MATH. SCAND. 79 (1996), 263–282

RANDOMLY WEIGHTED SERIES OF CONTRACTIONS IN HILBERT SPACES

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Conditions are given for the convergence of randomly weighted series of contractions in Hilbert spaces. It is shown that under these conditions the series converges in operator norm outside of a (universal) null set simultaneously for all Hilbert spaces and all contractions of them. The conditions obtained are moreover shown to be as optimal as possible. The method of proof relies upon the spectral lemma for Hilbert space contractions (which allows us to imbed the initial problem into a setting of Fourier analysis), the standard Gaussian randomization (which allows us further to transfer the problem into the theory of Gaussian processes), and finally an inequality due to Fernique [3] (which gives an estimate of the expectation of the supremum of the Gaussian (stationary) process over a finite interval in terms of the spectral measure associated with the process by means of the Bochner theorem). As a consequence of the main result we obtain: Given a sequence of independent and identically distributed mean zero random variables $\{Z_k\}_{k\geq 1}$ defined on $(\Omega,\mathcal{F},P)$ satisfying $E|Z_1|^2 < \infty$, and $\alpha > 1/2$, there exists a (universal) P-null set $N^* \in \mathcal{F}$ such that the series:

$$\sum_{k=1}^{\infty} \frac{Z_k(\omega)}{k^\alpha} T^k$$

converges in operator norm for all $\omega \in \Omega^*$, whenever H is a Hilbert space and T is a contraction in H.

1. Introduction.

The purpose of the paper is to investigate and establish conditions for the convergence in operator norm of the randomly weighted series of contractions in Hilbert spaces:

$$\sum_{k=1}^{\infty} W_k(\omega) T^{p_k} \tag{1.1}$$

where $\{W_k\}_{k\geq 1}$ is a sequence of independent mean zero square-integrable random variables defined on the probability space $(\Omega,\mathcal{F},P)$, and T is a (linear) contraction in the Hilbert space H, while $\{p_k\}_{k\geq 1}$ is a non-decreasing sequence of non-negative integers, and $\omega \in \Omega$. Our main aim is to find sufficient conditions (and in this context to prove that they are as optimal as

Received March 23, 1995.

possible) for the convergence of the series in (1.1) which is valid simultaneously for all Hilbert spaces H and all contractions T in H. More precisely, we find conditions (see (3.1) in Theorem 3.1 and (3.1′) in Remark 3.2) in terms of the numbers p_k and $E|W_k|^2$ for $k \geq 1$, under which there exists a (universal) P-null set $N^* \in \mathcal{F}$, such that the series in (1.1) converges in operator norm for all $\omega \in \Omega \backslash N^*$, whenever H is a Hilbert space and T is a contraction in H. (This is the main result of the paper.) Then we specialize and investigate this result when $W_k = Z_k/k^\alpha$ for $k \geq 1$ and $\alpha > 0$, where $\{Z_k\}_{k\geq 1}$ is a sequence of independent and identically distributed mean zero random variables satisfying $E|Z_1|^2 < \infty$ (see Theorem 3.4), and in particular we obtain that for $\alpha > 1/2$ the series:

$$\sum_{k=1}^{\infty} \frac{Z_k(\omega)}{k^\alpha} T^k \tag{1.2}$$

converges in operator norm for all $\omega \in \Omega \backslash N^*$, whenever H is a Hilbert space and T is a contraction in H (see Corollary 3.5). In the end (see Remark 3.6) it is shown that the conditions obtained throughout are as optimal as possible.

The method of proof may be described as follows. First, we use the spectral lemma for Hilbert space contractions (Lemma 2.1), and in this way imbed the initial problem about the series in (1.1) into a setting of Fourier analysis (see [4] and [7]), which concerns expressions of the form:

$$\sup_{-\pi<\lambda\leq\pi} \left| \sum_{k=1}^{N} W_k e^{ip_k\lambda} \right| \tag{1.3}$$

for some $W_1, \ldots, W_N \in \mathsf{R}$ with $0 \leq p_1 \leq \ldots \leq p_N$ being integers for $N \geq 1$. Second, by performing a standard procedure of Gaussian randomization we imbed the problem about (1.3) into the theory of Gaussian processes. The Gaussian process which appears as the product of the randomization procedure is given by:

$$G_\lambda(\omega', \omega'') = \sum_{k=1}^{N} W_k (g_k'(\omega') \cos(p_k\lambda) + g_k''(\omega'') \sin(p_k\lambda)) \tag{1.4}$$

for $\lambda \in \mathsf{R}$, $(\omega', \omega'') \in \Omega_g' \otimes \Omega_g''$, $W_1, \ldots, W_N \in \mathsf{R}$, and $0 \leq p_1 \leq \ldots \leq p_N$ with $N \geq 1$. (Here $g' = \{g_k'\}_{k\geq 1}$ and $g'' = \{g_k''\}_{k\geq 1}$ are (mutually independent) sequences of independent standard Gaussian ($\sim N(0,1)$) random variables defined on $(\Omega_g', \mathcal{F}_g', P_g')$ and $(\Omega_g'', \mathcal{F}_g'', P_g'')$ respectively.) The problem in this context is reduced to estimate the expression:

$$E\left(\sup_{-\pi<\lambda\leq\pi} |G_\lambda| \right). \tag{1.5}$$

For this, we apply an inequality due to Fernique [3] which estimates (1.5) in terms of the spectral measure associated with the process $G = \{G_\lambda\}_{\lambda\in\mathbf{R}}$ by means of the Bochner theorem (see Lemma 2.2). It turns out that the spectral measure can be found explicitly, thus allowing to obtain an accurate estimate of (1.5) which suffices for our purposes. The optimality of the conditions deduced is proved by using Kolmogorov's three series theorem.

To conclude the introduction, we find it convenient to mention that we are unaware of similar results, and to the best of our knowledge the sort of series which appears in (1.1) and (1.2) has not been studied previously. It should be also remarked that our emphasis is on the method of proof which seems to be flexible enough to admit applications in the study of various problems having a similar character (see [10]).

2. Preliminary facts.

In this section we find it convenient to recall and display some results and facts which will be used in the proof of the main result of this paper (Theorem 3.1).

