 of [DN], since we didn't realize that it is a consequence of 4.2.11 of [DN].) There is a canonical homomorphism (see [DN])

$$\chi : \mathrm{kk}(Y, X) \to \mathrm{KK}\big(C_0(X), C_0(Y)\big)\,.$$

The following result is Theorem 3.3 of [EG2] (also see [DN] for a slightly weaker result).

PROPOSITION 2.2. *Let X be a m_2-dimensional connected finite* CW *complex and Y be a m_1-connected finite* CW *complex. If $m_2 \leq m_1 + 3$, and $H^{m_1+3}(X)$ is a finite group, then χ is an isomorphism.*

2.3. The following spaces were used in [EG2]:

(0) $X = \{\text{point}\}$ (let us call this type 0);
(1) $X = S^1$ (call this type I);
(2) $X = T_{II,k}$ (call this type II), which is a 2-dimensional finite CW complex with $\tilde{H}^*(T_{II,k}) = H^2(T_{II,k}) = Z/k$;
(3) $X = T_{III,k}$ (call this type III), which is a 3-dimensional finite CW complex with $\tilde{H}^*(T_{III,k}) = H^3(T_{III,k}) = Z/k$;
(4) $X = S^2$ (call this type IV).

In the above and in all what follows, we shall use $\tilde{H}^*(X)$ to denote the reduced cohomology group $\bigoplus_{i=1}^{\infty} H^i(X, Z)$ with coefficients in Z. If $X = X_1 \vee X_2 \vee X_3 \ldots \vee X_k$ is a finite wedge of spaces of the above special forms, then we shall say X is a CW complex of special form. By the above-proposition, for any space X of special form and any connected finite CW complex Y, the map

$$\chi : \mathrm{kk}(Y, X) \to \mathrm{KK}\big(C_0(X), C_0(Y)\big)$$

is an isomorphism. (Notice that $H^3(X)$ is a finite group.)

For any finite CW complex X, one can find a CW complex $\bar{X}$ of special form such that

$$K^0(X) = K^0(\bar{X}) \qquad \text{and} \qquad K^1(X) = K^1(\bar{X}).$$

The main idea in this paper is to replace X by $\bar{X}$ which is a 3-dimensional CW complex. (Notice that $\bar{X}$ is not unique, but it is not important which $\bar{X}$ we choose.)

2.4. Suppose that $\phi : M_k(C(X)) \to PM_n(C(Y))P$ is a unital homomorphism. Then ϕ induces a homomorphism $\phi|_{e_{11}M_k(C(X))e_{11}} : C(X) \to pM_n(C(Y))p$, where e_{11} is the matrix unit of $M_k(C(X))$ corresponding to the upper left corner and $p = \phi(e_{11})$. Hence it induces an element $[\phi] \in \mathrm{kk}(Y, X)$. Also, ϕ induces a map

$$\phi_* : K_0(rM_k(C(X))) \to K_0\big(PM_n(C(Y))P\big) = K_0(C(Y)) \ ,$$

where $rM_k(C(X)) = M_k(C)$. (Notice that we use the notation that $A^0 = \bigoplus_{i=1}^{t} M_{k_i} C_0(X_i)$, and $rA = A/A^0$, if $A = \bigoplus_{i=1}^{t} M_{k_i}(C(X_i))$; see 1.6 of [EG2].) The following proposition is 4.2.11 of [DN].

PROPOSITION 2.5. *Let* $A = \bigoplus_{i=1}^{q} C(X_i) \otimes M_{n_i}$, $D = \bigoplus_{j=1}^{h} C(Y_j) \otimes M_{m_j}$, *where* X_i, Y_j *are connected finite* CW *complexes and* $\dim(Y_j) \leq n$ *for all* $1 \leq j \leq h$. *Let* φ, $\psi \in \mathrm{Hom}(A, D)$ *be* $3(n+3)/2$*-large (see* 2.1.8 *of* [DN] *for the definition of this). Then* φ *is homotopic to* $\mathrm{Ad}u \circ \psi$ *for some unitary* $u \in D$ *if and only if* $[\varphi^{i,j}] = [\psi^{i,j}] \in \mathrm{kk}(Y_j, X_i)$ *for all i,j and*

$$K_0\big(\varphi \mid r(A)\big) = K_0\big(\psi \mid r(A)\big) \ .$$

We shall say that two such homomorphisms define the same maps at the kk *level and at the* K_0 *level.*

LEMMA 2.6. *For any fixed finite set* $G \subset C(S^{2m})$ *and* $\varepsilon > 0$, *there exist an integer n, a unital homomorphism* $\phi : C(S^{2m}) \to M_n(C(S^{2m}))$ *and a finite dimensional* C^**-algebra* $B \subset M_n(C(S^{2m}))$ *such that*

(1) $[\phi] = \mathrm{id} \in \mathrm{kk}(S^{2m}, S^{2m})$;

(2) $\mathrm{dist}\big(\phi(g), B\big) < \varepsilon$ *for each* $g \in G$.

PROOF: If one allows B to be a direct sum of matrix algebras over $C(S^1)$, then this is a special case of Theorem 8 of [EGLP2] (see also [EG1]). Since we are dealing with an even sphere, we can change $C(S^1)$ to C as follows. First, we can construct a unital inductive limit sequence

$$C(S^{2m}) \xrightarrow{\phi_{1,2}} M_2(C(S^{2m})) \xrightarrow{\phi_{2,3}} M_4(C(S^{2m})) \to \cdots \to A ,$$

with the following properties:
(1) A is of real rank zero;
(2) $[\phi_{i,j}] = \mathrm{id} \in \mathrm{kk}(S^{2m}, S^{2m})$;
(3) $\phi_{i,j}$ takes trivial projections to trivial projections.

(The conditions (2) and (3) imply that

$$\big(K_0(A), K_0(A)_+, \mathbf{1}_A\big) = \big(\mathsf{Z}(\tfrac{1}{2}) \oplus \mathsf{Z}, \mathsf{Z}(\tfrac{1}{2})_+ \oplus \mathsf{Z} \cup \{(0,0\}, \{(1,0)\}\big) .)$$

From [EGLP2] and [EG2], we know that A is an AF-algebra. Therefore, for any $\varepsilon > 0$, and any finite $G \subset C(S^{2m})$, there are an n_1 and a finite dimensional C*-algebra $B \subset M_{2^{n_1-1}}(C(S^{2m}))$ such that $\phi_{1,n_1}(G)$ is approximately contained in B to within ε.

2.7. Suppose that X is an arbitrary finite CW complex. Let $2m \geq \dim(X)$. We will prove a result which can be roughly stated as the following: $\mathrm{id} \in \mathrm{kk}(X \wedge S^{2m}, X \wedge S^{2m})$ can be realized by a homomorphism

$$\phi : C(X \wedge S^{2m}) \to M_n(C(X \wedge S^{2m})) \qquad \text{(for } n \text{ large enough)}$$

such that the image of a given finite set $G \subset C(X \wedge S^{2m})$ via ϕ is approximately contained in a sub-algebra $A \subset M_n(C(X \wedge S^{2m}))$ to within an arbitrarily small number, where A is a direct sum of matrix algebras over $C(X)$.

2.8. Let X be as in 2.7. Consider the following short sequence:

$$X \vee S^{2m} \xrightarrow{i} X \times S^{2m} \xrightarrow{\pi} X \wedge S^{2m} .$$

It induces a short exact sequence

$$0 \to C_0(X \wedge S^{2m}) \xrightarrow{\pi^*} C_0(X \times S^{2m}) \xrightarrow{i^*} C_0(X \vee S^{2m}) \to 0 .$$

Using the Künneth formula, it is routine to prove that the above sequence induces the following split exact sequences:

$$0 \to K_i\big(C_0(X \wedge S^{2m})\big) \to K_i\big(C_0(X \times S^{2m})\big) \to K_i\big(C_0(X \vee S^{2m})\big) \to 0 .$$

This means that $C_0(X \wedge S^{2m}) \oplus C_0(X \vee S^{2m})$ is KK-equivalent to $C_0(X \times S^{2m})$. And there is a $\theta \in \mathrm{KK}\big(C_0(X \vee S^{2m}), C_0(X \times S^{2m})\big)$ such that

$\pi^* \oplus \theta$ induces a KK-equivalence, where $\pi^* : C_0(X \wedge S^{2m}) \to C_0(X \times S^{2m})$ is induced by $\pi : X \times S^{2m} \to X \wedge S^{2m}$, and $\theta \times i^* = \mathrm{id} \in KK\big(C_0(X \vee S^{2m}), C_0(X \vee S^{2m})\big)$, where $i^* : C_0(X \times S^{2m}) \to C_0(X \vee S^{2m})$ is induced by the inclusion map $X \vee S^{2m} \to X \times S^{2m}$. Hence, for any C*-algebra C (in particular, we let $C = C_0(Z)$ for a finite CW complex Z), the two boundary maps δ_0 and δ_1 in the following six term exact sequence are zero:

$$\begin{array}{ccccc} KK(C_0(X \vee S^{2m}), C_0(Z)) & \to & KK(C_0(X \times S^{2m}), C_0(Z)) & \to & KK(C_0(X \wedge S^{2m}), C_0(Z)) \\ \delta_1 \uparrow & & & & \downarrow \delta_0 \\ KK_1(C_0(X \wedge S^{2m}), C_0(Z)) & \leftarrow & KK_1(C_0(X \times S^{2m}), C_0(Z)) & \leftarrow & KK_1(C_0(X \vee S^{2m}), C_0(Z)). \end{array}$$

We also have the following long exact sequence of kk (see 3.2.10 of [DN]):

$$\cdots \to kk(Z, X \vee S^{2m}) \to kk(Z, X \times S^{2m}) \to kk(Z, X \wedge S^{2m}) \to$$
$$kk_{-1}(Z, X \vee S^{2m}) \to \cdots .$$

Combining this with the above exact sequence of KK fone has

$$\begin{array}{ccccccc} kk(Z, X \vee S^{2m}) & \to & kk(Z, X \times S^{2m}) & \to & kk(Z, X \wedge S^{2m}) & \to & kk_{-1}(Z, X \vee S^{2m}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ KK(C_0(X \vee S^{2m}), C_0(Z)) & \to & KK(C_0(X \times S^{2m}), C_0(Z)) & \to & KK(C_0(X \wedge S^{2m}), C_0(Z)) & \to & KK_1(C_0(X \vee S^{2m}), C_0(Z)) \end{array}$$

Let $Z = X \wedge S^{2m}$. Then Z is $(2m-1)$-connected. Hence $kk_{-1}(Z, X \vee S^{2m}) \to KK_1(C_0(X \vee S^{2m}), C_0(Z))$ is an isomorphism (see Theorem 3.4.5 of [DN] and notice that $2m \geq \dim(X)$). Since $KK(C_0(X \wedge S^{2m}), C_0(Z)) \xrightarrow{\delta} KK_1(C_0(X \vee S^{2m}), C_0(Z))$ is the zero map, we know that the map

$$kk(X \wedge S^{2m}, X \wedge S^{2m}) \to kk_{-1}(X \wedge S^{2m}, X \vee S^{2m})$$

is the zero map. Hence $kk(X \wedge S^{2m}, X \times S^{2m}) \to kk(X \wedge S^{2m}, X \wedge S^{2m})$ is a surjection.

2.9 There is a natural isomorphism T from $kk(X \wedge S^{2m}, X)$ to $kk(X \wedge S^{2m+1}, X \wedge S^1)$ (see 3.1.9 of [DN]). Let $b \in kk(S^{2m+1}, S^1)$ denote the Bott element (see 3.3.2 of [DN]). We call $T^{-1}(\mathrm{id}(\otimes b)$ the Bott element in $kk(X \wedge S^{2m}, X)$, where $\mathrm{id} \in kk(X, X)$ and hence $\mathrm{id} \otimes b \in kk(X \wedge S^{2m+1}, X \wedge S^1)$. For our convenience, we denote $T^{-1}(\mathrm{id} \otimes b)$ by b. (We avoid using S^0, since it is not connected.)

We still denote by $\pi \in kk(X \times S^{2m}, X \wedge S^{2m})$ the element corresponding to the canonical map $\pi : X \times S^{2m} \to X \wedge S^{2m}$. Also, $i \in kk(X \vee S^{2m}, X \times S^{2m})$ denotes the inclusion. We use $p_1 \in kk(X \times S^{2m}, X)$ and $p_2 \in kk(X \times S^{2m}, S^{2m})$ to denote the elements induced by the projection maps from $X \times S^{2m}$ to the first factor X and to the second factor S^{2m} respectively.

Recall that if $\alpha \in \mathrm{kk}(X, Y)$ and $\beta \in \mathrm{kk}(Z, X)$, then $\alpha \times \beta \in \mathrm{kk}(Z, Y)$. (Notice the change of order, since $\mathrm{kk}(X, Y)$ corresponds to $\mathrm{KK}(C_0(Y), C_0(X))$.)

LEMMA 2.10. *There is an* $\alpha \in \mathrm{kk}(X \wedge S^{2m}, X \times S^{2m})$ *such that:*
(i) $\pi \times \alpha = \mathrm{id} \in \mathrm{kk}(X \wedge S^{2m}, X \wedge S^{2m})$;
(ii) $p_1 \times \alpha = b \in \mathrm{kk}(X \wedge S^{2m}, X)$ *is the Bott element;*
(iii) $p_2 \times \alpha = 0 \in \mathrm{kk}(X \wedge S^{2m}, S^{2m})$.

PROOF. Let $i_1 \in \mathrm{kk}(X, X \times S^{2m})$ and $i_2 \in \mathrm{kk}(S^{2m}, X \times S^{2m})$ denote the compositions $X \hookrightarrow X \vee S^{2m} \xrightarrow{i} X \times S^{2m}$ and $S^{2m} \hookrightarrow X \vee S^{2m} \xrightarrow{i} X \times S^{2m}$, respectively. Then

$$p_1 \times i_1 = \mathrm{id} \in \mathrm{kk}(X, X) \quad \text{and} \quad p_2 \times i_2 = \mathrm{id} \in \mathrm{kk}(S^{2m}, S^{2m}) .$$

By 2.8, there is an $\alpha_1 \in \mathrm{kk}(X \wedge S^{2m}, X \times S^{2m})$ whose image under the canonical map $\mathrm{kk}(X \wedge S^{2m}, X \times S^{2m}) \to \mathrm{kk}(X \wedge S^{2m}, X \wedge S^{2m})$ is id. That is,

$$\pi \times \alpha_1 = \mathrm{id} \in \mathrm{kk}(X \wedge S^{2m}, X \wedge S^{2m}) .$$

Notice that

$$\mathrm{kk}(Z, X \vee S^{2m}) = \mathrm{kk}(Z, X) \oplus \mathrm{kk}(Z, S^{2m}) .$$

Set

$$\begin{aligned}\beta = (p_1 \times \alpha_1 - b) \oplus (p_2 \times \alpha_1) &\in \mathrm{kk}(X \wedge S^{2m}, X) \oplus \mathrm{kk}(X \wedge S^{2m}, S^{2m}) \\ &= \mathrm{kk}(X \wedge S^{2m}, X \vee S^{2m}) .\end{aligned}$$

Set $\alpha = \alpha_1 - i \times \beta$. Then

$$\begin{aligned}\pi \times \alpha &= \pi \times \alpha_1 - \pi \times i \times \beta = \mathrm{id} - 0 \times \beta = \mathrm{id} \in \mathrm{kk}(X \wedge S^{2m}, X \wedge S^{2m}) \\ p_1 \times \alpha &= p_1 \times \alpha_1 - p_1 \times i \times p_1 \times \alpha_1 + p_1 \times i \times b - p_1 \times i \times p_2 \times \alpha_1 \\ &= p_1 \times \alpha_1 - p_1 \times \alpha_1 + b - 0 = b \in \mathrm{kk}(X \wedge S^{2m}, X) , \quad \text{and} \\ p_2 \times \alpha &= p_2 \times \alpha_1 - 0 - p_2 \times i \times p_2 \times \alpha_1 = 0 \in \mathrm{kk}(X \wedge S^{2m}, S^{2m}) .\end{aligned}$$

LEMMA 2.11. *For any fixed finite subset* $F \subset C(X \wedge S^{2m})$ *and* $\varepsilon > 0$, *there are a unital homomorphism* $\phi : C(X \wedge S^{2m}) \to M_n(C(X \wedge S^{2m}))$ *(for n large enough), a* C^**-algebra* $B = \bigoplus_{i=1}^{t} M_{k_i}(C(X))$, *and a unital homomorphism* $\psi : \bigoplus_{i=1}^{t} M_{k_i}(C(X)) \to M_n(C(X \wedge S^{2m}))$, *with the following properties:*
(i) $\phi = \mathrm{id} \in \mathrm{kk}(X \wedge S^{2m}, X \wedge S^{2m})$;
(ii) *Each partial map* ψ^i *of* ψ *from* $M_{k_i}(C(X))$ *to* $M_n(C(X \wedge S^{2m}))$ *defines the Bott element* $b \in \mathrm{kk}(X \wedge S^{2m}, X)$;

(iii) $\psi^i(\mathbf{1}_{k_i})$ *are trivial projections in* $M_n(C(X \wedge S^{2m}))$;
(iv) $\operatorname{dist}(\phi(f), \psi(B)) < \varepsilon$ *for each* $f \in F$.