We begin by recalling a useful fact (called the *spectral lemma* for Hilbert space contractions) which allows us to transfer the norm operator problems into the setting of Fourier analysis. For this, we should first clarify that a linear operator T (defined in a (complex) Hilbert space H) is called a *contraction*, if $\|T(f)\| \leq \|f\|$ for a $f \in H$ (see [2]). Given a contraction T in H and $f \in H$, denote $P_n(f) = \langle T^n(f), f\rangle$ for $n \geq 0$, and put $P_n(f) = \overline{P_{-n}(f)}$ for $n < 0$. Then the sequence $\{P_n(f)\}_{n\in\mathbf{Z}}$ is non-negative definite ($\sum_k \sum_l z_k \bar{z}_l P_{k-l}(f) \geq 0$, for all $z_k \in \mathbf{C}$ with $|k| \leq N$ and $N \geq 1$), see [5] (p. 94–95). Thus by the Herglotz theorem [5] there exists a finite positive measure μ_f on $\mathscr{B}(]-\pi,\pi])$ (called the *spectral measure* of f) such that:

$$\langle T^n(f), f\rangle = \int_{-\pi}^{\pi} e^{in\lambda} \mu_f(d\lambda) \tag{2.1}$$

for all $n \geq 0$. From this fact, by induction in degree of the polynomial, one can deduce the following remarkable inequality. (This proof was communicated to us by M. Wierdl.)

LEMMA 2.1 (The spectral lemma). *If T is a contraction in a Hilbert space H, and f is an element from H with the spectral measure μ_f, then the inequality is satisfied:*

$$\|P(T)(f)\| \leq \left(\int_{-\pi}^{\pi} |P(e^{i\lambda})|^2 \mu_f(d\lambda)\right)^{1/2} \tag{2.2}$$

whenever $P(z) = \sum_{k=0}^{N} a_k z^k$ *is a complex polynomial of degree* $N \geq 0$.

PROOF. The proof is carried out by induction on the degree N of the polynomial $P(z)$. If $N = 0$, then we have equality in (2.2), since it follows from (2.1) that $\|f\|^2 = \mu_f(]-\pi, \pi])$. Suppose that the inequality (2.2) is true for $N-1 \geq 0$. Denote $Q(z) = \sum_{k=1}^{N} a_k z^k$ and $R(z) = \sum_{k=1}^{N} a_k z^{k-1}$. Then by (2.1) we find:

$$\begin{aligned} \|P(T)(f)\|^2 &= \|a_0 f + Q(T)(f)\|^2 = |a_0|^2\|f\|^2 + \langle a_0 f, Q(T)(f)\rangle \\ &+ \langle Q(T)(f), a_0 f\rangle + \|Q(T)(f)\|^2 = |a_0|^2\|f\|^2 + \int_{-\pi}^{\pi} a_0\overline{Q(e^{i\lambda})}\mu_f(d\lambda) \\ &+ \int_{-\pi}^{\pi} \overline{a_0} Q(e^{i\lambda})\mu_f(d\lambda) + \|Q(T)(f)\|^2. \end{aligned} \tag{2.3}$$

Since T is a contraction, then by the assumption we get:

$$\begin{aligned} \|Q(T)(f)\|^2 &= \|T(R(T))(f)\|^2 \leq \|R(T)(f)\|^2 \\ &\leq \int_{-\pi}^{\pi} |R(e^{i\lambda})|^2\mu_f(d\lambda) = \int_{-\pi}^{\pi} |Q(e^{i\lambda})|^2\mu_f(d\lambda). \end{aligned} \tag{2.4}$$

From (2.3) and (2.4) we conclude:

$$\|P(T)(f)\|^2 \leq \int_{-\pi}^{\pi} (|a_0|^2 + a_0\overline{Q(e^{i\lambda})} + \overline{a_0}Q(e^{i\lambda}) + |Q(e^{i\lambda})|^2)\mu_f(d\lambda)$$

$$= \int_{-\pi}^{\pi} |a_0 + Q(e^{i\lambda})|^2\mu_f(d\lambda) = \int_{-\pi}^{\pi} |P(e^{i\lambda})|^2\mu_f(d\lambda).$$

It should be noted from (2.4) in Lemma 2.1 that if T is an isometry in H ($\|T(f)\| = \|f\|$ for $f \in H$), then we have equality in (2.2). (It also follows more directly by (2.1).) It allows us to deduce the inequality (2.2) in a straightforward way by using the (oldest and best known) *dilation theorem* of Sz.-Nagy (see Theorem 1 in [9] (p. 2)): *If T is a contraction in a Hilbert space H, then there exists a Hilbert space K and a unitary operator U in K such that H is a subspace of K and $T^n(f) = Z(U^n)(f)$ for all $n \geq 0$ and all $f \in H$, where Z is the orthogonal projection from K into H.* (It remains to apply the equality in (2.2) to the isometry U, and then use the preceding identity and the fact that $\|Z\| \leq 1$.)

However, if $\|T\| < 1$, then the error appearing in the estimate (2.2) may be noticeably large. To see this more explicitly, take for instance $P(z) = z^n$ with $n \geq 1$. Then the left-hand side in (2.2) equals $\|T^n(f)\|$ which is bounded by

$\|T\|^n\|f\|$, and thus arbitrarily small when n increases. On the other hand, the right-hand side in (2.2) equals $\|f\|$, thus being arbitrarily large as f runs through H. We will come back to this sort of analysis later on in section 4.

Our next aim is to display an inequality due to Fernique [3] which is shown to be at the basis of our results in the next section. For this reason we shall first recall the (wide sense) stationary setting upon which this result relies (see [1]).

Let $G = \{G_\lambda\}_{\lambda\in\mathsf{R}}$ be a (wide sense) stationary process defined on the probability space $(\Omega, \mathcal{F}, P)$ and taking values in the set of complex numbers C. Thus, we have:

$$E|G_\lambda|^2 < \infty \tag{2.5}$$

$$E(G_\lambda) = E(G_0) \tag{2.6}$$

$$\operatorname{Cov}(G_{\lambda+h}, G_{\eta+h}) = \operatorname{Cov}(G_\lambda, G_\eta) \tag{2.7}$$

for all $\lambda, \eta, h \in \mathsf{R}$. As a matter of convenience, we suppose:

$$E(G_\lambda) = 0 \tag{2.8}$$

for all $\lambda \in \mathsf{R}$. Thus the *covariance function* of G is given by:

$$R(\lambda) = E(G_\lambda\overline{G_0}) \tag{2.9}$$

for all $\lambda \in \mathsf{R}$. By the *Bochner theorem* there exists a finite positive measure μ on $\mathcal{B}(\mathsf{R})$ such that:

$$R(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x}\mu(dx) \tag{2.10}$$

for all $\lambda \in \mathsf{R}$. The measure μ is called the *spectral measure* of G. The *spectral representation theorem* states if R is continuous (which is equivalent to the fact that G is continuous in quadratic mean), then there exists an *orthogonal stochastic meaure* Z on $\Omega \times \mathcal{B}(\mathsf{R})$ such that:

$$G_\lambda = \int_{-\infty}^{\infty} e^{i\lambda x}Z(dx) \tag{2.11}$$

for all $\lambda \in \mathsf{R}$. The fundamental identity in this context is as follows:

$$E\left|\int_{-\infty}^{\infty} \varphi(x)Z(dx)\right|^2 = \int_{-\infty}^{\infty} |\varphi(x)|^2\mu(dx) \tag{2.12}$$

whenever $\varphi : \mathsf{R} \to \mathsf{C}$ belongs to $L^2(\mu)$. Hence we find:

$$E|G_\lambda - G_\eta|^2 = 2\int_{-\infty}^{\infty} (1-\cos((\lambda-\eta)x))\mu(dx) \tag{2.13}$$

for all $\lambda, \eta \in \mathsf{R}$. The result may be now stated as follows.

LEMMA 2.2 (Fernique). *Let $G = \{G_\lambda\}_{\lambda\in\mathsf{R}}$ be a stationary mean zero Gaussian process with values in R and the spectral measure μ. Suppose that G is separable and continuous in quadratic mean. Then there exists a (universal) constant $K > 0$ (which does not depend on the Gaussian process G itself), such that the inequality is satisfied:*

$$E\Big(\sup_{0\le\lambda\le 1} G_\lambda\Big) \le K\left(\left|\int_{-\infty}^{\infty}(x^2\wedge 1)\mu(dx)\right|^{1/2} + \int_0^{\infty} |\mu(]e^{x^2},\infty[)|^{1/2}dx\right). \tag{2.14}$$

PROOF. It follows from (0.4.5) in Theorem 0.4 in [3] upon taking $c_1 = b_1 = 1$ and $c_k = b_k = 0$ in (0.4.4) for $k \ge 2$.

In order to make use of the inequality (2.14) we shall denote $I_k = [k, k+1]$ for $k \in \mathsf{Z}$. Let $I \subset \mathsf{R}$ be any bounded interval, then $I \subset \bigcup_{k\in A} I_k$ for some finite $A \subset \mathsf{Z}$. Let K_I denote the number of elements in A. Then by the separability, stationarity, and symmetry of the Gaussian process $G = \{G_\lambda\}_{\lambda\in\mathsf{R}}$, we easily obtain:

$$\begin{aligned} E\Big(\sup_{\lambda\in I}|G_\lambda|\Big) &= E\Big(\max_{k\in A}\sup_{\lambda\in I_k}|G_\lambda|\Big) \le \sum_{k\in A} E\Big(\sup_{\lambda\in I_k}|G_\lambda|\Big) \\ &= K_I E\Big(\sup_{0\le\lambda\le 1}|G_\lambda|\Big) \le K_I\Big(E|G_0| + 2E\Big(\sup_{0\le\lambda\le 1} G_\lambda\Big)\Big). \end{aligned} \tag{2.15}$$

We will apply Lemma 2.2 with (2.15) in the proof of Theorem 3.1 (next section) to the following Gaussian process (obtained by the randomization procedure described below);

$$G_\lambda(\omega',\omega'') = \sum_{k=1}^{N} W_k(g'_k(\omega')\cos(p_k\lambda) + g''_k(\omega'')\sin(p_k\lambda)) \tag{2.16}$$

with $\lambda \in \mathsf{R}$, $(\omega',\omega'') \in \Omega'_g \otimes \Omega''_g$, $W_1, \dots, W_N \in \mathsf{R}$, and $0 \le p_1 \dots \le p_N$ with $N \ge 1$. Here $g' = \{g'_k\}_{k\ge 1}$ and $g'' = \{g''_k\}_{k\ge 1}$ are (mutually independent) sequences of independent standard Gaussian ($\sim N(0,1)$) random variables defined on the probability spaces $(\Omega'_g, \mathcal{F}'_g, P'_g)$ and $(\Omega''_g, \mathcal{F}''_g, P''_g)$ respectively. It is easily verified that the orthogonal stochastic measure associated with $G = \{G_\lambda\}_{\lambda\in\mathsf{R}}$ is given by:

$$Z((\omega',\omega''),\Delta) = \sum_{k=1}^{N} W_k g'_k(\omega')(\delta_{\{-p_k\}}(\Delta) + \delta_{\{p_k\}}(\Delta))/2 + i\sum_{k=1}^{N} W_k g''_k(\omega'')(\delta_{\{-p_k\}}(\Delta) - \delta_{\{p_k\}}(\Delta))/2 \tag{2.17}$$

for $(\omega',\omega'') \in \Omega'_g \otimes \Omega''_g$ and $\Delta \in \mathscr{B}(\mathsf{R})$. From (2.12) and (2.17) it follows easily by independence that the spectral measure of G is given by:

$$\mu(\Delta) = \sum_{k=1}^{N} |W_k|^2(\delta_{\{-p_k\}}(\Delta) + \delta_{\{p_k\}}(\Delta))/2 \tag{2.18}$$

for $\Delta \in \mathscr{B}(\mathsf{R})$. Having this expression in mind, one might observe that the right-hand side of inequality (2.14) (applied to the Gaussian process (2.16)) gets a rather explicit form which is easily estimated in a rigorous way. This will be demonstrated in more detail in the proof of Theorem 3.1.

In the remaining part of this section we describe the procedure of Gaussian randomization. It allows us to imbed the problem under consideration (which involves expressions (1.3) and appears by applications of (2.2)) into the theory of Gaussian processes. It will be used below in the proof of our main result (Theorem 3.1). It should be observed that the Gaussian process in (2.16) appears precisely after performing Gaussian randomization as described in the next lemma.

LEMMA 2.3 (Gaussian randomization). *Let $W = \{W_k\}_{k\geq 1}$ be a sequence of independent mean zero random variables defined on the probability space $(\Omega, \mathscr{F}, P)$, let $g' = \{g'_k\}_{k\geq 1}$ and $g'' = \{g''_k\}_{k\geq 1}$ be (mutually independent and independent from W) sequences of independent standard Gaussian ($\sim N(0,1)$) random variables defined on the probability spaces $(\Omega'_g, \mathscr{F}'_g, P'_g)$ and $(\Omega''_g, \mathscr{F}''_g, P''_g)$ respectively, and let $p_k \geq 1$ be non-negative integers for $k \geq 1$. Then the inequality is satisfied:*

$$E\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=1}^{N} W_k e^{ip_k\lambda}\right|\right) \leq \sqrt{8\pi}E\left(\sup_{-\pi\lambda\leq\pi}\left|\sum_{k=1}^{N} W_k(g'_k\cos(p_k\lambda) + g''_k\sin(p_k\lambda))\right|\right) \tag{2.19}$$

for all $N \geq 1$.