Proof. Let $\phi_1 : C(X \wedge S^{2m}) \to C(X \times S^{2m})$ denote the homomorphism induced by $\pi : X \times S^{2m} \to X \wedge S^{2m}$. Consider $F_1 = \phi_1(F) \subset C(X \times S^{2m})$, $\varepsilon > 0$, and a finite set of generators $G \subset C(S^{2m})$. There is an $\eta > 0$ with the following property. If $\phi_2' : C(S^{2m}) \to M_{n_1}(C(S^{2m}))$ is a unital homomorphism and $B_1 \subset M_{n_1}(C(S^{2m}))$ is a finite dimensional sub C*-algebra with $\operatorname{dist}(\phi_2'(g), B_1) < \eta$, for each $g \in G$, then

$$\phi_2 = \mathrm{id}_{C(X)} \otimes \phi_2' : C(X \times S^{2m}) \to M_{n_1} C(X \times S^{2m}) (= C(X) \otimes M_{n_1}(C(S^{2m})))$$

and $B = C(X) \otimes B_1 \subset C(X) \otimes M_{n_1}(C(S^{2m}))$ satisfy the relation

$$\operatorname{dist}(\phi_2(f), B) < \varepsilon \qquad \text{for each} \quad f \in F_1 .$$

For such an η, by Lemma 2.6, one can find $\phi_2' : C(S^{2m}) \to M_{n_1}(C(S^{2m}))$ (for n_1 large enough) to satisfy the above condition and $[\phi_2'] = \mathrm{id} \in \mathrm{kk}(S^{2m}, S^{2m})$. Let $\phi_3' : C(X \times S^{2m}) \to M_{n_2}(C(X \wedge S^{2m}))$ represent $\alpha \in \mathrm{kk}(X \wedge S^{2m}, X \times S^{2m})$ in 2.10. And let $\phi_3 : M_{n_1} C(X \times S^{2m}) \to M_{n_1 n_2}(C(X \wedge S^{2m}))$ be defined by $\phi_3 = \phi_3' \otimes \mathrm{id}_{M_{n_1}}$. Finally, let $\phi : C(X \wedge S^{2m}) \to M_{n_1 n_2}(C(X \wedge S^{2m}))$ be defined by $\phi = \phi_3 \circ \phi_2 \circ \phi_1$ and $\psi : B(= C(X) \otimes B_1 := \bigoplus_{i=1}^{t} M_{k_i}(C(X))) \to M_{n_1 n_2} C(X \wedge S^{2m})$ be defined by the composition

$$B \hookrightarrow M_{n_1}(C(X \times S^{2m})) \xrightarrow{\phi_3} M_{n_1 n_2}(C(X \wedge S^{2m})) .$$

It is obvious that (iv) holds. (i) follows from (i) of Lemma 2.10, and $[\phi_2'] = \mathrm{id} \in \mathrm{kk}(S^{2m}, S^{2m})$. (ii) and (iii) follows from (ii) and (iii) of Lemma 2.10 respectively. (Notice that if $i : B \hookrightarrow M_{n_1} C(X \times S^{2m})$ is the inclusion, then $i(\mathbf{1}_{M_{k_i}}) \in M_{n_1} C(X \times S^m)$ corresponds to a vector bundle over $X \times S^{2m}$ which is a pull back of a vector bundle over S^{2m} via $p_2 : X \times S^{2m} \to S^{2m}$.)

Remark 2.12. In the above proof, one can replace $C(X)$ by $M_k(C(X))$. In this case, we can also require $\phi(e_{ij}) \in \psi(B)$. Furthermore, in the proof one can require that ϕ_1 and ϕ_3 are injective. Hence ϕ and ψ can be chosen to be injective. Finally, one can require that ψ^i takes any trivial projection to a trivial projection.

2.13. Let us say that two homomorphisms $\phi, \psi : A \to B$ are approximately unitarily equivalent if for any finite set $F \subset A$ and $\varepsilon > 0$, there is a unitary $u \in B$ such that $\|\phi(f) - \mathrm{Ad}u \circ \psi(f)\| < \varepsilon$ for each $f \in F$. Using the approximately intertwining argument, one can easily prove the following (see [Ell] and [R]).

PROPOSITION 2.14. *If there are sequences of homomorphisms $\phi_n : A \to B$, and $\psi_n : B \to A$ such that for each n, $\psi_n \circ \phi_n$ is approximately unitarily equivalent to* $\mathrm{id} \in \mathrm{Hom}(A, A)$ *and $\phi_{n+1} \circ \psi_n$ is approximately unitarily equivalent to* $\mathrm{id} \in \mathrm{Hom}(B, B)$, *then A is isomorphic to B. Furthermore, one can find an isomorphism $\phi : A \to B$, with ϕ_n approximately unitarily equivalent to ϕ and ψ_n approximately unitarily equivalent to ϕ^{-1}, for each n.*

The following lemma will often be used to deduce the isomorphism of two C^*-algebras from a one-sided intertwining.

LEMMA 2.15. *Suppose that A and B are separable nuclear C^*-algebras of real rank zero and stable rank one. If $\phi : A \to B$ is surjective and induces an isomorphism from $K_0(A)$ to $K_0(B)$, then ϕ is an isomorphism.*

To prove the above lemma, one only needs to notice that, every ideal of A is generated by the projections inside the ideal. Furthermore, it is known that, for each ideal $I \subset A$, the canonical map from $K_0(I)$ to $K_0(A)$ is injective, since A is of real rank zero and stable rank one.

We will quote some notations from [EfK]

DEFINITION 2.16 ([EfK]). Suppose that (A_n, φ_{nm}) and (B_n, ψ_{nm}) are two inductive limit systems. A *system map* $\underline{\alpha} : (A_n) \to (B_n)$ consists of a sequence of integers $l_1 < l_2 < l_3 \ldots$ and unital homomorphisms $\alpha_n : A_n \to B_{l_n}$ such that each square of the diagram

$$\begin{array}{ccccc} A_1 & \rightarrow & A_2 & \rightarrow & \cdots \\ {\scriptstyle\alpha_1}\downarrow & & {\scriptstyle\alpha_2}\downarrow & & \\ B_{l_1} & \rightarrow & B_{l_2} & \rightarrow & \cdots \end{array}$$

commutes at the level of homotopy.

We say the two system maps $\underline{\alpha} : (A_n) \to (B_n)$ and $\underline{\beta} : (A_n) \to (B_n)$ are homotopic if for each n, there is an m larger than $l_n(\alpha)$ and $l_n(\beta)$ (here the $l_n(\alpha)$ and $l_n(\beta)$ denote the above l_n for α and β respectively) such that $\psi_{l_n(\alpha),m} \circ \alpha_n$ is homotopic to $\psi_{l_n(\beta),m} \circ \beta_n$.

2.17. Suppose that there are two sequences $k_1 < k_2 < k_3 < \cdots$ and $l_1 < l_2 < l_3 < \cdots$ and a sequence of homomorphisms $\alpha_n : A_{k_n} \to B_{l_n}$ such that each square of the diagram

$$\begin{array}{ccccccc} A_{k_1} & \rightarrow & A_{k_2} & \rightarrow & A_{k_3} & \cdots \\ {\scriptstyle\alpha_1}\downarrow & & {\scriptstyle\alpha_2}\downarrow & & {\scriptstyle\alpha_3}\downarrow & \\ B_{l_1} & \rightarrow & B_{l_2} & \rightarrow & B_{l_3} & \cdots \end{array}$$

commutes, at the level of homotopy. Then the homomorphisms α_n also induce a system map $\underline{\alpha} = (\tilde{\alpha}_n)$, where $\tilde{\alpha}_n : A_n \to B_{l_n}$ is defined by $\tilde{\alpha}_n = \alpha_n \circ \phi_{n,k_n}$. (Notice that $k_n \geq n$.)

2.18. As pointed out in [EfK], system maps can be composed in the obvious way. Let $\underline{\mathrm{id}} : (A_n) \to (A_n)$ be given by $\mathrm{id} : A_n \to A_n$. We say two systems (A_n) and (B_n) are shape equivalent if there are system maps $\underline{\alpha} : (A_n) \to (B_n)$ and $\underline{\beta} : (B_n) \to (A_n)$ such that $\underline{\alpha} \circ \underline{\beta} = \underline{\mathrm{id}}_{(B_n)}$ and $\underline{\beta} \circ \underline{\alpha} = \underline{\mathrm{id}}_{(A_n)}$. In [EG2], George A. Elliott and the author proved that for certain inductive limit systems, shape equivalence implies that the limit algebras are isomorphic. We will quote the results here.

PROPOSITION 2.19. *Let* $A = \bigoplus_{i=1}^{s} M_{k_i}(C(X_i))$, $B = \bigoplus_{i=1}^{t} M_{l_i}(C(Y_i))$, *where* X_i, Y_i *are arbitrary connected finite* CW *complexes. Suppose that the two unital homomorphisms* $\phi, \psi : A \to B$ *are homotopic, and suppose that* $F \subset A$ *is weakly approximately constant to within* ε *(see 1.4 of* [EG2]; *for the definition of this). There exists* $\delta > 0$ *and an integer* N *with the following property: For any* $C = \bigoplus_{i=1}^{r} M_{m_i}(C(Z_i))$, *if a unital homomorphism* $\alpha : B \to C$ *satisfies*

(i) $\mathrm{SPV}(\alpha^{i,j}) \leq \delta$ *for each partial map* $\alpha^{i,j}$ *of* α *and*
(ii) $\mathrm{rank}(\mathbf{1}_{B_n^i}) \geq N(\dim(Z_j) + 1)$,

then there is a unitary $U \in C$ *such that*

$$\|U^* \alpha \circ \psi(f) U - \alpha \circ \phi(f)\| \leq 70\varepsilon$$

for all $f \in F$.

The above proposition did not appear in [EG2] in the above form. But it was proved in the proof of Theorem 2.29 of [EG2]. Notice that one needs to modify the statement and the proof of another theorem, Theorem 2.21 of [EG2] slightly. Namely, change "there is a δ" to "there is a δ and an integer N" and change "$\mathrm{SPV}(\phi) < \frac{\alpha}{\dim Y + 1}$" to "$\mathrm{SPV}(\phi) < \delta$, and $\mathrm{rank}(p) \geq N(\dim Y + 1)$". In the proof, one can choose $N = 2(L+1)(K+1)^n$, and $\delta = \delta_2$. (See [EG2] for details.)

2.20. In [EG2], we gave a definition of slow dimension growth as follows. We say an inductive limit system $A = \lim(A_n = \bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m})$ is of slow dimension growth if for each n,

$$(*)\qquad \lim_{m\to\infty} \min_{\operatorname{rank}\phi^{i,j}_{n,m}(\mathbf{1}_{A^i_n})\neq 0} \left\{ \frac{\operatorname{rank}\phi^{i,j}_{n,m}(\mathbf{1}_{A^i_n})}{\dim \mathrm{SP}A^j_m + 1} \right\} = +\infty .$$

(This definition also appeared in [BE].)

However, if A is not simple, then the uniform boundedness of the dimensions of the spaces $X_{n,i}$ does not imply the above slow dimension growth condition. In this paper, we will use the following more general slow dimension growth condition: for any n, there exists a positive integer M such that

$$(**)\qquad \lim_{m\to\infty} \min_{\substack{\dim X_{m,j}>M \\ \operatorname{rank}\phi^{i,j}_{n,m}(\mathbf{1}_{A^i_n})\neq 0}} \left\{ \frac{\operatorname{rank}\phi^{i,j}_{n,m}(\mathbf{1}_{A^i_n})}{\dim \mathrm{SP}A^j_m + 1} \right\} = +\infty .$$

Here we use the convention that the minimum of the empty set is $+\infty$. If the spaces $X_{m,j}$ have uniformly bounded dimension, then one can choose $M = \sup_{m,j} \dim(X_{m,j})$; then $(**)$ automatically holds.

2.21. Suppose that $A = \lim_{n}(A_n, \phi_{n,m})$ is an inductive limit with slow dimension growth as in $(**)$. Suppose that A is of real rank zero. For each n, there is a positive integer M with the property that for any positive integer N, there is an m such that for each block A^i_n and each block A^j_m, either

$$\operatorname{rank}\phi^{i,j}_{n,m}(\mathbf{1}_{A^i_n}) \le N\cdot(M+1)$$

or

$$\operatorname{rank}\phi^{i,j}_{n,m}(\mathbf{1}_{A^i_n}) \ge N\cdot(\text{dim of } \mathrm{SP}A^j_m + 1).$$

2.22. Let us say that a homomorphism $\phi : M_k(C(X)) \to PM_l(C(Y))P$ is defined by point evaluations if there exist finitely many points $x_1, x_2, \dots, x_m$ such that $\phi(f) = \phi(g)$ if and only if $f(x_i) = g(x_i)$. This means that $\phi(f)$ depends only on the values of f at finitely many points. We shall say that ϕ is defined by base point evaluation if $\phi(f) = \phi(g)$ if and only if $f(x_0) = g(x_0)$, where x_0 is the base point of X. (Notice that unital homomorphisms defined by base point evaluation may not be unitary equivalent to each other, since they may define different K-theory maps.) A homomorphism ϕ is defined by point evaluations if and only if image (ϕ) is a finite dimensional C*-algebra.

Combining 2.21, and Lemma 2.3 of [EG2] (or more precisely, the proof of it), one can prove

LEMMA 2.23. *Suppose that $A = \lim(A_n, \phi_{n,m})$ is an inductive limit satisfying the slow dimension growth condition $(**)$. Suppose that A is of real rank zero.*

For any finite subset $F_n^i \subset A_n^i \subseteq A_n$, any $\varepsilon > 0$, and any integer N, there is an $m > n$ such that each partial map $\phi_{n,m}^{i,j}$ satisfies either
(1) $\operatorname{rank}(\phi_{n,m}^{i,j}(\mathbf{1}_{A_n^i})) \geq N(\dim X_{m,j} + 1)$ *or*
(2) $\phi_{n,m}^{i,j}$ *is homotopic to a homomorphism ϕ defined by point evaluations and*

$$\|\phi_{n,m}^{i,j}(f) - \phi(f)\| < \varepsilon$$

for each $f \in F$.

Using the above lemma, one can prove the following analogy of Theorem 2.29 of [EG2].

PROPOSITION 2.24. *Let $C = \bigoplus_{i=1}^{s} M_{k_i}(C(X_i))$ and let A be a real rank zero unital inductive limit of a sequence($A_n = \bigoplus_{i=1}^{k_n} M_{[n,i]}\big((X_{n,i}), \phi_{n,m}\big)$ satisfying the slow dimension growth condition (**). Suppose that a finite set $F \subset C$ is weakly approximately constant to within ε in each block of C. If two homomorphisms $\phi, \psi : C \to A_n$ are homotopic, then there exist $m > n$ and a unitary $U \in A_m$ such that*

$$\|\phi_{n,m} \circ \phi(f) - U\phi_{n,m} \circ \psi(f) U^*\| \leq 70\varepsilon$$

DEFINITION 2.25. Suppose that $A = \lim(A_n, \phi_{n,m})$ and $B = \lim(B_n, \psi_{n,m})$ are inductive limit systems. Two system maps $\underline{\alpha}, \underline{\beta} : (A_n) \to (B_n)$ defined by

$$\begin{array}{ccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & \cdots \\
{\scriptstyle \alpha_1}\big\downarrow & & {\scriptstyle \alpha_2}\big\downarrow & & {\scriptstyle \alpha_3}\big\downarrow & & \\
B_{l_1} & \longrightarrow & B_{l_2} & \longrightarrow & B_{l_3} & \longrightarrow & \cdots
\end{array}$$

and

$$\begin{array}{ccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & \cdots \\
{\scriptstyle \beta_1}\big\downarrow & & {\scriptstyle \beta_2}\big\downarrow & & {\scriptstyle \beta_3}\big\downarrow & & \\
B_{k_1} & \longrightarrow & B_{k_2} & \longrightarrow & B_{k_3} & \longrightarrow & \cdots
\end{array}$$

are said to be *equivalent* if for each A_n, there exist $m \geq \max(k_n, l_n)$ and a unitary $u \in B_m$ such that $\operatorname{Ad}u \circ \psi_{k_n,m} \circ \beta_n$ is homotopic to $\phi_{l_n,m} \circ \alpha_n$. (Compare with 2.16.)

PROPOSITION 2.26. *Suppose that $A = \lim(A_n, \phi_{n,m})$ and $B = \lim(B_n, \psi_{n,m})$ are of real rank zero and satisfy the slow dimension growth condition (**). Any system map $\underline{\alpha} : (A_n) \to (B_n)$ is equivalent to a system map $\underline{\beta} : (A_n) \to (B_n)$ which makes the diagram*

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & \cdots & \longrightarrow & A \\ {\scriptstyle\beta_1}\downarrow & & {\scriptstyle\beta_2}\downarrow & & {\scriptstyle\beta_3}\downarrow & & & & \\ B_{k_1} & \longrightarrow & B_{k_2} & \longrightarrow & B_{k_3} & \longrightarrow & \cdots & \longrightarrow & B \end{array}$$

a one-sided approximately intertwining in the sense of [Ell], *and therefore defines a homomorphism* $\beta_\infty : A \to B$. *Furthermore, if* $\underline{\beta}'$ *is another such system with map* β'_∞, *then* β_∞ *and* β'_∞ *are approximately unitarily equivalent.*

This proposition can be proved by applying 2.24.

DEFINITION 2.27. A weak system map $\underline{\alpha} : (A_n) \to (B_n)$ is a system of homomorphisms $\alpha_n : A_n \to B_{l_n}$ such that

$$\begin{array}{ccc} A_n & \xrightarrow{\phi_{n,n+1}} & A_{n+1} \\ {\scriptstyle\alpha_n}\downarrow & & \downarrow{\scriptstyle\alpha_{n+1}} \\ B_{l_n} & \xrightarrow{\psi_{l_n,l_{n+1}}} & B_{l_{n+1}} \end{array}$$

commutes at the level of K_0 and kk (i.e. $\alpha_{n+1} \circ \phi_{n,n+1}$ and $\psi_{l_n,l_{n+1}} \circ \alpha_n$ define the same maps at the level of K_0 and kk in the sense of 2.5). Two weak system maps $\underline{\alpha}$ and $\underline{\beta}$ are equivalent if, for each n, there is an $m \geq l_n(\alpha)$, $l_n(\beta)$ such that

$$\phi_{l_n(\alpha),m} \circ \alpha_n \quad \text{and} \quad \psi_{l_n(\beta),m} \circ \beta_n$$

define the same map at the level of K_0 and kk. Note that a system map is a weak system map.

REMARK 2.28. Combining Proposition 2.5 and §1.6 of [EG2], one can generalize Proposition 2.26 to the case that $\underline{\alpha}$ is only a weak system map. Also, we can still require that $\underline{\beta}$ be a system map.

LEMMA 2.29. *Let* $A = \lim(A_n, \phi_{nm})$ *be an inductive limit system. Let* $u_n \in A_n$ $(1 \leq n < +\infty)$ *be a sequence of unitaries. Set* $\psi_{n-1,n} = \mathrm{Ad}u_n \circ \phi_{n-1,n}$. *Then* $\varinjlim(A_n, \phi_{n,m}) \cong \varinjlim(A_n, \psi_{n,m})$.

PROOF. Set $v_2 = u_2$, $v_3 = u_3 \cdot \phi_{23}(v_2)$, $v_4 = u_4 \cdot \phi_{34}(v_3), \ldots$. In general, set $v_n = u_n \cdot \phi_{n-1,n}(v_{n-1})$. The diagram

$$\begin{array}{ccccccc} A_1 & \xrightarrow{\phi_{1,2}} & A_2 & \xrightarrow{\phi_{2,3}} & A_3 & \longrightarrow & \cdots \\ {\scriptstyle\mathrm{id}}\uparrow & \overset{\mathrm{Ad}v_2\cdot\phi_{12}}{\searrow} & {\scriptstyle\mathrm{Ad}v_2^{-1}}\uparrow & \overset{\mathrm{Ad}v_3\cdot\phi_{23}}{\searrow} & \uparrow{\scriptstyle\mathrm{Ad}v_3^{-1}} & & \\ A_1 & \xrightarrow{\psi_{1,2}} & A_2 & \xrightarrow{\psi_{23}} & A_3 & \longrightarrow & \cdots \end{array}$$

commutes.

2.30. A weak system map $\underline{\alpha} : (A_n) \to (B_n)$ induces a system of maps on the K-theory which is commutative. So it induces a K-map $K_*\alpha : K_*A \to K_*B$.

3. Suspension and a special reduction.

First, we would like to prove an analogue of Theorem 2.29 of [EG2]; see also Proposition 2.24 above.

3.1. In this section, we will reserve the notation P, Q for polynomials. Therefore, projections will be denoted by E, e, or small letters p, q.

By a polynomial P, we mean a polynomial of n real variables which is defined on

$$[-1,1]^n = \underbrace{[-1,1] \times [-1,1] \times \cdots [-1,1]}_{n-\text{copies}} .$$

If P is a polynomial in n variables, and $x_1, x_2, \ldots, x_n \in A$ are self-adjoint elements with $\|x_i\| \le 1$, we define $P(x_1, x_2, \ldots, x_n) \in A$ to be the corresponding linear combination of

$$x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n} .$$

That is, such a term as $x_2 x_1$ does not appear in the linear combination. (Note that if $P(t_1, t_2) = t_1 t_2 = t_2 t_1$ then $P(x_1, x_2) = x_1 x_2$ (may not be equal to $x_2 x_1$).) It is well known, for any polynomial P and $\varepsilon > 0$, that there exists $\delta > 0$ such that $\|x_i - y_i\| \le \delta \Rightarrow \|P(x_1 \ldots x_n) - P(y_i \ldots y_n)\| < \varepsilon$.

3.2. Let X be a finite CW complex and let $\hat{X}$ denote the cone space CX of X. From [DN] (see 2.11 of [EG2] for details) it following, for any finite CW complex X, there is a homomorphism

$$\mathrm{id}^* : C(X) \to M_{k-1}(C(X))$$

such that the map $\mathrm{id} \oplus \mathrm{id}^* : C(X) \to M_k(C(X))$ can can be factored as

$$C(X) \xrightarrow{\theta} M_k(C(\hat{X})) \xrightarrow{\iota} M_k(C(X)),$$

where ι denotes the map induced by the inclusion of X into $\hat{X}$. We will fix the notation θ, ι and id^* in this section.

3.3. Suppose that $\hat{X}$ is a subspace of $[-1,1]^n$. Let $g_1, g_2, \ldots, g_n$ denote the coordinate functions of $\hat{X}$. That is, g_i is the composition

$$\hat{X} \hookrightarrow [-1,1]^n \xrightarrow{\text{project to the ith coordinate}} [-1,1] .$$

Then $g_1, g_2, \ldots, g_n$ generate $C(\hat{X})$.

For any finite set $F \subset M_a(C(\hat{X}))$ and $\varepsilon > 0$, one can find a set $\mathscr{P}_F^\varepsilon$ of finitely many polynomials in n variables $g_1, g_2, \dots, g_n$ with the following property. Let $\tilde{\mathscr{P}}_F^\varepsilon$ denote the set of all $a \times a$ matrices with entries polynomials of $g_1, g_2, \dots, g_n$ belonging to $\mathscr{P}_F^\varepsilon$. That is,

$$\tilde{\mathscr{P}}_F^\varepsilon = \{(P_{ij})_{a\times a} \mid P_{i,j} \in \mathscr{P}_F^\varepsilon \quad \text{for each} \quad i,j\} .$$

The condition we require is that, for any $f \in F$, there be an element $P \in \tilde{\mathscr{P}}_F^\varepsilon$ such that

$$\|f(x) - P(g_1(x), g_2(x), \dots, g_n(x))\| < \varepsilon$$

for all $x \in \hat{X}$. The choice of $\mathscr{P}_F^\varepsilon$ is certainly not unique. If one further supposes that F is weakly approximately constant to within ε, then one can choose a finite set $\mathring{\mathscr{P}}_F^\varepsilon \subset \tilde{\mathscr{P}}_F^\varepsilon$ such that

(i) $\operatorname{dist}(f, \mathring{\mathscr{P}}_F^\varepsilon) < \varepsilon$ for each $f \in F$,

(ii) $\mathring{\mathscr{P}}_F^\varepsilon$ is approximately constant to within 2ε.

The following result is an analogue of Theorem 2.8 of [EGLP1] which can be proved by the same method.

LEMMA 3.4. *Suppose that $\hat{X}$ is as above and $G = \{g_1, g_2, \dots, g_n\}$ is the generating set as above. For any $\varepsilon > 0$, there exist $\delta > 0$, and an integer N such that if a positive linear unital map $\phi : C(\hat{X}) \to A$ satisfies*

$$\|\phi(g_i g_j) - \phi(g_i) \cdot \phi(g_j)\| < \delta$$

for all $1 \le i,j \le n$, then there are $\xi_j \in \hat{X}$ and $\lambda_k \in \hat{X}$, $j = 1, 2, \dots, m$, $k = 1, 2, \dots, l$, and two sets of mutually orthogonal projections $p_j \in M_N(A)$ and $q_k \in M_{N+1}(A)$ such that

$$\Sigma p_j = \mathbf{1}_N , \quad \Sigma q_k = \mathbf{1}_{N+1} \quad \text{and} \quad \left\| \phi(g_i) \oplus \sum_{j=1}^{m} g_i(\xi_j) p_j - \sum_{k=1}^{l} g_i(\lambda_k) q_j \right\| < \varepsilon$$

for $1 \le i \le n$.

In Theorem 2.8 in [EGLP1], it is assumed that the map is a cut-down of a homomorphism. But, in the proof, we did not use any special property of a cutting down of a homomorphism. The property of $p\phi p$ used in Theorem 2.8 of [EGLP1] is that

$$\|p\phi(g_i)p \cdot p\phi(g_j)p - p\phi(g_j)p \cdot p\phi(g_i)p\| < \delta$$

for each $1 \le i, j \le n$. Note that if one replaces $\hat{X}$ by $[-1, 1]^n$, the above lemma was stated in the proof of Theorem 2.1 of [EGLP1].

3.5. Suppose that $\mathscr{P}$ is a finite set of polynomials in n real variables. For each $P \in \mathscr{P}$, one can regard P as a function from $\hat{X}$ to C by

$$P(x) = P\big(g_1(x), g_2(x), \ldots, g_n(x)\big).$$

For any $\varepsilon_1 > 0$, there are an $\varepsilon_2 > 0$ and a $\delta > 0$ such that, if

$$\|\phi(g_i \cdot g_j) - \phi(g_i) \cdot \phi(g_j)\| < \delta \quad \text{and}$$
$$\left\| \phi(g_i) \oplus \sum_{j=1}^{m} g_i(\xi_j) p_j - \sum_{k=1}^{l} g_i(\lambda_k) q_k \right\| < \varepsilon_2,$$

then

$$\left\| P\big(\phi(g_1), \phi(g_2), \ldots, \phi(g_n)\big) \oplus \sum_{j=1}^{m} P(\xi_j) p_j - \sum_{k=1}^{l} P(\lambda_k) q_k \right\| < \varepsilon_1$$

for all $P \in \mathscr{P}$.

If one further assumes that

$$\|P\big(\phi(g_1), \ldots, \phi(g_n)\big) - \phi(P)\|$$

is very small for every $P \in \mathscr{P}$, then one can conclude that

$$\left\| \phi(P) \oplus \sum_{j=1}^{m} P(\xi_j) \hat{p}_j - \sum_{k=1}^{l} P(\lambda_k) q_k \right\|$$

is very small for every $P \in \mathscr{P}$.

Therefore, the following corollary holds.

COROLLARY 3.6. *Let $\hat{X}$, $G = \{g_1, g_2, \ldots, g_n\}$, and $\mathscr{P}$ be as above. For any $\varepsilon > 0$, there exist $\delta > 0$ and an integer N such that if ϕ is a completely positive contractive linear map from $C(\hat{X})$ to an arbitrary* C*-*algebra A with*

$$\|\phi(g_i g_j) - \phi(g_i)\phi(g_j)\| < \delta \quad \textit{for any} \quad 1 \le i,\ j \le n\ ,$$

and

$$\|\phi(P(g_1, g_2, \ldots, g_n)) - P\big(\phi(g_1), \ldots, \phi(g_n)\big)\| < \delta \quad \textit{for any} \quad P \in \mathscr{P}\ ,$$

then there exist $\xi_j \in \hat{X}$, $\lambda_k \in \hat{X}$ $j = 1, 2, \ldots, m$, $k = 1, 2, \ldots, l$, and two sets of mutually orthogonal projections $p_j \in M_N(A)$, and $q_k \in M_{N+1}(A)$ such that

$$\Sigma p_j = \mathbf{1}_N \ , \quad \Sigma q_k = \mathbf{1}_{N+1}, \quad and$$

$$\left\| \phi(P) \oplus \sum_{j=1}^{m} P(\xi_j)p_j - \sum_{k=1}^{l} P(\lambda_k)q_k \right\| < \varepsilon$$

for all $P \in \mathscr{P}$, and

$$\left\| \phi(g) \oplus \sum_{j=1}^{m} g(\xi_i)p_j - \sum_{k=1}^{l} g(\lambda_k)q_k \right\| < \varepsilon$$

for all $g \in G = \{g_1, g_2, \dots, g_n\}$.

(Note that if $i \le j$, then the property

$$\|\phi(g_i g_j) - \phi(g_i)\phi(g_j)\| < \delta$$

can be considered to be a special case of the property

$$\|\phi(P(g_1, g_2, \dots, g_n)) - P\big(\phi(g_1), \dots, \phi(g_n)\big)\| < \delta,$$

by taking $P = x_i x_j$.)

REMARK 3.7. Consider the subset $\tilde{\mathscr{P}} \subset M_a(C(\hat{X}))$ defined by

$$\tilde{\mathscr{P}} = \left\{ (f_{ij}) \in M_a(C(\hat{X}))) \ ; \quad f_{ij} \in \mathscr{P} \quad \text{for all} \quad i,j \right\} .$$

As in the above corollary, one has that

$$\left\| (\phi \otimes \mathbf{1}_a)(P) \oplus \sum_{j=1}^{m} P(\xi_j)\hat{p}_j^{(a)} - \sum_{k=1}^{l} P(\lambda_k)\hat{q}_k^{(a)} \right\| < a^2 \cdot \varepsilon$$

for each $P \in \tilde{\mathscr{P}} \subset M_a(C(X))$. (The notation $\hat{p}_j^{(a)}$, $\hat{q}_k^{(a)}$ will be explained in 3.8.)

LEMMA 3.8. *Let $\varepsilon > 0$. Let a finite set $F \subset M_l(C(X))$ be weakly approximately constant to within ε. Consider the generating set $G = \{g_1, g_2, \dots, g_n) \subset C(\hat{X})$. Let $\theta : C(X) \to M_k(C(\hat{X}))$ be as in 3.2. There are a finite set of polynomials $\mathscr{P} \subset C(\hat{X})$, a $\delta > 0$, and an integer N such that if $\phi : C(\hat{X}) \to A$ is a completely positive linear contractive map with*

$$\|\phi(g_i g_j) - \phi(g_i)\phi(g_j)\| < \delta$$

for all $1 \le i,j \le n$, and

$$\|\phi\big(P(g_1, g_2, \dots, g_n)\big) - P\big(\phi(g_1), \dots, \phi(g_n)\big)\| < \delta,$$

then there is a unitary $U \in M_{(N+1)kl}(A)$ such that

$$\|(\phi \otimes \mathbf{1}_{kl}) \circ (\theta \otimes \mathrm{id}_l)(f) \oplus f(x_0)\hat{\mathbf{1}}_{Nk}^{(l)} - U^* f(x_0) \cdot \hat{\mathbf{1}}_{(N+1)k}^{(l)} U\| < 4\varepsilon$$

for all $f \in F$.

(Here we use the notations from 1.3.5 of [EG2], $\hat{p}^{(l)} = \underbrace{p \oplus p \oplus p \oplus \cdots p}_{l}$,

and for $a = (a_{ij}) \in M_l(C)$, $a \cdot \hat{p}^{(l)}$ is defined by

$$\begin{pmatrix} a_{1,1}p, & a_{1,2}p, & \dots & a_{1,l}p \\ \vdots & & & \vdots \\ a_{l,1}p, & a_{l,2}p, & \dots & a_{l,l}p \end{pmatrix}.$$

See 1.3.5 and 2.6 of [EG2] for details.)

This lemma is an analogue of Lemma 2.16 of [EG2]. The proof is exactly the same. Set $F_1 = \theta \otimes \mathbf{1}_l(F) \subset M_{kl}C(\hat{X})$. One can find $\mathscr{P}^{\varepsilon}_{F_1}$ as in 3.3. Set $\mathscr{P} = \mathscr{P}^{\varepsilon}_{F_1}$. With the aid of the sets $\mathscr{P}, \tilde{\mathscr{P}}$ (see Remark 3.7) and the contractivity of ϕ, one can use Corollary 3.6 and Remark 3.7 (instead of 1.3.5 of [EG2]) to prove the above lemma.

The following is a weak version of Theorem 2.29 of [EG2] for a path of almost homomorphisms, instead of a path of homomorphisms.

THEOREM 3.9. *Let $\varepsilon > 0$, and let the finite subset $F \subset M_{l_1}(C(X))$ be weakly approximately constant to within ε. Consider the generating subset $G = \{g_1, g_2, \dots, g_n\}$ of $C(\hat{X})$. (Also, one can regard g_i as $g_i|_X$, so that $\{g_1, g_2, \dots, g_n\}$ is also a generating subset of $C(X)$.) There is a $\delta > 0$, and a finite set of polynomials $\mathscr{P} \subset C(\hat{X})$ ($P \in \mathscr{P}$ can be regarded as $P|_X \in C(X)$) with the following property. If $\phi_t : C(X) \to M_{l_2}(C(Y))$ is a path of completely positive contractive maps satisfying*

$$\|\phi_t(g_i)\phi_t(g_j) - \phi_t(g_i g_i)\| < \delta \qquad \text{and}$$

$$\|\phi_t(g_1, g_2, \dots, g_n) - P(\phi_t(g_1), \phi_t(g_2), \dots, \phi_t(g_n))\| < \delta$$

for $1 \le i,j \le n$ and each $t \in [0,1]$, then there exists $N > 0$ such that if $l_3 \ge N$ and $\psi : C(X) \to M_{l_3}(C(Y))$ is defined by

$$\psi(f) = f(x_0) \in M_{l_3}(C(Y)) \quad \text{for each} \quad f \in C(X),$$

where x_0 is the base point of X, then

$$\|(\phi_0 \otimes \mathrm{id}_{l_1} \oplus \psi \otimes \mathrm{id}_{l_1})(f) - U^*(\phi_1 \otimes \mathrm{id}_{l_1} \oplus \psi \otimes \mathrm{id}_{l_1}(f))U\| \le 10\varepsilon,$$

for a certain unitary $U \in M_{l_1(l_2+l_3)}(C(Y))$, and all $f \in F$.

(In other words, one can write

$$\|\phi_0 \otimes \mathrm{id}_{l_1}(f) \oplus f(x_0)\hat{\mathbf{1}}_{l_3}^{(l_1)} - U^*(\phi_1 \otimes \mathrm{id}_{l_1} \oplus f(x_0)\hat{\mathbf{1}}_{l_3}^{(l_1)})U\| \le 10\varepsilon.)$$

PROOF. Applying Lemma 3.8 for F, let δ and $\mathscr{P} \subset C(\hat{X})$ and integer N_1 be as in Lemma 3.8. Let $0 = t_0 < t_1 < t_2 < \cdots < t_m = 1$ be such that

$$\|\phi_{t_i}(P) - \phi_{t_{i+1}}(P)\| \leq \frac{\varepsilon}{l_1^2}$$

$$\|((\phi_{t_i} \otimes \mathrm{id}_{k-1}) \circ \mathrm{id}^*)(P) - ((\phi_{t_{i+1}} \otimes \mathrm{id}_{k-1}) \circ \mathrm{id}^*)(P)\| \leq \frac{\varepsilon}{l_1^2}$$

for each $P \in \mathscr{P}$, where $\mathrm{id}^* : C(X) \to M_{k-1}(C(X))$ is as in 3.2. Denote the composed map

$$(\phi_t \otimes \mathrm{id}_{k-1}) \circ \mathrm{id}^* : C(X) \xrightarrow{\mathrm{id}^*} M_{k-1}(C(X)) \xrightarrow{\phi_t \otimes \mathrm{id}_{k-1}} M_{(k-1)l_2}(C(Y))$$

by ϕ_t^*.