(It should be clarified here that the sequences W, g' and g'', initially defined on Ω, Ω'_g and Ω''_g respectively, might be well-defined as the projections onto the first, second and third coordinate of the product probability space

$(\Omega \otimes \Omega'_g \otimes \Omega''_g, \mathcal{F} \otimes \mathcal{F}'_g \otimes \mathcal{F}''_g, P \otimes P'_g \otimes P''_g)$, thus being mutually independent. The second expectation sign in (2.19) denotes the $P \otimes P'_g \otimes P''_g$-integral. Moreover, this probability space is to be enlarged in the proof below as follows. The Rademacher sequence $\epsilon = \{\epsilon_k\}_{k\geq 1}$ (ϵ_k's are independent and take values ± 1 with probability $1/2$) appearing in the proof below, initially defined on the probability space $(\Omega_\epsilon, \mathcal{F}_\epsilon, P_\epsilon)$, is understood to be defined as the projection onto the fifth coordinate of the product probability space $(\Omega \otimes \Omega'_g \otimes \Omega''_g \otimes \Omega' \otimes \Omega_\epsilon, \mathcal{F} \otimes \mathcal{F}'_g \otimes \mathcal{F}''_g, \otimes \mathcal{F}' \otimes \mathcal{F}_\epsilon, P \otimes P'_g \otimes P''_g \otimes P' \otimes P_\epsilon)$, thus being independent of W, g', g'' and W', where $W' = \{W'_k\}_{k\geq 1}$ is an independent copy of W defined on $(\Omega', \mathcal{F}', P')$ and thus, as the projection onto the fourth coordinate, independent from W, g', g'' and ϵ.)

PROOF. First note that by the triangle inequality we have:

$$(2.20)\qquad E\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=1}^{N} W_k e^{ip_k\lambda}\right|\right) \leq E\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=1}^{N} W_k \cos(p_k\lambda)\right|\right) + E\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=1}^{N} W_k \sin(p_k\lambda)\right|\right).$$

In the remainder, as a matter of convenience, we shall write cs instead of either cos or sin function. Let $W' = \{W'_k\}_{k\geq 1}$ be an independent copy of W defined on $(\Omega', \mathcal{F}', P')$, and let $\epsilon = \{\epsilon_k\}_{k\geq 1}$ be a Rademacher sequence defined on the probability space $(\Omega_\epsilon, \mathcal{F}_\epsilon, P_\epsilon)$ and understood to be independent from both W and W'. Then $\{\epsilon_k(W_k - W'_k)\}_{k\geq 1}$ and $\{W_k - W'_k\}_{k\geq 1}$ are identically distributed for any given and fixed choice of signs $\epsilon_k = \pm 1$ with $k \geq 1$, and since W_k's are of mean zero, we get:

$$\begin{aligned}(2.21)\quad E\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=1}^{N} W_k \mathrm{cs}(p_k\lambda)\right|\right) &= E\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=1}^{N}(W_k - E'(W'_k))\mathrm{cs}(p_k\lambda)\right|\right)\\ &\leq EE'\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=1}^{N}(W_k - W'_k)\mathrm{cs}(p_k\lambda)\right|\right)\\ &= EE'\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=1}^{N}\epsilon_k(W_k - W'_k)\mathrm{cs}(p_k\lambda)\right|\right).\end{aligned}$$

Taking the P_ϵ-integral on both sides in (2.21), and using the triangle inequality, we obtain:

$$(2.22)\qquad E\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=1}^{N} W_k \mathrm{cs}(p_k\lambda)\right|\right) \leq 2EE_\epsilon\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=1}^{N}\epsilon_k W_k \mathrm{cs}(p_k\lambda)\right|\right).$$

On the other hand, since $\{\epsilon_k|g_k|\}_{k\geq 1}$ and $\{g_k\}_{k\geq 1}$ are identically distributed, we have:

$$(2.23)\qquad EE_\epsilon\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=1}^{N}\epsilon_k W_k\mathrm{cs}(p_k\lambda)\right|\right)$$

$$= \sqrt{\frac{\pi}{2}}EE_\epsilon\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=1}^{N}\epsilon_k E_{g'}(|g'_k|)\,W_k\mathrm{cs}(p_k\lambda)\right|\right)$$

$$\leq \sqrt{\frac{\pi}{2}}EE_\epsilon E_{g'}\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=1}^{N}\epsilon_k|g'_k|\,W_k\mathrm{cs}(p_k\lambda)\right|\right)$$

$$= \sqrt{\frac{\pi}{2}}EE_{g'}\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=1}^{N}g'_k\,W_k\mathrm{cs}(p_k\lambda)\right|\right).$$

Now from (2.20), (2.22) and (2.23) we easily conclude:

$$E\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=1}^{N}W_k e^{ip_k\lambda}\right|\right) \leq \sqrt{2\pi}EE_{g'}\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=1}^{N}g'_k\,W_k\cos(p_k\lambda)\right|\right)$$

$$+\sqrt{2\pi}EE_{g''}\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=1}^{N}g''_k\,W_k\sin(p_k\lambda)\right|\right)$$

$$\leq 2\sqrt{2\pi}EE_{g'}E_{g''}\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=1}^{N}g'_k\,W_k\cos(p_k\lambda)+g''_k\,W_k\sin(p_k\lambda)\right|\right).$$

This is precisely (2.19), and the proof is complete.

For convenience of the reader we shall conclude this section by recalling the Lévy's inequality (for proof see [6] p. 47–48) which is used in the proof of our main result in the next section.

LEMMA 2.4 (Lévy's inequalities). *Let $\{X_k\}_{k\geq 1}$ be a sequence of independent and symmetric random variables with values in a separable Banach space B. Denote $S_n = \sum_{i=1}^{n} X_i$ for all $n \geq 1$. Then we have:*

$$(2.24)\qquad P\left\{\max_{1\leq k\leq n}\|S_k\| > t\right\} \leq 2P\{\|S_n\| > t\}$$

$$(2.25)\qquad E\left(\max_{1\leq k\leq n}\|S_k\|^p\right) \leq 2E(\|S_n\|^p)$$

for all $t > 0$, all $0 < p < \infty$, and all $n \geq 1$.

3. The main results.

In this section we present the main results of the paper. We begin by the next fundamental theorem which should be read together with Remark 3.2 following it. In the subsequent part of this section we pass to a more elaborate analysis of this result.

THEOREM, 3.1. *Let $W = \{W_k\}_{k\geq 1}$ be a sequence of independent mean zero random variables defined on the probability space $(\Omega, \mathcal{F}, P)$, and let $\{p_k\}_{k\geq 1}$ be a non-decreasing sequence of non-regative integers (with $p_1 > 1$). Suppose that there exist integers $0 := n_0 < n_1 < n_2 \ldots$, such that the following condition is satisfied:*

$$\sum_{i=0}^{\infty}\left(\sqrt{\log(p_{n_{i+1}})}\left(\sum_{k=n_i+1}^{n_{i+1}} E|W_k|^2\right)^{1/2}\right) < \infty. \tag{3.1}$$

*Then there exists a (universal) sequence of P-integrable random variables $M = \{M_N\}_{N\geq 1}$ defined on $(\Omega, \mathcal{F}, P)$ which converges to zero P-*a.s. *and in P-mean, such that for any Hilbert space H and any contraction T in H we have:*

$$\sup_{R>n_N}\left\|\sum_{k=n_N+1}^{R} W_k(\omega)T^{p_k}\right\| \leq M_N(\omega) \tag{3.2}$$

for all $\omega \in \Omega$ and all $N \geq 1$. In particular, there exists a (universal) P-null set $N^* \in \mathcal{F}$ such that the series:

$$\sum_{k=1}^{\infty} W_k(\omega)T^{p_k} \tag{3.3}$$

converges in operator norm for all $\omega \in \Omega \backslash N^*$, whenever H is a Hilbert space and T is a contraction in H.

PROOF. Given $R > n_N \geq 1$ for some $N \geq 1$, there exists (a unique) $l_R \geq 0$ such that:

$$n_{N+l_R} < R \leq n_{N+l_R+1}. \tag{3.4}$$

Let $f \in H$ be given and fixed, and let μ_f be the spectral measure of f. Then by the triangle inequality and the spectral lemma (see (2.2) in Lemma 2.1) we get:

$$
\begin{aligned}
(3.5)\qquad & \left\| \sum_{k=n_N+1}^{R} W_k(\omega) T^{p_k}(f) \right\| \le \sum_{i=N}^{N+l_R-1} \left\| \sum_{k=n_i+1}^{n_{i+1}} W_k(\omega) T^{p_k}(f) \right\| \\
& + \left\| \sum_{k=n_{N+l_R}+1}^{R} W_k(\omega) T^{p_k}(f) \right\| \le \sum_{i=N}^{\infty} \left\| \sum_{k=n_i+1}^{n_{i+1}} W_k(\omega) T^{p_k}(f) \right\| \\
& + \sum_{j=0}^{\infty} \sup_{n_{N+j} < R \le n_{N+j+1}} \left\| \sum_{k=n_{N+j}+1}^{R} W_k(\omega) T^{p_k}(f) \right\| \\
& \le \sum_{i=N}^{\infty} \left(\int_{-\pi}^{\pi} \left| \sum_{k=n_i+1}^{n_{i+1}} W_k(\omega) e^{ip_k\lambda} \right|^2 \mu_f(d\lambda) \right)^{1/2} \\
& + \sum_{j=0}^{\infty} \left(\sup_{n_{N+j} < R \le n_{N+j+1}} \left(\int_{-\pi}^{\pi} \left| \sum_{k=n_{N+j}+1}^{R} W_k(\omega) e^{ip_k\lambda} \right|^2 \mu_f(d\lambda) \right)^{1/2} \right) \\
& \le \sum_{i=N}^{\infty} \left(\sup_{-\pi < \lambda \le \pi} \left| \sum_{k=n_i+1}^{n_{i+1}} W_k(\omega) e^{ip_k\lambda} \right| \right) \|f\| \\
& + \sum_{j=0}^{\infty} \left(\sup_{n_{N+j} < R \le n_{N+j+1}} \left(\sup_{-\pi < \lambda \le \pi} \left| \sum_{k=n_{N+j}+1}^{R} W_k(\omega) e^{ip_k\lambda} \right| \right) \right) \|f\|
\end{aligned}
$$

for all $\omega \in \Omega$. In order to control the last term in (3.5), we shall generate $W' = \{W'_k\}_{k\ge 1}$ an independant copy of $W = \{W_k\}_{k\ge 1}$ and apply Lévy's inequality (2.25) with $p = 1$ to the sequence of independent and symmetric random variables $\{(W_k - W'_k)e^{ip_k}\}_{k\ge 1}$ with values in the separable Banach space B of all bounded continuous complex valued functions on $]-\pi,\pi]$ with respect to the sup-norm. In this way we obtain:

$$E\left(\sup_{n_{N+j}<R\leq n_{N+j+1}}\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=n_{N+j}+1}^{R} W_k e^{ip_k\lambda}\right|\right)\right) \tag{3.6}$$

$$= E\left(\sup_{n_{N+j}<R\leq n_{N+j+1}}\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=n_{N+j}+1}^{R} \big(W_k - E'(W_k')\big) e^{ip_k\lambda}\right|\right)\right)$$

$$\leq EE'\left(\sup_{n_{N+j}<R\leq n_{N+j+1}}\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=n_{N+j}+1}^{R} \big(W_k - W_k'\big) e^{ip_k\lambda}\right|\right)\right)$$

$$\leq 2EE'\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=n_{N+j}+1}^{n_{N+j+1}} \big(W_k - W_k'\big) e^{ip_k\lambda}\right|\right)$$

$$\leq 4E\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=n_{N+j}+1}^{n_{N+j+1}} W_k e^{ip_k\lambda}\right|\right)$$

for all $j \geq 0$. Taking supremums in (3.5) first over all $f \in H$ with $||f|| \leq 1$ and then over all $R > n_N$, it is clear that the proof of (3.2) will be completed as soon as we show that:

$$\sum_{i=0}^{\infty} E\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=n_i+1}^{n_{i+1}} W_k e^{ip_k\lambda}\right|\right) < \infty. \tag{3.7}$$

Note that once (3.7) being proved, the variables M_N appearing in (3.2) may be defined as follows:

$$M_N(\omega) = \sum_{i=N}^{\infty}\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=n_i+1}^{n_{i+1}} W_k(\omega) e^{ip_k\lambda}\right|\right) \tag{3.8}$$

$$+\sum_{j=0}^{\infty}\left(\sup_{n_{N+j}<R\leq n_{N+j+1}}\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=n_{N+j}+1}^{R} W_k(\omega) e^{ip_k\lambda}\right|\right)\right)$$

for all $\omega \in \Omega$ and all $N \geq 1$. The proof of (3.7) is carried out in two steps.