From the above, we know that

$$\|(\phi_{t_i} \otimes \mathrm{id}_{l_1})(P) - (\phi_{t_{i+1}} \otimes \mathrm{id}_{l_1})(P)\| < \varepsilon \qquad \text{and}$$

$$\|(\phi_{t_i}^* \otimes \mathrm{id}_{l_1})(P) - (\phi_{t_{i+1}}^* \otimes \mathrm{id}_{l_1})(P)\| < \varepsilon$$

for all $P \in \tilde{\mathscr{P}}$ (see Remark 3.7 for the definition of $\tilde{\mathscr{P}}$). Therefore,

$$\|(\phi_{t_i} \otimes \mathrm{id}_{l_1})(f) - (\phi_{t_{i+1}} \otimes \mathrm{id}_{l_1})(f)\| \leq 2\varepsilon \qquad \text{and}$$

$$\|(\phi_{t_i}^* \otimes \mathrm{id}_{l_1})(f) - (\phi_{t_{i+1}}^* \otimes \mathrm{id}_{l_1})(f)\| \leq 2\varepsilon$$

for each $f \in F$, since each f can be approximated by an element in $\tilde{\mathscr{P}}$ to within ε. Denote $\phi_{t_i} \otimes \mathrm{id}_{l_1}$ by Φ_{t_i} and $\phi_{t_i}^* \otimes \mathrm{id}_{l_1}$ by $\Phi_{t_i}^*$. Set $\phi_t' = \phi_t \circ \iota :$ $C(\hat{X}) \xrightarrow{\iota} C(X) \xrightarrow{\phi_t} M_{l_2}(C(Y))$, where ι is as in 3.2. Then

$$(\phi_t' \otimes \mathrm{id}_{kl_1}) \circ (\theta \otimes \mathrm{id}_{l_1})(f) \;=\; \phi_t \otimes \mathrm{id}_{l_1}(f) \oplus \phi_t^* \otimes \mathrm{id}_{l_1}(f)\ ,$$

where θ is as in 3.2. Therefore,

$$\|\Phi_{t_i}(f) \oplus \Phi_{t_i}^*(f) \oplus f(x_0)\hat{\mathbf{1}}_{N_1k}^{(l_1)} - U_i^* f(x_0)\hat{\mathbf{1}}_{(N_1+1)k}^{(l_1)} U_i\| < 4\varepsilon$$

for certain U_i by Lemma 3.8. Set $N = k(m+1)(N_1+1)$ (where $m+1$ is the cardinal of $\{t_0, t_1, \ldots, t_m\}$). If $l_3 > N$, then $f(x_0)\hat{\mathbf{1}}_{l_3}^{(l_1)}$ is unitarily equivalent to

$$\begin{aligned} (1) \qquad & \Phi_{t_0}(f) \oplus \Phi_{t_0}^*(f) \oplus \Phi_{t_1}(f) \oplus \Phi_{t_1}^*(f) \oplus \cdots \oplus \\ & \Phi_{t_m}(f) \oplus \Phi_{t_m}^*(f) \oplus f(x_0)\hat{\mathbf{1}}_{l_3-k(m+1)}^{(l_1)}, \end{aligned}$$

by a single unitary for all $f \in F$ (the unitary does not depend on f), to within 4ε. So $\Phi_1(f) \oplus f(x_0)\hat{\mathbf{1}}_{l_3}^{l_1}$ is unitarily equivalent to

$$\Phi_{t_0}(f) \oplus \Phi^*_{t_0}(f) \oplus \Phi_{t_1}(f) \oplus \Phi^*_{t_1}(f) \oplus \cdots \oplus \tag{2}$$
$$\Phi_{t_m}(f) \oplus \Phi^*_{t_m}(f) \oplus \Phi_1(f) \oplus f(x_0)\hat{\mathbf{1}}^{(l_1)}_{l_3-k(m+1)}$$

to within 4ε for all $f \in F$ (by a single unitary).

On the other hand, (1) is unitarily equivalent to

$$\Phi^*_{t_0}(f) \oplus \Phi_{t_0}(f) \oplus \Phi^*_{t_1}(f) \oplus \Phi_{t_1}(f) \oplus \cdots \oplus \Phi^*_{t_m}(f) \oplus \Phi_{t_m}(f) \oplus f(x_0)\hat{\mathbf{1}}^{(l_1)}_{l_3-k(m+1)}$$

(by a single unitary). Hence, $\Phi_{t_0}(f) \oplus f(x_0)\hat{\mathbf{1}}^{(l_1)}_{l_3}$ is unitarily equivalent to

$$\Phi_{t_0}(f) \oplus \Phi^*_{t_0}(f) \oplus \Phi_{t_0}(f) \oplus \Phi^*_{t_1}(f) \oplus \cdots \oplus \tag{3}$$
$$\Phi^*_{t_m}(f) \oplus \Phi_{t_m}(f) \oplus f(x_0)\hat{\mathbf{1}}^{(l_1)}_{l_3-k(m+1)},$$

to within 4ε (by a single unitary). But

$$\|(3) - (2)\| \leq 2\varepsilon \ ,$$

since $\|\Phi_{t_i}(f) - \Phi_{t_{i+1}}(f)\| \leq 2\varepsilon$.

So we have proved that $\Phi_{t_0}(f) \oplus f(x_0)\mathbf{1}^{(l_1)}_{l_3}$ is unitarily equivalent to $\Phi_{t_1}(f) \oplus f(x_0)\hat{\mathbf{1}}^{(l_1)}_{l_3}$, by a single unitary, to within $4\varepsilon + 4\varepsilon + 2\varepsilon$.

REMARK 3.10. Chris Phillips [Phi] first used paths of unitaries u_t and u_t^* to study exponential rank. The clever idea of shifting the index (i.e., changing (3) to (2)) is due to him. This idea is also used in [GL]. The generalization of the idea to the case of paths of homomorphisms and the "adjoints" of homomorphisms first appeared in [EGLP1]. The idea of using the property of being approximately constant to change all the base point evaluations to base point evaluation first appeared in [EG2].

3.11. Let X be a finite CW complex. Let $2m \geq \dim X$. We can construct a real rank zero inductive limit $A(X)$ (of a sequence of algebras $A_n = M_{k_n}(C(X))$) with the following properties;

(i) each connecting homomorphism $\phi_{n,n+1}$ satisfies

$$[\phi_{n,n+1}] = \mathrm{id} \in \mathrm{kk}(X, X)$$

and $\phi_{n,n+1}$ takes any trivial projection in A_n to a trivial projection in A_{n+1};

(ii) $(K_*(A), K_*(A)_+, \mathbf{1}_A) = \big(\mathbb{Q} \oplus \tilde{K}^*(X), \mathbb{Q}_+ \oplus \tilde{K}^*(X) \cup \{(0,0)\}, (\mathbf{1}_\mathbb{Q} \oplus 0)\big)$.

Let $A = \lim(A_n, \phi_{n,m})$ and $A' = \lim(A'_n, \phi'_{n,m})$ be two systems satisfying the above condition for the same X. Using §4 of [DN] (also see §3 of [EG2]), one can prove that there is an intertwining between the two systems at the level of homotopy, and therefore $A \cong A'$. (See 2.23, 2.24, 2.25, 2.26, 2.27, and 2.28.)

In this section, we will prove that $A(X \wedge S^{2m})$ (where $2m > \dim X$) can be written as an inductive limit of algebras $M_{l_n}(C(X))$. More precisely, we will prove $A(X \wedge S^{2m}) \cong A(X)$. (We will make use of 2.11.) By using this fact, we will further prove that $A(X)$ is an inductive limit of direct sums of matrix algebras over 3-dimensional finite CW complexes of special type as defined above.

LEMMA 3.12. *For any finite subset $F \subset C(X \wedge S^{2m})$ and any $\varepsilon > 0$, there are unital homomorphisms $\phi : C(X \wedge S^{2m}) \to M_l(C(X \wedge S^{2m}))$ (for l large enough) and $\psi : M_{l_1}(C(X)) \to M_l(C(X \wedge S^{2m}))$ and a finite subset $F_1 \subset M_{l_1}(C(X))$ with the following properties;*

(i) $[\phi] = \mathrm{id} \in \mathrm{kk}(X \wedge S^{2m}, X \wedge S^{2m})$;

(ii) $[\psi] = b \in \mathrm{kk}(X \wedge S^{2m}, X)$ *is the Bott element;*

(iii) $\psi(e_{11})$ *is a trivial projection in* $M_l(C(X \wedge S^{2m}))$ *for the matrix unit* $e_{11} \in M_{l_1}(C(X))$;

(iv) $\mathrm{dist}(\phi(f), \psi(F_1)) < \varepsilon$, *for each* $f \in F$;

(v) F_1 *is approximately constant to within* ε.

PROOF. Let $\phi_1 : C(X \wedge S^{2m}) \to M_n(C(X \wedge S^{2m}))$, $B = \bigoplus_{i=1}^{k} M_{k_i}(C(X))$, and $\lambda : B \to M_n(C(X \wedge S^{2m}))$ be as in Lemma 2.11 (ϕ_1 is in place of ϕ and λ is in place of ψ). From the proof of the lemma, one can see that (iii) of Lemma 2.11 can be changed to the property that $\psi^i(e^i_{11})$ are trivial projections for all matrix units $e^i_{11} \in M_{k_i}(C(X))$. One can find finite sets $F^i \subset B^i = M_{k_i}(C(X))$ such that $\mathrm{dist}(\phi_1(f), \lambda(\oplus F^i)) < \varepsilon$ for each $f \in F$. Let us show that we may choose B carefully such that the following assertion holds.

Assertion: There are finite subsets $F^i \subset B^i$ ($1 \le i \le k$) such that each F^i is weakly approximately constant to within ε in B^i and

$$\mathrm{dist}(\phi_1(f), \lambda(\oplus_{i=1}^{t} F^i)) < \varepsilon$$

for each $f \in F$.

There is $\delta > 0$, such that for any $f \in F_i$ $(1 \le i \le k)$ and $x_1, x_2 \in X$ with $\mathrm{dist}(x_1, x_2) < \delta$, one has $\|f(x_1) - f(x_2)\| < \varepsilon$. Set $l_i = \frac{n}{k_i}$. For each $y \in X \wedge S^{2m}$,

$$\mathrm{SP}\lambda_y = \mathrm{SP}\lambda_y^1 \cup \mathrm{SP}\lambda_y^2 \cup \cdots \cup \mathrm{SP}\lambda_y^k \subset \underbrace{X \coprod X \coprod \cdots \coprod X}_{k} .$$

Furthermore, $\#\ \mathrm{SP}\lambda_y^i = l_i$, counting multiplicity.

One can choose a subset $\{y_1, y_2, \ldots, y_t\}$ dense enough in $X \wedge S^{2m}$ such that $\bigcup_{s=1}^{t} \mathrm{SP}\lambda^i_{y_s}$ is $\dfrac{\delta}{2l_i}$ dense in X (note that λ is injective by Remark 2.12). Hence one can choose $\{x_1^i, x_2^i, \ldots, x_t^i\} \subset \mathrm{SP}B^i = X$ such that $\{x_1^i, x_2^i, \ldots, x_t^i\}^{l_i}$ can be paired with $\bigcup_{s=1} \mathrm{SP}\lambda^i_{y_s}$ to within δ, where $\{\ \}^{l_i}$ denotes the set (or family) consisting of the original set with multiplicity l_i. Define a homomorphism $\nu : B \to M_{t+1}(B)$ by

$$\nu(f)(x) = \begin{pmatrix} f(x) & & & \\ & f(x_1^i) & & \\ & & \ddots & \\ & & & f(x_t^i) \end{pmatrix}.$$

Also, define $\mu : M_n(C(X \wedge S^{2m})) \to M_{n(t+1)}(C(X \wedge S^{2m}))$ by

$$\mu(f)(y) = \begin{pmatrix} f(y) & & & \\ & f(y_1) & & \\ & & \ddots & \\ & & & f(y_t) \end{pmatrix}.$$

One can easily prove (by using $\{x_1^i \ldots x_t^i\}^{l_i}$ paired with $\bigcup_{s=1}^{t} \mathrm{SP}\lambda^i_{y_s}$ to within ε) that

$$\begin{array}{ccc} M_n(C(X \wedge S^{2m})) & \xrightarrow{\mu} & M_{n(t+1)}(C_0(X \wedge S^{2m})) \\ \lambda\uparrow & & \uparrow \lambda\otimes\mathbf{1}_{t+1} \\ B & \xrightarrow[\nu]{} & M_{t+1}(B) \end{array}$$

commutes approximately to within ε on $\oplus F^i$, up to conjugating by a unitary $U \in M_{n(t+1)}(C(X \wedge S^{2m}))$. Note that $\nu(F^i) \subset M_{t+1}(B)$ is weakly approximately constant to within ε. One can change ϕ_1 to $\mu \circ \phi_1$, change λ to $\mathrm{Ad}U \circ (\lambda \otimes \mathbf{1}_{t+1})$ and change B to $M_{t+1}(B)$ to prove the assertion. (Note that one only gets $\mathrm{dist}(\phi_1(f), \lambda(\oplus F^i)) \leq 2\varepsilon$ instead of $\mathrm{dist}(\phi_1(f), \lambda(\oplus F^i)) \leq \varepsilon$.)

It follows that we may suppose that our assertionis true for the original $\lambda : B \to M_n(C(X \wedge S^{2m}))$ and $\oplus F^i$.

Also, we can enlarge the size of B (replace B by $M_{t+1}(B)$ as above for larger t) such that

$$\frac{\mathrm{rank}(\lambda^i(\mathbf{1}_{k_i}))}{k_i} \geq 3\dim(X) + 1.$$

Hence there is a map $\alpha : B \to M_n(C(X))$ such that $\mathrm{rank}(\alpha(\mathbf{1}_{k_i})) = \mathrm{rank}(\lambda^i(\mathbf{1}_{k_i}))$ and the partial map α^i defines $\mathrm{id} \in \mathrm{kk}(X, X)$. Finally, one can construct homomorphisms $\psi_2 : M_n(C(X)) \to M_{nt}(C(X \wedge S^{2m}))$ and $\phi_2 : M_n(C(X)) \to M_{nt}(C(X \wedge S^{2m}))$ with the properties $[\psi_2] = b \in \mathrm{kk}(X \wedge S^{2m},$

X), $[\phi_2] = \mathrm{id} \in \mathrm{kk}(X \wedge S^{2m}, X \wedge S^{2m})$, and $\psi_2(e_{11}), \phi_2(e_{11})$ are trivial projections. Hence, we have the following diagram

$$\begin{array}{ccccc} C(X \wedge S^{2m}) & \xrightarrow{\phi_1} & M_n(C(X \wedge S^{2m})) & \xrightarrow{\phi_2} & M_{nt}(C(X \wedge S^{2m})) \\ & & \lambda \Big\uparrow & & \Big\uparrow \psi_2 \\ & & B & \xrightarrow[\alpha]{} & M_n(C(X)) \end{array},$$

and $\psi_2 \circ \alpha$ and $\phi_2 \circ \lambda$ are homotopic since they define the same maps at the kk level and at the K_0 level. One can choose a map $\phi_3 : M_{nt}(C(X \wedge S^{2m})) \to M_{ntt_1}(C(X \wedge S^{2m}))$ such that $[\phi_3] = \mathrm{id} \in \mathrm{kk}(X \wedge S^{2m}, X \wedge S^{2m})$ and $\mathrm{SPV}(\phi_3) < \delta$ for arbitrary given δ. By Proposition 2.19, if δ is small enough, then there is a unitary $u \in M_{ntt_1} C(X \wedge S^{2m})$ such that

$$\|\phi_3 \circ \phi_2 \circ \lambda(f) - \mathrm{Ad}u \circ \phi_3 \circ \psi_2 \circ \alpha(f)\| \le 70\varepsilon$$

for each $f \in \oplus F^i$.

Hence one can choose $\phi = \phi_3 \circ \phi_2 \circ \phi_1$, $F_1 = \alpha(\oplus F^i)$, and $\psi = \mathrm{Ad}u \circ \phi_3 \circ \psi_2$ to finish the proof.

We also need the following lemma.

LEMMA 3.13. *Suppose that* $\alpha : C(X) \to M_{k_1}(C(X \wedge S^{2m}))$ *is a homomorphism with* $[\alpha] = b \in \mathrm{kk}(X \wedge S^{2m}, X)$ *(Bott element). Let* $G_1 \subset C(X)$, $G_2 \subset C(X \wedge S^{2m})$ *be finite subsets of self-adjoint elements of* $C(X)$ *and* $C(X \wedge S^{2m})$, *respectively. (We may assume that* G_1 *and* G_2 *consist of the restrictions of generators of* $C(\hat{X})$ *and* $C(\widetilde{X \wedge S^{2m}})$ *to* $X \subset \hat{X}$, *and* $X \wedge S^{2m} \subset \widetilde{X \wedge S^{2m}}$, *respectively, where* $\hat{X}$ *and* $\widetilde{X \wedge S^{2m}}$ *are cones of* X *and* $X \wedge S^{2m}$, *respectively.) Let* $\mathscr{P}_1 \subset C(X)$, *and* $\mathscr{P}_2 \subset C(X \wedge S^{2m})$ *be finite subsets consisting of polynomials in elements in* G_1 *and* G_2, *respectively. Let* $\delta > 0$. *There are homomorphisms* $\beta : C(X) \to M_{k_1k_2}(C(X))$, $\alpha_1 : C(X) \to M_{k_3}(C(X \wedge S^{2m}))$, *and* $\beta_1 : C(X \wedge S^{2m}) \to M_{k_2k_3}(C(X \wedge S^{2m})$, *a completely positive contractive linear map* $\gamma : C(X \wedge S^{2m}) \to M_{k_2}(C(X))$, *and paths of completely positive linear contractive maps*

$$\Delta^s : C(X) \to M_{k_1k_2}(C(X)) \quad \text{for} \quad 0 \le s \le 1 \qquad \text{and}$$

$$\Omega^s : C(X \wedge S^{2m}) \to M_{k_2k_3}(C(X \wedge S^{2m}))$$

such that

(i) $\Delta^\circ = \beta$, $\Delta^1 = (\gamma \otimes \mathrm{id}_{k_1}) \circ \alpha$,

(ii) $\Omega^\circ = \beta_1$, *and* $\Omega^1 = (\alpha_1 \otimes \mathrm{id}_{k_2}) \circ \gamma$,

(iii) $\|\Delta^s(g_ig_j) - \Delta^s(g_i)\Delta^s(g_j)\| \le \delta$ *for all* $g_i, g_j \in \{g_1, g_2, \dots, g_{n_1}\} := G_1$,
$\|\Omega^s(h_ih_j) - \Omega^s(h_i)\Omega^s(h_j)\| \le \delta$ *for all* $h_i, h_j \in \{h_1, h_2, \dots, h_{n_2}\} := G_2$,

(iv) $\|\Delta^s(P(g_1, g_2, \dots, g_{n_1})) - P(\Delta^s(g_1), \Delta^s(g_2), \dots, \Delta^s(g_{n_1}))\| < \delta$ *for all*

$P \in \mathscr{P}_1$, *and*
$\|\Omega^s(P(h_1, h_2, \ldots, h_{n_2})) - P(\Omega^s(h_1), \Omega^s(h_2), \ldots, \Omega^s(h_{n_2}))\| < \delta$ *for all*
$P \in \mathscr{P}_2$.

PROOF. By Lemma 3.14 of [EG2], without loss of generality, we may suppose that $\alpha : C(X) \to M_{k_1}(C(X \wedge S^{2m}))$ satisfies $\alpha(C_0(X)) \subset M_k(C_0(X \wedge S^{2m}))$. Let us still denote $\alpha|_{C_0(X)}$ by α. By Theorem 4.3 of [DL], (see also [CH], [D1], [D2]), there is an asymptotic homomorphism

$$\gamma_t : C_0(X \wedge S^{2m}) \to C_0(X) \otimes \mathscr{K}$$

which represents the inverse of the Bott element $b^{-1} \in E(C_0(X \wedge S^{2m}), C_0(X)) = \mathrm{KK}(C_0(X \wedge S^{2m}), C_0(X))$. (Notice that γ_t can be chosen as $\mathrm{id}_{C_0(X)} \otimes \gamma'_t$, where $\gamma'_t : C_0(S^{2m}) \to \mathscr{K}$ is the representation of the inverse of Bott element.)

Let $\chi : C_0(X) \to C_0(X) \otimes \mathscr{K} \otimes M_{k_1}$ be defined by sending a to $a \otimes e_{11}$ (where e_{11} is a minimal projection in $\mathscr{K} \otimes M_{k_1}$). Then

$$[\chi] \;=\; [\gamma_t \otimes \mathrm{id}_{k_1} \circ \alpha] \in E(C_0(X), C_0(X)) \,.$$

By [DL] again, there is $\Phi_t^s : C_0(X) \to C_0(X) \otimes \mathscr{K} \otimes M_{k_1}$, such that $\Phi_t^0 = \chi$ and $\Phi_t^1 = \gamma_t \otimes \mathrm{id}_{k_1} \circ \alpha$. By [D], and Lemma 1 of section 2 of [CH], one can assume the above γ_t and Φ_t^s are completely positive linear contractive maps.

Similarly, let $\lambda : C_0(X \wedge S^{2m}) \to C_0(X \wedge S^{2m}) \otimes M_{k_1} \otimes \mathscr{K}$ be defined by sending a to $a \otimes e_{11}$. Then

$$\begin{aligned}(\alpha \otimes \mathrm{id}_{\mathscr{K}}) \circ \gamma_t : C_0(X \wedge S^{2m}) &\to C_0(X) \otimes \mathscr{K} \\ &\to C_0(X \wedge S^{2m}) \otimes M_{k_1} \otimes \mathscr{K}\end{aligned}$$

represents the same element as that of λ in $E(C_0(X \wedge S^{2m}), C_0(X \wedge S^{2m}))$. Hence there is a path of asymptotic homomorphisms (completely positive linear contractive maps)

$$\Psi_t^s : C_0(X \wedge S^{2m}) \to C_0(X \wedge S^{2m}) \otimes M_{k_1} \otimes \mathscr{K}$$

such that

$$\Psi_t^0 = \lambda \qquad \text{and} \qquad \Psi_t^1 = (\alpha \otimes \mathrm{id}_{\mathscr{K}}) \circ \gamma_t \,.$$

The diagrams

$$\begin{tikzcd}
M_{k_1}(C_0(X\wedge S^{2m})) \arrow[dr, "\gamma_t\otimes \mathrm{id}_{k_1}"] & \\
C_0(X) \arrow[u, "\alpha"] \arrow[r, "\chi"'] \arrow[ur, phantom, "\Phi_t^s"] & C_0(X)\otimes\mathcal{K}\otimes M_{k_1}
\end{tikzcd}$$

and

$$\begin{tikzcd}
C_0(X\wedge S^{2m}) \arrow[r, "\lambda"] \arrow[dr, "\gamma_t"'] \arrow[r, phantom, shift right=6ex, "\Psi_t^s"] & C_0(X\wedge S^{2m})\otimes\mathcal{K}\otimes M_{k_1} \\
 & C_0(X)\otimes\mathcal{K} \arrow[u, "\alpha\otimes\mathrm{id}_{\mathcal{K}}"']
\end{tikzcd}$$

commutes up to homotopy via the paths Φ_t^s, Ψ_t^s, $0 \le s \le 1$.

We may suppose that $G_1 \subset C_0(X)$ and $G_2 \subset C_0(X \wedge S^{2m})$. (To do this, we need to choose a special embedding of X (or $\hat{X}$) into I^{n_1} which takes x_0 into $(0,0\dots0)$, see 3.3). Let $\mathscr{P}_1^0 \subset C_0(X)$ be defined by

$$\mathscr{P}_1^0 = \{P - P(x_0) \mid P \in \mathscr{P}_1\}\ .$$

That is, we delete the constant term from the polynomial. One can define $\mathscr{P}_2^0 \subset C_0(X \wedge S^{2m})$ corresponding to $\mathscr{P}_2 \subset C(X \wedge S^{2m})$.

Since Φ_t^s is a path of asymptotic homomorphisms, there is a t_1 such that

$$\|\Phi_t^s(g_ig_j) - \Phi_t^s(g_i)\Phi_t^s(g_j)\| \le \frac{\delta}{8}$$

and

$$\|\Phi_t^s\big(P(g_1,g_2,\dots,g_{n_1})\big) - \big(P(\Phi_t^s(g_1),\Phi_t^s(g_2),\dots,\Phi_t^s(g_{n_1})\big)\| \le \frac{\delta}{8}$$

for all $g_i, g_j \in G_1$, $P \in \mathscr{P}_1^0$, $s \in [0,1]$, and $t \ge t_1$. Similarly, one has t_2 such that if $t \ge t_2$ then

$$\|\Psi_t^s(h_ih_j) - \Psi_t^s(h_i)\Psi_t^s(g_j)\| \le \frac{\delta}{8}$$

and

$$\|\Psi_t^s\big(P(h_1,h_2,\dots,h_{n_2})\big) - \big(P(\Psi_t^s(h_1),\Psi_t^s(h_2),\dots,\Psi_t^s(h_{n_2})\big)\| \le \frac{\delta}{8}$$

for all $h_i, h_j \in G_2$ and $P \in \mathscr{P}_2^0$.

For each fixed s, we can choose a projection $R_s \in \mathscr{K}$ such that

$$\|\Phi_{t_0}^s(g_i)\hat{R}_s - \hat{R}_s\Phi_{t_0}^s(g_i)\| \leq \frac{\delta}{8}$$

for each $g_i \in G_1$ and

$$\|\Phi_{t_0}^s(P)\cdot\hat{R}_s - \hat{R}_s\Phi_{t_0}^s(P)\| \leq \frac{\delta}{8}$$

for each $P \in \mathscr{P}_1^0$, where $\hat{R}_s = \mathrm{id}_{k_1} \otimes R_s \in M_{k_1} \otimes \mathscr{K}$. Choose $0 = s_0 < s_1 < s_2 < \cdots < s_l = 1$ such that

$$\|\Phi_{t_0}^{s_i}(g) - \Phi_{t_0}^s(g)\| \leq \frac{\delta}{8}$$

for each $s \in [s_i, s_{i+1}]$ and $g \in G_1 \cup \mathscr{P}_1^0$.

Finally, we choose a projection $R \in \mathscr{K}$ which dominates $R_{s_0}, R_{s_1}, \ldots, R_{s_l}$. Hence for all $s \in [0,1]$,

$$\|\Phi_{t_0}^s(g)\hat{R} - \hat{R}\Phi_{t_0}^s(g)\| \leq \frac{\delta}{4}$$

for all $f \in G_2 \cup \mathscr{P}_1^0$, where $\hat{R} = \mathbf{1}_{k_1} \otimes R \in M_{k_1} \otimes \mathscr{K}$.

Similarly, one can find $R' \in \mathscr{K}$ such that for all $s \in [0,1]$

$$\|\Psi_{t_0}^s(g)\hat{R}' - \hat{R}'\Psi_{t_0}^s(g)\| \leq \frac{\delta}{4}$$

for all $g \in G_1 \cup \mathscr{P}_2^0$. We can suppose that $R = R'$. We know that $R\mathscr{K}R = M_{k_2}$ for some k_2. By cutting all the images of the maps by $\hat{R}$ or R, we have diagrams

$$\begin{tikzcd}
M_{k_1}(C_0(X \wedge S^{2m})) \arrow[drr, "\hat{R}(\gamma_{t_0} \otimes \mathrm{id}_{k_1})\hat{R}"] & & \\
C_0(X) \arrow[u, "\alpha"] \arrow[rr, "\hat{R}\chi\hat{R}"'] \arrow[ur, phantom, "\hat{R}\Phi_{t_0}^s\hat{R}"] & & C_0(X) \otimes M_{k_1} \otimes M_{k_2}
\end{tikzcd}$$

$$
\begin{tikzcd}
C_0(X\wedge S^{2m}) \arrow[r, "\hat{R}\lambda\hat{R}"] \arrow[dr, "R\gamma_{t_0}R"'] \arrow[dr, phantom, shift left=4ex, "\hat{R}\Psi^s_{t_0}\hat{R}"] & C_0(X\wedge S^{2m})\otimes M_{k_1}\otimes M_{k_2} \\
 & C_0(X)\otimes M_{k_2} \arrow[u, "\alpha\otimes \mathrm{id}_{k_2}"']
\end{tikzcd}
$$

which commute up to homotopy via the paths $\hat{R}\Phi^s_{t_0}\hat{R}$ and $\hat{R}\Psi^s_{t_0}\hat{R}$, respectively, ($0 \leq s \leq 1$).

Let $\gamma : C(X\wedge S^{2m}) \to C(X)\otimes M_{k_2}$ be defined by

$$\gamma(f) = R\gamma_{t_0}(f - f(y_0))R + f(y_0)R \in C(X)\otimes M_{k_2}$$

which is a unitalization of $R\gamma_{t_0}R$. Let Δ^s denote the unitalization of $\hat{R}\Phi^s_{t_0}\hat{R}$, and $\beta : C(X) \to C(X)\otimes M_{k_1}\otimes M_{k_2}$ the unitalization of $\hat{R}\chi\hat{R}$. One can obtain the following diagram

$$
\begin{tikzcd}
M_{k_1}(C(X\wedge S^{2m})) \arrow[dr, "\gamma\otimes\mathrm{id}_{k_1}"] \arrow[dr, phantom, shift right=4ex, "\Delta^s"] & \\
C(X) \arrow[u, "\alpha"] \arrow[r, "\beta"] & C(X)\otimes M_{k_1}\otimes M_{k_2}
\end{tikzcd}
$$

which commutes up to homotopy via the path Δ^s. Similarly, one can obtain another diagram,

$$
\begin{tikzcd}
C(X\wedge S^{2m}) \arrow[r, "\beta_1"] \arrow[dr, "\gamma"'] \arrow[dr, phantom, shift left=4ex, "\Omega^s"] & C(X\wedge S^{2m})\otimes M_{k_2}\otimes M_{k_1} \\
 & C(X)\otimes M_{k_2} \arrow[u, "\alpha\otimes\mathrm{id}_{k_2}"']
\end{tikzcd}
$$

which commutes up to homotopy via the path Ω^s. (Notice that we let $k_3 = k_1$ and $\alpha_1 = \alpha$.) This ends the proof.

Combining Lemma 3.13 and Theorem 3.9, we can prove the following lemma.

LEMMA 3.14. *Let $\varepsilon, \varepsilon_1, \eta$ be positive numbers. Suppose that $\alpha : C(X) \to M_{k_1}(C(X\wedge S^{2m}))$ is a unital homomorphism with $[\alpha] = b \in \mathrm{kk}(X\wedge S^{2m}, X)$*

(the Bott element). Let $H \subset H' \subseteq M_l(C(X))$ be finite sets and suppose that H is weakly approximately constant to within ε. Let $F \subset M_{lk_1}(C(X \wedge S^{2m}))$ be any finite set. There exists a diagram

$$\begin{tikzcd}
M_{lk_1}(C(X \wedge S^{2m})) \arrow[r, "\phi"] & M_{lk_1k_2}(C(X \wedge S^{2m})) \\
M_l(C(X)) \arrow[u, "\alpha\otimes\mathrm{id}_l"] \arrow[r, "\psi"'] & M_{lk_2}(C(X)) \supset H_1 \arrow[u, "\alpha_1"']
\end{tikzcd}$$

where each of α_1, ϕ, ψ is a unital homomorphism, such that:

(i) *α_1, ϕ, ψ take any trivial projection to a trivial projection; (Notice that $\alpha \otimes \mathrm{id}_l$ takes trivial projections to trivial projections.)*

(ii) $[\alpha_1] = b \in \mathrm{kk}(X \wedge S^{2m}, X)$, $[\phi] = \mathrm{id} \in \mathrm{kk}(X \wedge S^{2m}, X \wedge S^{2m})$, $[\psi] = \mathrm{id} \in \mathrm{kk}(X, X)$;

(iii) $\|\alpha_1 \circ \psi(f) - \phi \circ (\alpha \otimes \mathrm{id}_l)(f)\| \le 24\varepsilon$ *for each* $f \in H$;

(iv) *There is a finite set $H_1 \subset M_{lk_2}(C(X))$ (see the diagram) which is approximately constant to within ε_1, and $\psi(H') \subset H_1$;*

(v) $\mathrm{dist}\big(\phi(f), \alpha_1(H_1)\big) < \varepsilon_1$ *for each* $f \in F \subset M_{lk_1}(C(X \wedge S^{2m}))$;

(vi) $\mathrm{SPV}(\phi) \le \eta$, $\quad \mathrm{SPV}(\psi) \le \eta$.

Proof. Without loss of generality, we may suppose that $\alpha \otimes \mathrm{id}_l(H') \subseteq F$ and that F is weakly approximately constant to within $\min\left(\frac{\varepsilon}{12}, \frac{\varepsilon_1}{12}\right)$ (otherwise we can compose $\alpha : C(X) \to M_{k_1}(C(X \wedge S^{2m}))$ with a certain map from $M_{k_1}(C(X \wedge S^{2m}))$ to $M_{k_1'}(C(X \wedge S^{2m}))$) with small spectral variation to make F satisfy the condition).

Let G_1, G_2 be the finite sets of self-adjoint generators of $C(X)$ and $C(X \wedge S^{2m})$, respectively. Considering $G_1 \subset C(X)$, $H \subset M_l(C(X))$ and ε, one can find $\delta_1 > 0$ and a set of polynomials $\mathscr{P}_1 \subset C(X)$ as in Theorem 3.9. Similarly, for $G_2 \subset C(X \wedge S^{2m})$ $F \subset M_{lk_1}(C(X \wedge S^{2m}))$ and $\min\left(\frac{\varepsilon}{12}, \frac{\varepsilon_1}{12}\right)$, one can find $\delta_2 > 0$, and a set of polynomials $\mathscr{P}_2 \subset C(X \wedge S^{2m})$. Set $\delta = \min(\delta_1, \delta_2)$. Applying Lemma 3.13 to $G_1, G_2, \mathscr{P}_1, \mathscr{P}_2$, $\alpha : C(X) \to M_{k_1}(C(X \wedge S^{2m}))$, and $\delta > 0$, one can find the following two diagrams

$$\begin{tikzcd}
M_{k_1}(C(X \wedge S^{2m})) \arrow[dr, "\gamma\otimes\mathrm{id}k_1"] & \\
C(X) \arrow[u, "\alpha"] \arrow[r, "\beta"'] \arrow[ur, phantom, "\Delta'"] & M_{k_1 k_2'}(C(X))
\end{tikzcd}$$

and

$$\begin{tikzcd}
C(X\wedge S^{2m}) \arrow[r, "\beta_1"] \arrow[dr, "\gamma"'] \arrow[dr, phantom, shift left=3ex, "\Omega'"] & M_{k_2'k_1}(C(X\wedge S^{2m})) \\
& M_{k_2'}(C(X)) \arrow[u, "\alpha\otimes\mathrm{id}_{k_2'}"']
\end{tikzcd}$$

satisfying the conditions (i)–(iv) of Lemma 3.13. (Notice that γ is a completely positive linear contractive map.) Let $\chi : C(X \wedge S^{2m}) \to M_{k_2''}(C(X))$ denote the homomorphism defined by point evaluation. That is,

$$\chi(f)(x) = \begin{pmatrix} f(y_1) & & \\ & \ddots & \\ & & f(y_{k_2''}) \end{pmatrix}$$

for certain $y_1, y_2, \ldots, y_{k_2''} \in X \wedge S^{2m}$.

Let $\chi_1 : C(X) \to M_{k_1k_2''}(C(X))$ be defined by $\chi_1 = \chi \otimes \mathrm{id}_{k_1} \circ \alpha$ and $\chi_2 : C(X \wedge S^{2m}) \to M_{k_2''k_1}(C(X \wedge S^{2m}))$ by $\chi_2 = \alpha \otimes \mathrm{id}_{k_2''} \circ \chi$. Let $k_2 = k_1k_2' + k_1k_2''$. It is obvious that both χ_1 and χ_2 are defined by point evaluation. Hence $\chi_1 \otimes \mathrm{id}_l(f)$ is unitarily equivalent to $f(x_0)\hat{\mathbf{1}}^{(l)}_{k_1k_2}$ to within ε for all $f \in H$ by means of some single unitary (see the proof of Lemma 2.16 of [EG2] for details, and recall that H is weakly approximately constant to within ε). By Theorem 3.9, if k_2'' is large enough, then there is a unitary $u \in M_{lk_2}(C(X))$ ($= M_{lk_1(k_2'+k_2'')}(C(X))$) such that

$$\begin{aligned}(1)\qquad &\|\big((\gamma \otimes \mathrm{id}_{k_1} \circ \alpha) \otimes \mathrm{id}_l\big)(f) \oplus \chi_1 \otimes \mathrm{id}_l(f) \\ &\quad - u\big((\beta \otimes \mathrm{id}_l)(f) \oplus (\chi_1 \otimes \mathrm{id}_l)(f)\big)u^*\| \le 12\varepsilon\end{aligned}$$

for all $f \in H$. At the same time, since $F \subset M_{lk_1}(C(X \wedge S^{2m}))$ is weakly approximately constant to within $\min\left(\frac{\varepsilon}{12}, \frac{\varepsilon_1}{12}\right)$, if k_2'' is large enough, there is a unitary $v \in M_{lk_2}(C(X \wedge S^{2m}))$ such that

$$\begin{aligned}(2)\ &\|\big((\alpha \otimes \mathrm{id}_{k_2'} \circ \gamma) \otimes \mathrm{id}_{lk_1} \oplus (\chi_2 \otimes \mathrm{id}_{lk_1})\big)(f) - v\big((\beta_1 \otimes \mathrm{id}_{lk_1} \oplus \chi_2 \otimes \mathrm{id}_{lk_1})(f)\big)v^*\| \\ &\qquad \le 12 \cdot \min\left(\frac{\varepsilon}{12}, \frac{\varepsilon_1}{12}\right) = \min(\varepsilon, \varepsilon_1)\end{aligned}$$

for all $f \in F$.