First, by performing the standard Gaussian randomization (see (2.19) in Lemma 2.3) we get:

$$(3.9)\qquad E\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=n_i+1}^{n_{i+1}} W_k e^{ip_k\lambda}\right|\right)$$
$$\leq \sqrt{8\pi}E\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=n_i+1}^{n_{i+1}} W_k\left(g'_k\cos(p_k\lambda)+g''_k\sin(p_k\lambda)\right)\right|\right)$$

for all $i \geq 0$, where $g' = \{g'_k\}_{k\geq 1}$ and $g'' = \{g''_k\}_{k\geq 1}$ are (mutually independent and independent from W) sequences of independent standard Gaussian ($\sim N(0,1)$) random variables defined on $(\Omega'_g, \mathscr{F}'_g, P'_g)$ and $(\Omega''_g, \mathscr{F}''_g, P''_g)$ respectively.

Second, we apply Lemma 2.2 to the stationary Gaussian process:

$$G_\lambda(\omega',\omega'') = \sum_{k=n_i+1}^{n_{i+1}} W_k(\omega)\left(g'_k(\omega')\cos(p_k\lambda)+g''_k(\omega'')\sin(p_k\lambda)\right)$$

for $\lambda \in R$ and $(\omega',\omega'') \in \Omega'_g \otimes \Omega''_g$, where $\omega \in \Omega$ and $i \geq 0$ are given and fixed. Thus by (2.15) and (2.18) we get:

$$(3.10)\qquad E\left(\sup_{-\pi<\lambda\geq\pi}\left|\sum_{k=n_i+1}^{n_{i+1}} W_k(\omega)(g'_k\cos(p_k\lambda)+g''_k\sin(p_k\lambda))\right|\right)$$
$$\leq 8\left(E\left|\sum_{k=n_i+1}^{n_{i+1}} W_k(\omega)g'_k\right| + 2K\left(\left(\sum_{k=n_i+1}^{n_{i+1}} |W_k(\omega)|^2(p_k^2\wedge 1)\right)^{1/2}\right.\right.$$
$$\left.\left.+\int_0^\infty\left(\sum_{k=n_i+1}^{n_{i+1}} |W_k(\omega)|^2 1_{\{y\,|\,p_k>\exp(y^2)\}}(x)\right)^{1/2} dx\right)\right)$$

where $K > 0$ is the (universal) numerical constant from (2.14) not depending on the given and fixed $\omega \in \Omega$ and $i \geq 0$. The first term on the right-hand side of the inequality in (3.10) is easily controlled by Jensen's inequality and independence:

$$(3.11)\qquad E\left|\sum_{k=n_i+1}^{n_{i+1}} W_k(\omega)g'_k\right| \leq \left(E\left|\sum_{k=n_i+1}^{n_{i+1}} W_k(\omega)g'_k\right|^2\right)^{1/2}$$
$$= \left(\sum_{k=n_i+1}^{n_{i+1}} |W_k(\omega)|^2\right)^{1/2}$$

To estimate the last term on the right-hand side of the inequality in (3.10), we shall again use Jensen's inequality. In this way we get:

$$(3.12)\qquad \int_0^\infty \Bigg(\sum_{k=n_i+1}^{n_{i+1}} |W_k(\omega)|^2 1_{\{y\,|p_k>\exp(y^2)\}}(x)\Bigg)^{1/2} dx$$

$$\le \sqrt{\log(p_{n_{i+1}})}\Bigg(\int_0^{\sqrt{\log(p_{n_{i+1}})}} \sum_{k=n_i+1}^{n_{i+1}} |W_k(\omega)|^2 1_{\{y\,|\,p_k>\exp(y^2)\}}(x)\frac{dx}{\sqrt{\log(p_{n_{i+1}})}}\Bigg)^{1/2}$$

$$= \big(\log(p_{n_{i+1}})\big)^{1/4}\Bigg(\sum_{k=n_i+1}^{n_{i+1}} |W_k(\omega)|^2\sqrt{\log p_k}\Bigg)^{1/2}$$

for the given and fixed $\omega \in \Omega$ and $i \ge 0$. From (3.10), (3.11) and (3.12) we obtain the estimate:

$$(3.13)\qquad E\Bigg(\sup_{-\pi<\lambda\le\pi}\Bigg|\sum_{k=n_i+1}^{n_{i+1}} W_k(\omega)\big(g_k'\cos(p_k\lambda)+g_k''\sin(p_k\lambda)\big)\Bigg|\Bigg)$$

$$\le 8(1+2K)\Bigg(\Bigg(\sum_{k=n_i+1}^{n_{i+1}} |W_k(\omega)|^2\Bigg)^{1/2} + \sqrt{\log(p_{n_{i+1}})}\Bigg(\sum_{k=n_i+1}^{n_{i+1}} |W_k(\omega)|^2\Bigg)^{1/2}\Bigg)$$

for the given and fixed $\omega \in \Omega$ and $i \ge 0$. Now, taking the P-integral in (3.13) and applying Jensen's inequality to the second term on the right-hand side, we see by (3.1) and (3.9) that (3.7) holds. From this, by (3.6) and (3.8), we see that $E(M_N) < \infty$ for all $N \ge 1$, and hence from the specific form of M_N's we conclude $M_N \to 0$ P-a.s. and in P-mean as $N \to \infty$. This completes the proof of (3.2). The last statement about the series in (3.3) follows clearly from (3.2). □

REMARK 3.2. The proof of Theorem 3.1 shows (see (3.12) above) that condition (3.1) might be weakened to the following condition:

$$(3.1')\qquad \sum_{i=0}^{\infty}\Bigg(\big(\log(p_{n_{i+1}})\big)^{1/4} E\Bigg(\sum_{k=n_i+1}^{n_{i+1}} \sqrt{\log p_k}\,|W_k|^2\Bigg)^{1/2}\Bigg) < \infty.$$

Under (3.1′) we have again (3.2) and (3.3). In this context it should be observed that (3.1) follows straightforward from (3.1′) by Jensen's inequality. While apparently weaker as stated in (3.1′), the condition seems to be most convenient in the form given in (3.1).