Using $\chi_1 = \chi \otimes \mathrm{id}_{k_1} \circ \alpha$ and $\chi_2 = \alpha \otimes \mathrm{id}_{k_2''} \circ \chi$, one can easily verify that

$$
\begin{aligned}
(\alpha \otimes \mathrm{id}_{lk_1k_2''}) \circ (\chi_1 \otimes \mathrm{id}_l) &= (\alpha \otimes \mathrm{id}_{lk_1k_2''}) \circ [(\chi \otimes \mathrm{id}_{k_1} \circ \alpha) \otimes \mathrm{id}_l] \\
&= (\alpha \otimes \mathrm{id}_{lk_1k_2''}) \circ (\chi \otimes \mathrm{id}_{lk_1}) \circ (\alpha \otimes \mathrm{id}_l) \\
&= (\chi_2 \otimes \mathrm{id}_{lk_1}) \circ (\alpha \otimes \mathrm{id}_l).
\end{aligned}
$$

Also notice that

$$(\alpha \otimes \mathrm{id}_{lk_1k_2'}) \circ \big((\gamma \otimes \mathrm{id}_{k_1} \circ \alpha) \otimes \mathrm{id}_l\big) = ((\alpha \otimes \mathrm{id}_{k_2'} \circ \gamma) \otimes \mathrm{id}_{lk_1}) \circ (\alpha \otimes \mathrm{id}_l).$$

Direct summing the above two equations, we get

$$
\begin{aligned}
(3) \qquad & (\alpha \otimes \mathrm{id}_{lk_2}) \circ \big((\gamma \otimes \mathrm{id}_{k_1} \circ \alpha) \otimes \mathrm{id}_l \oplus \chi_1 \otimes \mathrm{id}_l\big) \\
&= \big((\alpha \otimes \mathrm{id}_{k_2'} \circ \gamma) \otimes \mathrm{id}_{lk_1} \oplus \chi_2 \otimes \mathrm{id}_{lk_1}\big) \circ (\alpha \otimes \mathrm{id}_l)
\end{aligned}
$$

Let us define $\psi : M_l(C(X)) \to M_{lk_2}(C(X))$ by $\psi = \mathrm{Ad}u(\beta \otimes \mathrm{id}_l \oplus \chi_1 \otimes \mathrm{id}_l)$ and $\phi : M_l(C(X \wedge S^{2m})) \to M_{lk_2}(C(X \wedge S^{2m}))$ by $\phi = \mathrm{Ad}v(\beta_1 \otimes \mathrm{id}_{lk_1} \oplus \chi_2 \otimes \mathrm{id}_{lk_1})$, and set $\alpha_1 = \alpha \otimes \mathrm{id}_{lk_2}$. Then by (1), (2), and (3), we have

$$\|\alpha_1 \circ \psi(f) - \phi \circ (\alpha \otimes \mathrm{id}_l)(f)\| \le 12\varepsilon + 12\varepsilon \le 24\varepsilon \quad \text{for all } f \in H.$$

Let $H_1 \subset M_{lk_2}(C(X))$ be defined by $H_1 = (\gamma \otimes \mathrm{id}_{lk_1} \oplus \chi \otimes \mathrm{id}_{lk_1})(F) \cup \psi(H')$. Then from the above, we see that

$$\mathrm{dist}\big(\phi(f), \alpha_1(H_1)\big) < \varepsilon_1 \quad \text{for all} \quad f \in F\ .$$

As in 3.12, one can make H_1 weakly approximately constant to within ε_1. Also, if k_2'' is large enough, and the set $\{y_1, y_2, \ldots, y_{k_2''}\}$ is dense enough, one has that $\mathrm{SPV}(\phi) < \eta$, $\mathrm{SPV}(\psi) < \eta$.

3.15. We will use 3.12 and 3.14 to construct two inductive limits $A(X \wedge S^{2m}) = \lim(M_{k_n}(C(X \wedge S^{2m})), \phi_{n,n+1})$ and $A(X) = \lim(M_{l_n}(C(X)), \psi_{n,n+1})$ with the property of 3.11. At the same time, we will construct a one-sided intertwining from $A(X)$ to $A(X \wedge S^{2m})$ which we will show induces an isomorphism between them. We will write $A(X \wedge S^{2m}) = \lim(A_n, \phi_{n,m})$ and

$$A(X) \;=\; \lim(B_n, \psi_{n,m})\ .$$

Let $A_0 = C(X \wedge S^{2m})$. And let $F_0 \subset C(X \wedge S^{2m})$ be a set of finite generators of $C(X \wedge S^{2m})$. Let $\varepsilon_0 > \varepsilon_1 > \cdots$ be a sequence of positive numbers with $\Sigma\varepsilon_i < +\infty$. By Lemma 3.12, one can construct the following diagram

$$
\begin{array}{ccc}
A_0 & \xrightarrow{\phi_{0,1}} & M_{k_1}(C(X \wedge S^{2m})) \\
 & & \Big\uparrow{\scriptstyle \alpha_1} \\
 & & M_{l_1}(C(X)) \supset H_1
\end{array}
$$

with the following properties:

(i) $[\phi_{0,1}] = \mathrm{id} \in \mathrm{kk}(X \wedge S^{2m}, X \wedge S^{2m})$;
(ii) $[\alpha_1] = b \in \mathrm{kk}(X \wedge S^{2m}, X)$;
(iii) α_1 takes trivial projections to trivial projections;
(iv) $\mathrm{dist}(\phi_{0,1}(f), \alpha_1(H_1)) < \varepsilon_0$, for all $f \in F_0$;
(v) H_1 is approximately constant to within ε_0.

Notice that α_1 takes trivial projections to trivial projections, so that α_1 can be written as $\alpha \otimes \mathrm{id}_{l_1}$, for a certain $\alpha : C(X) \to M_{k_1/l_1}(C(X \wedge S^{2m}))$. Let H_1' be a generating subset of $M_{l_1}(C(X))$, with $H_1' \supset H_1$. Let $F_1 \supset \phi_{0,1}(F_0) \cup \alpha_1(H_1')$ be a finite set of generators in $M_{k_1}(C(X \wedge S^{2m}))$. We can construct the following diagram (by Lemma 3.14),

$$\begin{array}{ccc} F_1 \subset M_{k_1}(C(X \wedge S^{2m})) & \xrightarrow{\phi_{1,2}} & M_{k_2}(C(X \wedge S^{2m})) \\ \uparrow{\scriptstyle \alpha_1} & & \uparrow{\scriptstyle \alpha_2} \\ H_1 \subset M_{l_1}(C(X)) & \xrightarrow[\psi_{1,2}]{} & M_{l_2}(C(X)) \supset H_2, \end{array}$$

with the following properties:

(i′) $\alpha_2, \phi_{1,2}$, and $\psi_{1,2}$ take trivial projections to trivial projections;
(ii′) $[\alpha_2] = [b]$, $[\phi_{1,2}] = \mathrm{id}$, $[\psi_{1,2}] = \mathrm{id}$ at the level of kk;
(iii′) $\|\alpha_2 \circ \psi_{1,2}(f) - \phi_{1,2} \circ \alpha_1(f)\| \le 24\varepsilon_0$ for $f \in H_1$;
(iv′) $\psi_{1,2}(H_1') \subset H_2$, and H_2 is approximately constant to within ε_1;
(v ′) $\mathrm{dist}(\phi_{1,2}(f), \alpha_2(H_2)) \le \varepsilon_1$ for all $f \in F_1$;
(vi′) $\mathrm{SPV}(\phi_{1,2}) \le \eta_1$, $\mathrm{SPV}(\psi_{1,2}) \le \eta_1$ (η_1 is a certain given small number).

Furthermore, we can require k_2 and l_2 to be multiples of $2! = 2$.
In general we can construct the diagram

$$\begin{array}{ccc} F_i \subset M_{k_i}(C(X \wedge S^{2m})) & \xrightarrow{\phi_{i,i+1}} & M_{k_{i+1}}(C(X \wedge S^{2m})) \\ \uparrow{\scriptstyle \alpha_i} & & \uparrow{\scriptstyle \alpha_{i+1}} \\ H_i \subset H_i' \subset M_{l_i}(C(X)) & \xrightarrow[\psi_{i,i+1}]{} & M_{l_{i+1}}(C(X)) \supset H_{i+1} \end{array}$$

to have the above properties (i′)–(vi′) by changing the indices $0, 1, 2$ to $i-1, i, i+1$. Hence, we obtain the diagram

$$\begin{array}{ccccccc} M_{k_1}(C(X \wedge S^{2m})) & \xrightarrow{\phi_{1,2}} & M_{k_2}(C(X \wedge S^{2m})) & \xrightarrow{\phi_{2,3}} & M_{k_3}(C(X \wedge S^{2m})) & \to & \cdots \\ \uparrow{\scriptstyle \alpha_1} & & \uparrow{\scriptstyle \alpha_2} & & \uparrow{\scriptstyle \alpha_3} & & \\ M_{l_1}(C(X)) & \xrightarrow[\psi_{1,2}]{} & M_{l_2}(C(X)) & \xrightarrow[\psi_{2,3}]{} & M_{l_3}(C(X)) & \to & \cdots \end{array}$$

By (iii′) and $H_{i+1} \supset H_i'$ (notice that H_i' is a generating subset of $M_{l_i}(C(X))$). We know the above diagram is one-sided approximately intertwining. By (v′) and the fact that each $F_i \subset M_{k_i}(C(X \wedge S^{2m}))$ is a generating subset, we know that the homomorphism defined by the above one-sided approximate intertwining is surjective. By suitable choice of η_i, we can make the two inductive limits in the above diagram to be of real rank zero. Hence both limit algebras are of real rank zero and simple. Therefore the homomorphism is also injective. In other words, we have proved

THEOREM 3.16. *There are two inductive limits and a one-sided intertwining between them,*

$$\begin{array}{ccccccccc}
A_1^2 & \xrightarrow{\phi_{1,2}} & A_2^2 & \xrightarrow{\phi_{2,3}} & A_3^2 & \to & \cdots & \to & A(X\wedge S^{2m}) \\
{\scriptstyle\alpha_1}\uparrow & & {\scriptstyle\alpha_2}\uparrow & & {\scriptstyle\alpha_3}\uparrow & & & & \\
A_1^1 & \xrightarrow[\psi_{1,2}]{} & A_2^1 & \xrightarrow[\psi_{2,3}]{} & A_3^1 & \to & \cdots & \to & A(X)
\end{array},$$

such that:

(i) $A_i^2 = M_{k_i}(C(X\wedge S^{2m}))$, $A_i^1 = M_{l_i}(C(X))$, *and* k_i, l_i *are both multiples of* $i! = 1\times 2\times 3\times\cdots\times i$;

(ii) *The above diagram defines an isomorphism* $\alpha : A(X) \to A(X\wedge S^{2m})$;

(iii) $[\alpha_i] = b \in \mathrm{kk}(X\wedge S^{2m}, X)$ *and* α_i *takes trivial projections to trivi projections.*

REMARK 3.17. Suppose that $p_1 < p_2 < \ldots$ and $q_1 < q_2 < \ldots$ are sequences of positive integers and $\beta_i : A^1_{p_i} \to A^2_{q_i}$ are unital homomorphisms making the following diagram one-sided approximately intertwining

$$\begin{array}{ccccccc}
A^2_{q_1} & \to & A^2_{q_2} & \to & \cdots & \to & A(X\wedge S^{2m}) \\
{\scriptstyle\beta_1}\uparrow & & {\scriptstyle\beta_2}\uparrow & & & & \uparrow \\
A^1_{p_1} & \to & A^1_{p_2} & \to & \cdots & \to & A(X)
\end{array}.$$

If we further suppose that $[\beta_i] = b$ and that the maps β_i take trivial projections to trivial projections, then by Proposition 2.26, β is approximately unitarily equivalent to α.

3.18. Let X be any finite CW complex. Denote by $A(X)$ the C*-algebra A constructed in 3.11. Then we know

$$\left(K_*(A(X)), K_*(A(X))_+, \mathbf{1}_{A(X)}\right) \cong \left(\mathbb{Q}\oplus \tilde{K}^*(X), \mathbb{Q}_+\oplus\tilde{K}^*(X)\cup\{(0,0)\}, \mathbf{1}_{\mathbb{Q}_+}\right),$$

where $\tilde{K}^*(X)$ is the reduced K-theory of X. Also, we have proved that $A(X)\cong A(X\wedge S^{2m})$, whenever $2m > \dim X$.

For any space X, let $\bar{X}$ be the 3-dimensional connected CW complex in 2.3 with $K^0(\bar{X})\cong K^0(X)$ and $K^1(\bar{X})\cong K^1(X)$. We will prove that $A(\bar{X})\cong A(X)$.

Note that $\mathrm{KK}(C_0(\bar{X}), C_0(X)) = \mathrm{kk}(X,\bar{X})$, and that $\mathrm{KK}(C_0(X), C_0(\bar{X}\wedge S^{2m})) = \mathrm{kk}(\bar{X}\wedge S^{2m}, X)$. (See Proposition 2.2.) One can find a

$u \in \mathrm{KK}(C_0(\bar{X}), C_0(X))$ which induces the isomorphism from $\tilde{K}^*(\bar{X})$ to $\tilde{K}^*(X)$. Let $U \in \mathrm{KK}(C_0(X), C_0(\bar{X} \wedge S^{2m}))$ denote the element satisfying $u \times U = b \in \mathrm{KK}(C_0(\bar{X}), C_0(\bar{X} \wedge S^{2m}))$, the Bott element. That is, $U = u^{-1} \times b \in \mathrm{KK}(C_0(X), C_0(\bar{X} \wedge S^{2m}))$.

For each n, one can construct a unital homomorphism $u_n : A_n(\bar{X})$ $(:= M_{n!}(C(\bar{X})) \to A_{l_n}(X)$ (for l_n large enough) such that $[u_n] = u \in \mathrm{kk}(X, \bar{X})$. This gives a diagram

$$\begin{array}{ccccccc} A_1(\bar{X}) & \to & A_2(\bar{X}) & \to & \cdots & \to & A(\bar{X}) \\ {\scriptstyle u_1}\big\downarrow & & {\scriptstyle u_2}\big\downarrow & & & & \big\downarrow \\ A_{l_1}(X) & \to & A_{l_2}(X) & \to & \cdots & \to & A(X) \end{array}$$

which defines a homomorphism $\phi_0 : A(\bar{X}) \to A(X)$ (by 2.26 and 2.28). Also, for each n, one can construct a unital homomorphism $U_n : A_n(X) \to A_{k_n}(\bar{X} \wedge S^{2m})$ such that $[U_n] = U \in \mathrm{kk}(\bar{X} \wedge S^{2m}, X)$. This gives a diagram which defines a homomorphism ψ_0 from $A(X)$ to $A(\bar{X} \wedge S^{2m}) = A(\bar{X})$. (Notice that the composition of weak system maps $\underline{U_n} \circ \underline{u_n}$ is equivalent to the system map $b_n : A_n(\bar{X}) \to A_?(C(\bar{X} \wedge S^{2m}))$ represented by the Bott element, where we use "?" to denote any possible integer. Hence if we identify $A(\bar{X} \wedge S^{2m})$ with $A(\bar{X})$ by 3.16, then $\psi_0 \circ \phi_0 : A(\bar{X}) \to A(\bar{X})$ is approximately unitarily equivalent to id by 3.17. Similarly there is a $u^1 \in \mathrm{KK}(C_0(\bar{X} \wedge S^{2m}), C_0(X \wedge S^{2m})) = \mathrm{kk}(X \wedge S^{2m}, \bar{X} \wedge S^{2m})$ with $U \times u^1 = b \in \mathrm{KK}(C_0(X), C_0(X \wedge S^{2m}))$, the Bott element. Then one can construct the homomorphism $\phi_1 : A(\bar{X} \wedge S^{2m}) \to A(X \wedge S^{2m})$ as above by using u^1. When we identify $A(X \wedge S^{2m})$ with $A(X)$, we know that $\phi_1 \circ \psi_0 : A(X) \to A(X)$ is approximately unitarily equivalent to the identity. Furthermore, one can choose

$$U^1 \in \mathrm{KK}(C_0(X \wedge S^{2m}), C_0(\bar{X} \wedge S^{4m})) = \mathrm{kk}(\bar{X} \wedge S^{4m}, X \wedge S^{2m}),$$

with

$$u^1 \times U^1 = b \in \mathrm{KK}(C_0(\bar{X} \wedge S^{2m}), C_0(\bar{X} \wedge S^{4m}))$$

the Bott element. Then one can construct $\psi_1 : A(X \wedge S^{2m})(= A(X)) \to A(\bar{X} \wedge S^{4m})(= A(\bar{X}))$ by using U^1. Similarly $\psi_1 \circ \phi_1 : A(\bar{X}) \to A(\bar{X})$ is approximately unitarily equivalent to the identity. (Notice that if $b_1 \in \mathrm{KK}(C_0(X), C_0(X \wedge S^{2m}))$, and $b_2 \in \mathrm{KK}(C_0(X \wedge S^{2m}), C_0(X \wedge S^{4m}))$ are the Bott elements, then $b_1 \times b_2 \in \mathrm{KK}(C_0(X), C_0(X \wedge S^{4m}))$ is the Bott element.)