REMARK 3.3. In the context of condition (3.1′) in Remark 3.2 it is worth recalling that $(\sum_{i=1}^{\infty} |x_i|^2)^{1/2} \le \sum_{i=1}^{\infty} |x_i|$ whenever $x_i \in \mathsf{R}$ for $i \ge 1$. Hence we see that if the conditions (3.1) and (3.1′) are aimed to go beyond the straightforward estimate (which is easily obtained by the triangle inequality):

$$(3.14) \qquad \left\| \sum_{k=1}^{\infty} W_k T^{p_k}(f) \right\| \leq \sum_{k=1}^{\infty} |W_k| \|T\|^{p_k} \|f\| \leq \left(\sum_{k=1}^{\infty} |W_k| \right) \cdot \|f\|$$

then $i \mapsto n_i$ must be of a considerable growth. (The estimate (3.14) also indicates that the main emphasis of the result in Theorem 3.1 is on the cases where $\|T\| = 1$; see also Remark 3.6 below). Moreover, if (3.1′) is satisfied, then we have:

$$(3.15) \qquad \sum_{i=0}^{\infty} \left(\sqrt{\log(p_{n_{i+1}})} E|W_{n_{i+1}}| \right) < \infty.$$

This indicates that in order to apply (3.1) or (3.1′) we must have $E|W_{n_{i+1}}| \to 0$ as $i \to \infty$.

Motivated by the last fact in Remark 3.3, in the remaining part of this section we invetigate:

$$(3.16) \qquad W_k = Z_k / r_k$$

for $k \geq 1$, where Z_k's are independent and identically distributed mean zero random variables satisfying $E|Z_1|^2 < \infty$, and r_k's are positive real numbers tending to infinity as $k \to \infty$. More explicitly, we take $r_k = k^\alpha$ with $\alpha > 0$, but other choices are possible as well. The result of Theorem 3.1 may be then refined as follows.

THEOREM 3.4. *Let $Z = \{Z_k\}_{k \geq 1}$ be a sequence of independent and identically distributed mean zero random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ such that $E|Z_1|^2 < \infty$, and let $\{p_k\}_{k \geq 1}$ be a non-decreasing sequence of non-negative integers (with $p_1 > 1$) satisfying condition:*

$$(3.17) \qquad \sum_{i=1}^{\infty} \frac{\sqrt{\log(p_{n_{i+1}})}}{(n_i)^{\alpha - 1/2}} < \infty$$

*for some integers $1 \leq n_1 < n_2 < \ldots$, and some $\alpha > 1/2$. Then there exists a (universal) sequence of P-integrable random variables $M = \{M_N\}_{N \geq 1}$ defined on $(\Omega, \mathcal{F}, P)$ which converges to zero P-*a.s. *and in P-mean, such that for any Hilbert space H and any contraction T in H we have:*

$$(3.18) \qquad \sup_{R > n_N} \left\| \sum_{k=n_N+1}^{R} \frac{Z_k(\omega)}{k^\alpha} T^{p_k} \right\| \leq M_N(\omega)$$

for all $\omega \in \Omega$ and all $N \geq 1$. In particular, there exists a (universal) P-null set $N^* \in \mathcal{F}$ such that the series:

$$(3.19) \qquad \sum_{k=1}^{\infty} \frac{Z_k(\omega)}{k^\alpha} T^{p_k}$$

converges in operator norm for all $\omega \in \Omega \backslash N^*$, whenever H is a Hilbert space and T is a contraction in H.

PROOF. We apply Theorem 3.1 with $W_k = Z_k/k^\alpha$ for $k \geq 1$. By the identical distibution of Z_k's we see that in order to verify condition (3.1) we have to estimate the expression:

$$(3.20) \qquad \sum_{i=0}^{\infty} \left(\sqrt{\log(p_{n_{i+1}})} \left(\sum_{k=n_i+1}^{n_{i+1}} E|W_k|^2 \right)^{1/2} \right)$$

$$= (E|Z_1|^2)^{1/2} \sum_{i=0}^{\infty} \left(\sqrt{\log(p_{n_{i+1}})} \left(\sum_{k=n_i+1}^{n_{i+1}} \frac{1}{k^{2\alpha}} \right)^{1/2} \right).$$

For this, a simple integral comparison shows that:

$$(3.21) \qquad \sum_{k=n_i+1}^{n_{i+1}} \frac{1}{k^{2\alpha}} \leq \int_{n_i}^{n_{i+1}} \frac{1}{x^{2\alpha}} dx = \frac{1}{2\alpha - 1} \left(\frac{1}{(n_i)^{2\alpha-1}} - \frac{1}{(n_{i+1})^{2\alpha-1}} \right)$$

$$\leq \frac{1}{2\alpha - 1} \cdot \frac{1}{(n_i)^{2\alpha-1}}$$

for all $i \geq 1$. Inserting (3.21) into (3.20), and using (3.17), we get:

$$(3.22) \qquad \sum_{i=1}^{\infty} \left(\sqrt{\log(p_{n_{i+1}})} \left(\sum_{k=n_i+1}^{n_{i+1}} E|W_k|^2 \right)^{1/2} \right)$$

$$\leq \left(\frac{E|Z_1|^2}{2\alpha - 1} \right)^{1/2} \sum_{i=1}^{\infty} \frac{\sqrt{\log(p_{n_{i+1}})}}{(n_i)^{\alpha - 1/2}} < \infty.$$

Thus (3.1) is satisfied, and (3.18) and (3.19) follow from (3.2) and (3.3). This completes the proof.

COROLLARY 3.5. *Let $Z = \{Z_k\}_{k \geq 1}$ be a sequence of independent and identically distributed mean zero random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ such that $E|Z_1|^2 < \infty$, and let $\alpha > 1/2$ be a given number. Then there exists a (universal) P-null set $N^* \in \mathcal{F}$ such that the series:*

$$(3.23) \qquad \sum_{k=1}^{\infty} \frac{Z_k(\omega)}{k^\alpha} T^k$$

converges in operator norm for all $\omega \in \Omega \backslash N^*$, *whenever* H *is a Hilbert space and* T *is a contraction in* H.