Hence we can construct $\phi_i : A(\bar{X}) \to A(X)$ and $\psi_i : A(X) \to A(\bar{X})$ such that $\psi_i \circ \phi_i$ are approximately unitarily equivalent to $\mathrm{id}_{A(\bar{X})}$ and $\phi_{i+1} \circ \psi_i$ are approximately unitarily equivalent to $\mathrm{id}_{A(X)}$. Therefore $A(X) \cong A(\bar{X})$ by Proposition 2.14. We have proved

THEOREM 3.19. *Suppose that X is a finite CW complex, and that $\bar{X}$ is a finite CW complex of special type with $K^0(\bar{X}) = K^0(X)$ and $K^1(\bar{X}) = K^1(X)$. Suppose that $A = \varinjlim(M_{n!}(C(X)), \phi_{nm})$, $B = \varinjlim(M_{n!}(C(\bar{X})), \psi_{nm})$ are of real rank zero and $\phi_{n,m} = \mathrm{id} \in \mathrm{kk}(X, X)$, $\psi_{n,m} = \mathrm{id} \in \mathrm{kk}(\bar{X}, \bar{X})$. Furthermore, suppose that ϕ_{nm}, ψ_{nm} take trivial projections to trivial projections. Then A and B are isomorphic. In particular, there is a system map $\alpha_n : B_n \to A_{l_n}$ such that*

$$\begin{array}{ccccccc}
B_1 & \to & B_2 & \to & B_3 & \to & \cdots \\
\alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow & & \\
A_{l_1} & \to & A_{l_2} & \to & A_{l_3} & \to & \cdots
\end{array}$$

is a one-sided approximate intertwining and induces the isomorphism.

4. The General Theorem.

In the last section, for each space X, we proved that one special inductive limit of $M_{k_n}(C(X))$ can be written as an inductive limit of algebras $M_{k_n}(C(\bar{X}))$, where $\bar{X}$ is a three-dimensional CW complex of special type. A local version of this result says that for any $F \subset M_k(C(X))$, there exist homomorphisms $\phi : M_k(C(X)) \to M_l(C(X))$ and $\psi : M_{l_1}(C(\bar{X})) \to M_l(C(X))$ such that $[\phi] = \mathrm{id} \in \mathrm{kk}(X, X)$ and $\phi(F)$ is approximately contained in the image of ψ to within an arbitrary given small number. We will use this local result and some related results to reduce a general inductive limit to an inductive limit of matrix algebras over three-dimensional finite CW complexes of special type. We will make those local results more precise.

LEMMA 4.1. *For any finite CW complex X, and any finite set $F \subset C(X)$ and $\varepsilon > 0$, there exist a homomorphism $\phi : C(X) \to M_k(C(X))$ (for certain large k), a unital homomorphism $\alpha : M_l(C(\bar{X})) \to M_k(C(X))$ (for certain l), and a finite set $F_1 \subset M_l(C(\bar{X}))$ with the following properties:*
(i) $[\phi] = \mathrm{id} \in \mathrm{kk}(X, X)$;
(ii) $\mathrm{dist}(\phi(f), \alpha(F_1)) < \varepsilon$ *for each* $f \in F$;
(iii) F_1 *is weakly approximately constant to within* ε *in* $M_l(C(\bar{X}))$.

PROOF. Consider two inductive limit systems in 3.19. We can suppose that in the diagram

$$\begin{array}{ccccccc} A_{l_1} & \to & A_{l_2} & \to & \cdots & \to & A \\ {\scriptstyle\alpha_1}\uparrow & & {\scriptstyle\alpha_2}\uparrow & & & & \\ B_1 & \to & B_2 & \to & \cdots & \to & B\,, \end{array}$$

(see 3.19), the nth square is approximately commutative, to within $\frac{1}{2^n}$ on the nth set in a sequence of increasing subsets $H_n \subset B_n$ with union dense in B. (This can be done by passing to a subsequence.) Consider $F \subset C(X) = A_1$, and $\varepsilon > 0$; there is an n with $\frac{1}{2^n} < \frac{\varepsilon}{4}$ such that $\phi_{1,l_n}(F) \subset A_n$ is approximately contained in $\alpha_n(B_n)$ to within $\frac{\varepsilon}{4}$. That is, one can find a subset $F_1' \subset B_n$ with $\operatorname{dist}\big(\phi_{1,l_n}(f), \alpha_n(F_1')\big) < \frac{\varepsilon}{4}$. Choose an $m > n$, such that $\psi_{n,m}(F_1')$ is weakly approximately constant to within ε and is also contained in the $\frac{\varepsilon}{4}$ -neighbourhood of H_m. And, finally, set $\phi = \phi_{1,l_m} : A_1 \to A_{l_m}$, $\alpha = \alpha_m : B_m \to A_{l_m}$, and $F_1 = \psi_{n,m}(F_1')$ to end the proof.

One should note that the above α induces a KK-equivalence in $\mathrm{KK}(C_0(\bar{X}), C_0(X))$.

REMARK 4.2. The above result is also true, if one replaces $C(X)$ by $M_t(C(X))$ for some integer t, and replaces $F \subset C(X)$ by $F \subset M_t(C(X))$. Namely, one can consider a generating set $G \subset C(X)$ and then for any $\varepsilon > 0$, there is a $\delta > 0$, such that if $\phi : C(X) \to C$ and $\psi : D \to C$ satisfy $\operatorname{dist}\big(\phi(f), \psi(D)\big) < \delta$ for each $f \in G$, then $\phi \otimes \mathrm{id}_t : M_t(C(X)) \to M_t(C)$ and $\psi \otimes \mathrm{id}_t : M_t(D) \to M_t(C)$ satisfy $\operatorname{dist}\big(\phi \otimes \mathbf{1}_t(f), \psi \otimes \mathbf{1}_t(M_t(D))\big) < \varepsilon$ for each $f \in F$.

LEMMA 4.3. *Let X be a connected finite CW complex. Let $F \subset M_t(C(X))$ be a finite set. Let D be a direct sum of matrix algebras over 3-dimensional* CW *complexes of special type. Let $G_1 \subset G_2$ be finite subsets of D. Let $\alpha : D \to M_t(C(X))$ be a unital homomorphism. And suppose that $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, and that G_1 is approximately constant to within ε_1.*

Then it follows that there exists a diagram

$$\begin{array}{ccc} F \subset M_t(C(X)) & \xrightarrow{\phi} & M_k(C(X)) \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\alpha'} \\ D & \xrightarrow[\psi]{} & M_l(C(\bar{X})) \supset F_1 \end{array}$$

(i.e. *there exist unital homomorphisms ϕ, ψ, α' as in the diagram for certain k and l) such that:*

(i) $\|\alpha' \circ \psi(g) - \phi \circ \alpha(g)\| \le 72\varepsilon_1$ *for each $g \in G_1$;*

(ii) $\operatorname{dist}(\phi(f), \alpha'(F_1)) < \varepsilon_2$ *for each* $f \in F$;
(iii) $\psi(G_2) \subset F_1$, *and* F_1 *is approximately constant to within* ε_2;
(iv) $[\phi] = \mathrm{id} \in \mathrm{kk}(X, X)$.

Proof. Suppose that $A = \varinjlim(A_n, \phi_{nm})$ and $B = \varinjlim(B_n, \psi_{nm})$ are the inductive limits in 3.19. From 3.11 and 3.19, we have the following diagram,

$$\begin{tikzcd}
M_t(A_1) \arrow[r] \arrow[dr, dashed, "\beta_1"'] & M_t(A_{l_1}) \arrow[r] \arrow[dr, dashed, "\beta_2"'] & M_t(A_{l_2}) \arrow[r] & \cdots & M_t(A) \\
 & M_t(B_{k_1}) \arrow[r] \arrow[u, "\alpha_1"] & M_t(B_{k_2}) \arrow[r] \arrow[u, "\alpha_2"] & \cdots & M_t(B). \arrow[u]
\end{tikzcd}$$

where the α_i are homomorphisms which induce an isomorphism from $M_t(B)$ to $M_t(A)$, and the β_i are only KK elements, and each triangle commutes at the level of KK.

Now we will use ϕ_{nm} (and ψ_{nm}) to denote also the map

$$\phi_{nm} \otimes \mathrm{id}_t : M_t(A_n) \to M_t(A_m) (\text{and } \psi_{nm} \otimes \mathrm{id}_t : M_t(B_n) \to M_t(B_m)).$$

As in 4.1, for $F \subset M_t(A_1) = M_t(C(X))$, there are k_i and l_i and $F_1' \subset M_t(B_{k_i})$ such that:
(i) $\operatorname{dist}(\phi_{1,l_i}(f), \alpha_i(F_1')) < \varepsilon_2$ for all $f \in F$;
(ii) F_1' is approximately constant to within ε_2.

Consider $[\alpha] \times \beta_1 \times [\psi_{k_1 k_i}] \in \mathrm{KK}(D, M_t(B_{k_i}))$. If k_i is large enough, then this element can be realized by a homomorphism $\psi : D \to M_t(B_{k_i})$, such that $\psi(G_2)$ is approximately constant to within ε_2. (Here we use the fact that D is a direct sum of matrix algebras over a 3-dimensional finite CW complex of special type.) Hence we have the following diagram

$$\begin{tikzcd}
M_t(A_1) \arrow[r, "\phi_{1,l_i}"] & M_t(A_{l_i}) \\
D \arrow[u, "\alpha"] \arrow[r, "\psi"'] & M_t(B_{k_i}), \arrow[u, "\alpha_i"']
\end{tikzcd}$$

which is commutative at the level of homotopy. One can choose an l_j and $U \in M_t(A_{l_j})$ such that

$$\|\phi_{1,l_j} \circ \alpha(f) - \mathrm{Ad}\, U \circ \phi_{l_i l_j} \circ \alpha_i \circ \psi(f)\| \leq 70\varepsilon_1$$

for each $f \in G_1$. (Notice that G_1 is weakly approximately constant to within ε_1.) Furthermore, there is an $l_p > l_j$ such that

$$\phi_{l_j l_p}(U) \in M_t(A_{l_p})$$

can be approximated by $V \in M_t(B_{k_p})$, that is,

$$\|\alpha_p(V) - \phi_{l_t l_p}(U)\| < \varepsilon_1 \ .$$

Hence we have the following diagram:

$$\begin{tikzcd}
M_t(A_1) \arrow[r, "\phi_{1,l_t}"] & M_t(A_{l_t}) \arrow[r, "\phi_{l_t l_p}"] & M_t(A_{l_p}) \\
D \arrow[u, "\alpha"'] \arrow[r, "\psi"'] & M_t(B_{k_t}). \arrow[u, "\alpha_t"'] \arrow[r, "\psi_{k_t k_p}"'] & M_t(B_{k_p}) \arrow[u, "\alpha_p"']
\end{tikzcd}$$

The larger rectangle is commutative up to $\mathrm{Ad}(\alpha_p(V))$ within 72 ε_1 on G_1. One can set $M_k(C(X)) = M_t(A_{l_p})$, $\alpha' = \alpha_p$, $M_l(C(\bar{X})) = M_t(B_{k_p})$, $\phi = \mathrm{Ad}(\alpha_p(V))\phi_{1,l_p}$, and, importantly, $F_1 = \mathrm{Ad}V(\psi_{k_t k_p}(F_1'))$, to end the proof.

The following is the main result of this paper.

THEOREM 4.4. *Suppose that a C^*-algebra A of real rank zero is the inductive limit of a sequence*

$$A_1 \to A_2 \to A_3 \to \cdots \ ,$$

with $A_n = \bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i}))$, where each space $X_{n,i}$ is a finite CW complex, and each $[n,i]$ is a positive integer. Suppose that the inductive limit system satisfies the slow dimension growth condition $(**)$ of 2.20. *Then A can be written as an inductive limit of direct sums of matrix algebras over* 3*-dimensional* CW *complexes.*

4.5. For any $A_n = \bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i}))$, and integer l large enough, suppose that $\tilde{A}_n := \bigoplus_{i=1}^{k_n} \left(M_{l[n,i]}(C(X_{n,i})) \oplus M_{[n,i]}(\mathbf{C})\right) := \bigoplus_{i=1}^{k_n} C_n^i \oplus D_n^i$. Define a unital homomorphism $A_n \xrightarrow{\alpha\oplus\beta} \tilde{A}_n$ with $\alpha : A_n \to \bigoplus_{i=1}^{k_n} C_n^i$ and $\beta : A_n \to \bigoplus_{i=1}^{k_n} D_n^i$ as follows: $\alpha^{i,j} = 0$, and $\beta^{i,j} = 0$ if $i \neq j$. $\alpha^{i,i}$ satisfies $[\alpha^{i,i}] = \mathrm{id} \in \mathrm{kk}(X_{n,i}, X_{n,i})$ and takes trivial projections to trivial projections, and $\beta^{i,i}$ satisfies $\beta^{i,i}(f) = f(x_0)$, where x_0 is the base point of $X_{n,i}$.

LEMMA 4.6. *With the above notation, if m is large enough, then there is a $\gamma : \tilde{A}_n \to A_m$ such that $\gamma \circ (\alpha \oplus \beta)$ is homotopic to $\phi_{n,m}$.*

PROOF. By Lemma 2.23, there is an $m > 0$, such that each $\phi_{nm}^{ij} : M_{[n,i]}(C(X_{n,i})) \to M_{[m,j]}(C(X_{m,j})) = A_m^j$, satisfies either of the following conditions:

(1) ϕ^{ij}_{nm} is homotopic to a homomorphism which can be factored as

$$M_{[n,i]}(C(X_{n,i})) \xrightarrow{\text{evaluated at } x_0} M_{[n,i]}(\mathbf{C}) \xrightarrow{\tau} \phi^{i,j}_{nm}(\mathbf{1}_{A^i_n}) \cdot A^j_m \phi^{ij}_{nm}(\mathbf{1}_{A^i_n}) \; ;$$

(2) $\mathrm{rank}\phi^{i,j}_{nm}(\mathbf{1}_{A^i_n}) \geq 3\dim(X_{[m,j]} + 1)(l+1)[n,i]$.

We can define $\gamma^{i,j} : \tilde{A}^i_n = C^i_n \oplus D^i_n \to A^j_m$ as follows. If (1) holds, set $\gamma^{i,j}|_{D^i_n} = \tau$ and $\gamma^{i,j}|_{C^i_n} = 0$. Suppose that (2) holds. Let $P = \phi^{ij}_{nm}(\mathbf{1}_{A^i_n})$ and $p = \phi^{i,j}_{nm}(e_{1,1})$, where $e_{1,1}$ is the matrix unit of $M_{[n,i]}(C(X_{n,i}))$ corresponding to the upper left corner. One can identify $PM_{[m,j]}(C(X_{m,j}))P$ with $pM_{[m,j]}(C(X_{m,j}))p \otimes M_{[n,i]}$, and identify ϕ^{ij}_{nm} with $\phi \otimes \mathrm{id}_{M_{[n,i]}}$, where

$$\phi : C(X_{n,i}) \to pM_{[m,j]}(C(X_{m,j}))p$$

is the restriction of ϕ^{ij}_{nm} to the upper left corner. Since $\mathrm{rank}(p) \geq 3(\dim X_{[m,j]} + 1)(l+1)$, we can write $p = p_0 + p_1$ with p_0 a trivial projection, and with

$$\mathrm{rank}(p_0) = 3(\dim(X_{[m,j]} + 1) \cdot l \qquad \text{and}$$

$$\mathrm{rank}(p_1) \geq 3 \cdot (\dim X_{[m,j]} + 1).$$

Write $p_0 = \underbrace{q_0 \oplus q_0 \cdots \oplus q_0}_{l}$ with $\mathrm{rank}(q_0) = 3 \cdot (\dim(X_{[m,j]} + 1))$. Let $\xi_1 : C(X_{n,i}) \to q_0 M_{[m,j]}(C(X_{m,j}))q_0$ be a unital homomorphism with $[\xi_1] = [\phi^{ij}_{nm}] \in \mathrm{kk}(X_{m,j}, X_{n,i})$. (Note that this can be done since $\mathrm{rank}(q_0) \geq 3(\dim X_{m,j} + 1)$.) Let $\xi_2 : \mathbf{C} \to p_1 M_{[m,j]} C(X_{m,j}) p_1$ be defined by $\xi_2(f)(x) = f(x_0)p_1$, where x_0 is the base point of $X_{n,i}$. Finally, let $\gamma^{i,j}|_{C^i_n}$ be defined by

$$\begin{aligned}\xi_1 \otimes \mathrm{id}_{M_{l[n,i]}} : C(X_{n,i}) \otimes M_{l[n,i]} &\to q_0 M_{[m,j]}(C(X_{m,j}))q_0 \otimes M_{l[n,i]} \\ &= p_0 M_{[m,j]} C(X_{m,j}) p_0 \otimes M_{[n,i]} \subset PM_{[m,j]}(C(X_{m,j}))P \; ,\end{aligned}$$

and let $\gamma^{i,j}|_{D^i_n}$ be defined by

$$\xi_2 \otimes \mathrm{id}_{M_{[n,i]}} : M_{[n,i]}(\mathbf{C}) \to p_1 M_{[m,j]}(C(X_{m,j}))p_1 \otimes M_{[n,i]} \subset PM_{[m,j]}(C(X_{m,j}))P \; .$$

Obviously, $\gamma \circ (\alpha \oplus \beta)$ and ϕ_{nm} are the same at the level of kk and K-theory, and therefore ϕ_{nm} is homotopic to $\mathrm{Ad}u \circ \gamma \circ (\alpha \oplus \beta)$ for a certain unitary $u \in A_m$. By changing γ to $\mathrm{Ad}u \circ \gamma$, we have proved the lemma.

PROOF OF THEOREM 4.4. We will construct an inductive limit $B = \varinjlim(B_n, \psi_{n,m})$ and a one-sided intertwining from $\{B_n\}$ to $\{A_n\}$ which defines an isomorphism, where the B_n are direct sums of matrix algebras over 3-dimensional finite CW complexes of special type. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \ldots$ be positive numbers with $\Sigma\varepsilon_n < +\infty$. Let $\{a_{1n}\}_{n=1}^{\infty}$ be a dense subset of A_1, and let $F_1 = \{a_{11}\}$. There is a k_1 such that $\phi_{1,k_1}(F_1)$ is approximately constant to within ε_1 in A_{k_1}. For each block of A_{k_1}, $M_{[k_1,i]}(C(X_{k_1,i}))$, regard $\phi_{1,k_1}(F_1)$ as a subset of $M_{k_1,i}(C(X_{k_1,i}))$ by projecting $\phi_{1,k_1}(F_1)$ into the algebra. For ε_1, one can find an l, l_1 and homomorphisms

$$\phi : M_{[k_1,i]}C(X_{k_1,i}) \to M_{l[k_1,i]}(C(X_{k_1,i})) \qquad \text{and}$$
$$\alpha : M_{l_1[k_1,i]}C(\bar{X}_{k_1,i}) \to M_{l[k_1,i]}(C(X_{k_1,i}))$$

as in Lemma 4.1 (or more precisely in Remark 4.2). Let l denote the product of all the numbers l for the various blocks. Then there exist homomorphisms

$$\alpha_1 : \oplus M_{[k_1,i]}(C(X_{k_1,i})) \to \oplus M_{l[k_1,i]}(C(X_{k_1,i}))$$

and

$$\xi_1 : \oplus M_{l_i[k_1,i]}(C(\bar{X}_{k_1,i})) \to \oplus M_{l[k_1,i]}(C(X_{k_1,i})),$$

and a subset $F_1' \subset \oplus M_{l_i[k_1,i]}(C(\bar{X}_{k_1,i}))$ weakly approximately constant to within ε_1, such that $\operatorname{dist}(\alpha_1(f), \xi_1(F_1')) < \varepsilon_1$ for each $f \in \phi_{1,k_1}(F)$. Also, as pointed out in the proof of Lemma 4.3, there is

$$\eta_1 \in \mathrm{KK}(A_{k_1}, \oplus M_{l_i[k_1,i]}(C(\bar{X}_{k_1,i}))) \qquad \text{with} \qquad \eta_1 \times [\xi_1] = [\alpha_1] \in \mathrm{KK}\,.$$

(Notice that α_1 and ξ_1 satisfy $\alpha_1^{i,j} = 0$ and $\xi_1^{ij} = 0$ whenever $i \neq j$, and $[\alpha_1^{i,i}] = \mathrm{id} \in \mathrm{kk}$.) Set $\tilde{A}_{k_1} = C_{k_1} \oplus D_{k_1} = \oplus M_{l[k_1,i]}(C(X_{k_1,i})) \oplus M_{[k_1,i]}(\mathbb{C})$ and $B_1 = \bigoplus_i M_{l_i[k_1,i]}(C(\bar{X}_{k_1,i})) \oplus \bigoplus_i M_{[k_1,i]}(\mathbb{C})$. Define $\beta_1 : A_{k_1} \to D_{k_1}$ by evaluation at the base point. One can extend ξ_1 to

$$\mathrm{new}\xi_1 = \mathrm{old}\xi_1 \oplus \mathrm{id} : B_1 \to \tilde{A}_{k_1}\,.$$

Furthermore, define $\mathrm{new}\eta_1 \in \mathrm{KK}(A_{k_1}, B_1)$ by $\mathrm{new}\eta_1 = (\mathrm{old}\eta_1, \beta_1)$. Hence we have the following diagram:

$$\begin{tikzcd}
A_{k_1} \arrow[r, "\alpha_1 \oplus \beta_1"] \arrow[dr, dashed, "\eta_1"'] & \tilde{A}_{k_1} \\
 & B_1 \arrow[u, "\xi_1"']
\end{tikzcd}$$

which commutes at the level of KK. There exists a subset $F_1' \subset B_1$ such that $\operatorname{dist}(\alpha_1 \oplus \beta_1(f), \xi_1(F_1')) < \varepsilon_1$ for each $f \in \phi_{1,k_1}(F_1)$. Applying Lemma 4.6, we obtain a homomorphism $\gamma_1 : \tilde{A}_{k_1} \to A_t$ such that $\gamma_1 \circ (\alpha_1 \oplus \beta_1)$ is homotopic to $\phi_{k_1,t}$. By Proposition 2.24, there are an m_2 and a unitary $U \in A_{m_1}$ such that

$$\|\mathrm{Ad}u \circ \phi_{t,m_2} \circ \gamma_1 \circ (\alpha_1 \oplus \beta_1)(f) - \phi_{k_1,m_2}(f)\| \leq 70\varepsilon_1$$

for all $f \in \phi_{1,k_1}(F_1)$. Set new $\gamma_1 = \mathrm{Ad}u \circ \phi_{t_1 m_2} \circ \gamma_1$; we have the following diagram:

$$\begin{tikzcd}
F_1 \subset A_1 \arrow[r] & A_{k_1} \arrow[rr, "\phi_{k_1,m_2}"] \arrow[dr, "\alpha_1 \oplus \beta_1"] \arrow[ddr, dashed, "\eta_1"'] & & A_{m_2} \\
& & \tilde{A}_{k_1} \arrow[ur, "\gamma_1"'] & \\
& & B_1 \arrow[u, "\xi_1"'] &
\end{tikzcd}.$$

We would like to simplify the above diagram. Set $\theta_1 = [\phi_{1,k_1}] \times \eta_1 \in \mathrm{KK}(A_1, B_1)$ and $\tau_1 = \gamma_1 \circ \xi_1$. Then we obtain the following diagram:

$$\begin{tikzcd}
F_1 \subset A_1 \arrow[r, "\phi_{1,m_2}"] \arrow[dr, dashed, "\theta_1"'] & A_{m_2} \\
& B_1 \supset F_1' \arrow[u, "\tau_1"']
\end{tikzcd},$$

with the properties

(i) $\theta_1 \times [\tau_1] = \phi_{1,m_2} \in \mathrm{KK}$,

(ii) $\mathrm{dist}(\phi_{1,m_2}(f), \tau_1(F_1')) < 71\varepsilon_1$ for each $f \in F_1$,

(iii) F_1' is approximately constant to within ε_1.

We will now construct the next diagram. Let $\{a_{2n}\}_{n=1}^{\infty}$ be a dense subset of A_{m_2} and $\{b_{1n}\}_{n=1}^{\infty}$ be a dense subset of B_1. Set $G_1 = F_1' \cup \{b_{11}\} \subset B_1$, and set

$$F_2 = \phi_{1,m_2}\{a_{11}, a_{12}\} \cup \tau_1(G_1) \cup \tau_1\{b_{12}\} \cup \{a_{21}, a_{22}\} \subset A_{m_2} .$$

That is, F_2 contains all the images of $\{a_{ij}\}_{\substack{1 \le i \le 2 \\ 1 \le j \le 2}}$ and $\{b_{11}, b_{12}\}$.

For this F_2 and ε_2, one can find a k_2 such that $\phi_{m_2 k_2}(F_2)$ is approximately constant to within ε_2.

Consider the map $\sigma = \phi_{m_2 k_2} \circ \tau_1 : B_1 \to A_{k_2}$. For each partial map $\sigma^i : B_1 \to A_{k_2}^i = M_{[k_2,i]}(C(X_{k_2,i}))$, we can apply Lemma 4.3 to construct a diagram

$$\begin{tikzcd}
A_{k_2}^i \arrow[r] & M_{l[k_2,i]}(C(X_{k_2,i})) \\
B_1 \arrow[u, "\sigma^i"] \arrow[r, "\psi"'] & M_{l_1[k_2,i]}(C(\bar{X}_{k_2,i})) \supset \tilde{F}_2' \arrow[u]
\end{tikzcd}$$

having the given properties with F_1' in place of G_1, G_1 in place of G_2, and $\phi_{m_1 k_1}^{,i}(F_2) \subset A_{k_2}^i$ in place of F. Similarly, one can find a common l for each block. So, there exists a diagram

$$\begin{array}{ccc} A_{k_2} & \xrightarrow{\alpha_2} & \oplus M_{l[k_{2,i}]}(C(X_{k_{2,i}})) \\ \sigma\uparrow & & \uparrow \xi_2 \\ B_1 & \xrightarrow[\psi]{} & \oplus M_{l_i[k_{2,i}]}(C(\bar{X}_{k_{2,i}})) \supset F_2' \end{array}$$

with the following properties:
(i) the diagram commutes on F_1' to within $72\varepsilon_1$;
(ii) $\mathrm{dist}(\alpha_2(f), \xi_2(F_2')) < \varepsilon_2$, for each $f \in \phi_{m_2,k_2}(F_2)$;
(iii) $\psi(G_1) \subseteq F_2'$ and F_2' is weakly approximately constant to within ε_2.

Set $B_2 = \bigoplus_i M_{l_i[k_{2,i}]}(C(\bar{X}_{k_{2,i}})) \bigoplus \bigoplus_i M_{[k_{2,i}]}(\mathrm{C})$, $\tilde{A}_{k_2} = \bigoplus_i M_{l_i[k_{2,i}]}(C(X_{k_{2,i}})) \oplus \bigoplus_i M_{[k_{2,i}]}(\mathrm{C})$, and new$\xi_2 = \xi_2 \oplus \mathrm{id}_{\bigoplus_i M_{[k_{2,i}]}(\mathrm{C})}$. Finally, define

$$\psi_{1,2} : B_1 \to B_2 \quad \text{to be} \quad \psi_{1,2} = \psi \oplus (\beta_2 \circ \sigma) .$$

Hence we have the following diagram:

$$\begin{tikzcd} A_{k_2} \arrow[r, "\alpha_2\oplus\beta_2"] \arrow[dr, dashed, "\eta_2"] & \tilde{A}_{k_2} \\ B_1 \arrow[u, "\sigma"] \arrow[r, "\psi_{1,2}"'] & B_2 \arrow[u, "\xi_2"'] \end{tikzcd},$$

where $\eta_2 \in \mathrm{KK}(A_{k_2}, B_2)$.

Write newF_2' = old$F_2' \oplus (\beta_2 \circ \phi_{m_2k_2})(F_2)$. Then new$F_2'$ is still approximately constant to within ϵ_2. Furthermore,

$$\mathrm{dist}((\alpha_2 \oplus \beta)(f), \xi_2(F_2')) < \epsilon_2 \qquad \text{for all} \quad f \in \phi_{m_2k_2}(F_2).$$

One can construct m_3 similarly, i.e., find m_3 and $\gamma_2 : \tilde{A}_{k_2} \to A_{m_3}$ such that

$$\|\gamma_2 \circ (\alpha_2 \oplus \beta_2)(f) - \phi_{k_2,m_3}(f)\| \le 70\varepsilon_2 \quad \text{for each} \quad f \in \phi_{m_2k_2}(F_2) .$$

(We put the conjugation by a unitary as part of γ_2.)

Set $\tau_2 = \gamma_2 \circ \xi_2$. We have a diagram

$$\begin{tikzcd} A_1 \arrow[r] & A_{m_2} \arrow[r] \arrow[dr, dashed, "\theta_2"] & A_{m_3} \\ & B_1 \arrow[u, "\tau_1"] \arrow[r, "\psi_{1,2}"'] & B_2 \supset F_2' \arrow[u, "\tau_2"'] \end{tikzcd}$$

with the following properties:
(i) The square commutes within $142\varepsilon_1$ on F_1';
(Notice that $\varepsilon_2 < \varepsilon_1$, and $\tau_1(F_1') \subseteq F_2$.)
(ii) $\phi_{m_2,m_3}(F_2)$ is approximately contained in $\tau_2(F_2')$ to within $71\varepsilon_2$;

(Notice that $(\alpha_2 \oplus \beta_2)\phi_{m_2k_2}(F_2)$ is approximately contained in $\xi_2(F_2')$ to within ε_2.)

(iii) $\psi_{1,2}(G_1) \subseteq F_2'$, and F_2' is approximately constant to within ε_2.

In such a way, we can construct a diagram

$$\begin{tikzcd}
F_1 & F_2 & F_3 & F_4 & & \\
\cap & \cap & \cap & \cap & & \\
A_1 \arrow[r] \arrow[dr, dashed, "\theta_1"] & A_{m_2} \arrow[r] \arrow[dr, dashed, "\theta_2"] & A_{m_3} \arrow[r] \arrow[dr, dashed, "\theta_3"] & A_{m_4} \arrow[r] & \cdots & A \\
 & B_1 \arrow[u, "\tau_1"] \arrow[r, "\psi_{1,2}"'] & B_2 \arrow[u, "\tau_2"] \arrow[r, "\psi_{2,3}"'] & B_3 \arrow[u, "\tau_3"] \arrow[r] & \cdots & B \\
 & \cup & \cup & \cup & & \\
 & G_1 & G_2 & G_3 & & \\
 & \cup & \cup & \cup & & \\
 & F_1' & F_2' & F_3' & &
\end{tikzcd}$$

with the following properties:

(i) $\tau_{i+1} \circ \psi_{i,i+1}$ and $\phi_{m_{i+1},m_{i+2}} \circ \tau_i$ are approximately equal on F_i' to within $142\varepsilon_{i-1}$;

(ii) $\phi_{m_i m_{i+1}}(F_i)$ is approximately contained in $\tau_i(F_i')$ to within $71\varepsilon_i$;

(iii) $F_{i+1}' \supset \psi_{i,i+1}(G_i)$, $F_i \supseteq \tau_{i-1}(G_{i-1}) \cup \phi_{m_{i-1}m_i}(F_{i-1})$;

(iv) G_i contains the image of $\{b_{kj}\}_{\substack{k\le i\\ j\le i}}$, and F_i contains the image of $\{a_{kj}\}_{\substack{k\le i\\ j\le i}}$;

(v) $\{b_{kj}\}_{i=1}^{\infty}$ is dense in B_k, and $\{a_{kj}\}_{i=1}^{\infty}$ is dense in A_{m_k}.

From (i) and (iii), one can prove that τ_i defines one-sided approximately intertwining (certainly we also need the properties (iv) and (v)). So it defines a homomorphism $\tau : B \to A$.

Combining (i) and (ii), one knows that $F_i \subset A_{m_i} \subset A$ is approximately contained in $\tau(F_i')$ to within $71\varepsilon_i + 142(\varepsilon_{i-1} + \varepsilon_i + \varepsilon_{i+1} + \cdots)$. This proves that τ is surjective. Finally, from the fact that the maps θ_i and τ_i form an intertwining at the KK level, we know that $\tau : B \to A$ defines an isomorphism at the level of K-theory. By Lemma 2.15, τ is an isomorphism.

REMARK 4.7. As pointed out in Remark 4.24 of [EG2], any inductive limit of finite direct sums of algebras of the form $P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}$ is a corner sub-algebra of an inductive limit of finite direct sums of matrix algebras over $C(X_{n,i})$. Hence Theorem 4.4 can be generalized to the case of inductive limits of finite direct sums of algebras of the above form.

5. Some Remarks.

5.1. In the above section, we proved that a general real rank zero inductive limit A (with slow dimension growth) can be expressed as an inductive limit of matrix algebras over $X_1 \vee X_2 \vee X_3 \ldots \vee X_n$, with each X_i of the form S^1, S^2, $T_{II,k}$, or $T_{III,k}$. By 5.9, 5.10, and 5.11 of [EG2], we know that the wedge is not necessary. That is, A can be written as an inductive limit of finite direct sums of matrix algebras over $S^1, S^2, T_{II,k}$, and $T_{III,k}$. In §4 of [EG2], we proved that the graded scaled ordered K-groups of those inductive limits which involve only the spaces $S^1, T_{II,k}$, and $T_{III,k}$ (i.e., without S^2) exhaust all possible graded scaled ordered K-groups of real rank zero inductive limits with slow dimension growth. One may wonder, can we obtain all inductive limit algebras without using space the S^2? That is, can all all inductive limit C^*-algebras with slow dimension growth be written as inductive limits of matrix algebras over $S^1, T_{II,k}$, and $T_{III,k}$ (without S^2). However, the answer is negative — the use of S^2 is essential (see below).

5.2. In §3 of [G1], we constructed two non-isomorphic inductive limits of finite direct sums of matrix algebras over 2-dimensional finite CW complexes with the same scaled ordered K-groups. One can construct similar examples by using the space $Y = S^2 \vee T_{III,2}$ to replace $X = S^1 \vee \mathscr{P}^2 = S^1 \vee T_{II,2}$ (in [G1]).That is, we can construct two inductive limits

$$\begin{array}{ccccc} A_1 & \longrightarrow & A_2 & \longrightarrow \cdots \longrightarrow & A \\ B_1 & \longrightarrow & B_2 & \longrightarrow \cdots \longrightarrow & B \end{array}$$

such that $\big(K_*(A), K_*(A)_+, 1_A\big) \cong \big(K_*(B), K_*(B)_+, 1_B\big)$ and A_i, B_i are finite direct sums of matrix algebras over Y, and $A \ncong B$. (We can copy the whole construction in §3 of [G1], replacing S^1 by S^2 and $\mathscr{P}^2$ by $T_{III,k}$, with only one revision:we should assume that each l_n in 3.6 of [G1] is a multiple of 2.)

From Proposition 4.18 of [G1] we know that, if A is a real rank zero inductive limit of matrix algebras over 3-dimensional CW complexes $X_{n,i}$ with $\tilde{K}^0(X_{n,i})$ finite, and $K_0(A)$ is torsion free, then A is completely determined by its scaled ordered K-group. So if both A and B constructed above could be written as inductive limits of finite direct sums of matrix algebras over $S^1, T_{II,k}$, and $T_{III,k}$, then they would be isomorphic. This proves that it is essential to use S^2 (or a space with infinite H^2 group) to produce all real rank zero AH algebras.

5.3. One can also prove that the above two examples are inductive limits of finite direct sums of dimension drop algebras (see [G3]). Therefore, they provide a counter example for the classification in terms of the graded ordered group K_* of real rank zero inductive limits of dimension drop C*-algebras. This is joint work with G. A. Elliott and H. Su (see [EGS]). (Note that G. A. Elliott proved the classification theorem for simple such C*-algebras, and H. Su generalized the result to the case of graphs with dimension drop.) In general, if we use dimension drop C*-algebras, we can remove both $T_{III,k}$ and S^2. Based on this idea one can prove the following:

THEOREM 5.4. *If A and B are real rank zero inductive limits satisfying the slow dimension growth condition* $(**)$, *then A is isomorphic to B if and only if A is unsuspended E-equivalent to B* (*or asymptotically isomorphic to B*).

The details of this theorem will appear elsewhere. This theorem is a generalization of Theorem 2.21 of [G1]. But the proof is similar. We conjecture that any separable nuclear C*-algebra of real rank zero and stable rank one is determined up to isomorphism by its unsuspended E-equivalence type.

[Added Note] Theorem 4.4 is proved independently by M. Dadarlat in Reduction to dimension three of local spectra of real rank zero C*-algebras, J. Reine Angew. Math. 460(1995), 189-212. A generalized version of Theorem 5.4 has been proved in a joint work of M. Dadarlat and the author: A classification result for approximately homogeneous C*-algebras of real rank zero, preprint.

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