PROOF. We apply Theorem 3.4 with $p_k = k$ and $n_i = 2^i$. For this, note that we have:

$$\sum_{i=1}^{\infty} \frac{\sqrt{\log(p_{n_{i+1}})}}{(n_i)^{\alpha-1/2}} = \sqrt{\log 2} \sum_{i=1}^{\infty} \frac{\sqrt{i+1}}{2^{i(\alpha-1/2)}} < \infty.$$

Thus condition (3.17) is fulfilled, and (3.23) follows from (3.19). This completes the proof. □

REMARK 3.6. The conclusion of Theorem 3.4 (and Corollary 3.5) fails for $0 < \alpha \leq 1/2$, unless $Z_k = 0$ P-a.s. for all $k \geq 1$. Indeed, if the conclusion would be true with $\alpha = 1/2$ for instance, then upon taking $T = I$ we would have:

$$\left\| \sum_{k=1}^{\infty} \frac{Z_k(\omega)}{\sqrt{k}} T^{p_k}(f) \right\| = \left| \sum_{k=1}^{\infty} \frac{Z_k(\omega)}{\sqrt{k}} \right| \cdot \|f\| < \infty$$

for P-a.s. $\omega \in \Omega$. Thus by Kolmogorov's three series theorem we would obtain:

$$\sum_{k=1}^{\infty} \mathrm{Var}\left(\frac{Z_k}{\sqrt{k}} 1_{\{|Z_k| \leq C\sqrt{k}\}} \right) = \sum_{k=1}^{\infty} \frac{1}{k} \mathrm{Var}\left(Z_k 1_{\{|Z_k| \leq C\sqrt{k}\}} \right) < \infty$$

for all $C > 0$. Hence (with $C = 1$) we would easily get:

$$\mathrm{Var}(Z_1 1_{\{|Z_1| \leq \sqrt{k}\}}) \to 0$$

as $k \to \infty$, so that we could conclude $Z_1 = 0$ P-a.s. The case $0 < \alpha < 1/2$ is treated in exactly the same manner. This completes the proof of the claim.

It should also be observed by triangle inequality that the result of Theorem 3.4 (and Corollary 3.5) follows easily either for $\alpha > 1$ or $\|T\| < 1$ (provided that $\alpha > 0$ and p_k's do not tend to infinity too slowly when $k \to \infty$). This shows that Theorem 3.4 (with Corollary 3.5) treats essentially the cases (up to the simultaneity) where $\|T\| = 1$ and $1/2 < \alpha \leq 1$.

4. Some remarks on the proof.

In the remaining part of the paper we shortly analyze optimality of the inequalities used in the proof of our main result in Theorem 3.1. Our motivation for this direction relies upon two facts. Firstly, we feel that the method of proof presented is instructive and can be modified to treat similar questions. Secondly, we see from Remark 3.6 that the conditions obtained throughout are as optimal as possible, but up to the fact that the assertions implied are valid simultaneously for all Hilbert spaces and all contractions of them. Thus the information to be given below might clarify the scope and lead to an improvement when something similar (in such a generality) is not required.

The first inequality in (3.5) is a consequene of the triangle inequality and reflects our basic idea of passing to the blocks of random elements under consideration. Since the length of the blocks is not fixed and may vary, the error appearing in (3.5) can be made arbitrarily small.

The third inequality in (3.5) may contain a noticeable large error (when $\|T\| < 1$) as already recorded following Lemma 2.1. It is also mentioned there that this inequality becomes an equality if T is an isometry. Thus, in the context of our results where the convergence is to be obtained simultaneously for all contractions of Hilbert spaces, the third inequality in (3.5) is as optimal as possible. It should be kept in mind, however, that this is not the case in general.

In the context of the fourth inequality in (3.5) it could be worthwhile to record the following. By Jensen's inequality (applied twice) we find:

$$\text{(4.1)} \qquad \frac{1}{n_{i+1}-n_i}\sum_{k=n_i+1}^{n_{i+1}} E|W_k| \le \left(\frac{1}{n_{i+1}-n_i}\sum_{k=n_i+1}^{n_{i+1}} (E|W_k|)^2\right)^{1/2} \le \left(\frac{1}{n_{i+1}-n_i}\sum_{k=n_i+1}^{n_{i+1}} E|W_k|^2\right)^{1/2}$$

This can be rewritten in the form:

$$\text{(4.2)} \qquad \sum_{k=n_i+1}^{n_{i+1}} E|W_k| \le \sqrt{n_{i+1}-n_i}\left(\sum_{k=n_i+1}^{n_{i+1}} E|W_k|^2\right)^{1/2}.$$

Consider moreover the case when the sequence $\{p_k\}_{k\ge 1}$ is *lacunary*, which means $p_{k+1}/p_k > \lambda$ for $k \ge 1$ with some $\lambda > 1$ (taken to be strictly less than $p_1 > 1$). Then $p_k > \lambda^k$, and thus $\log(p_k) > k\log(\lambda)$ for all $k \ge 1$. Hence from (4.2) we easily get:

$$\text{(4.3)} \qquad \sum_{k=n_i+1}^{n_{i+1}} E|W_k| \le \frac{1}{\sqrt{\lambda}}\sqrt{\log(p_{n_{i+1}})}\left(\sum_{k=n_i+1}^{n_{i+1}} E|W_k|^2\right)^{1/2}.$$

The inequality (4.3) shows that condition (3.1) in Theorem 3.1 implies that:

$$E\left(\sum_{k=1}^{\infty} |W_k|\right) = \sum_{k=1}^{\infty} |W_k| < \infty. \tag{4.4}$$

This clearly indicates that the result of Theorem 3.1 (and Theorem 3.4 with Coroallary 3.5) does not go beyond a triangle inequality argument in the case when the sequence $\{p_k\}_{k\geq 1}$ is lacunary. We think that this fact is by itself of theoretical interest.

To explore the Gaussian randomization inequality (3.9) (see Lemma 2.3), we could note that by (3.27) + (3.31) in [8] (together with Lévy's inequality and a passage to $W_k - W_k'$, where $\{W_k'\}_{k\geq 1}$ is an independent copy of $\{W_k\}_{k\geq 1}$, independent of $\{g_k'\}_{k\geq 1}$ as well) we get the inequality:

$$\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=n_i+1}^{n_{i+1}} g_k' W_k \mathrm{cs}(p_k\lambda)\right|\right) \tag{4.5}$$

$$\leq 32\sqrt{\frac{2}{\pi}}\,\frac{n_{i+1}}{n_i+1} E\left(\sup_{-\pi<\lambda\leq\pi}\left|\sum_{k=n_i+1}^{n_{i+1}} W_k \mathrm{cs}(p_k\lambda)\right|\right)$$

where cs stands for either cos or sin. This shows optimality of (3.9). Finally, the inequality appearing in (3.10) (see Lemma 2.2) is known to be sharp (see [3]).

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