ORTHOGONAL POLYNOMIALS, $L^2$-SPACES AND ENTIRE FUNCTIONS.

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Abstract.

We show that for determinate measures $\mu$ having moments of every order and finite index of determinacy, (i.e., a polynomial $p$ exists for which the measure $|p|^2 \mu$ is indeterminate) the space $L^2(\mu)$ consists of entire functions of minimal exponential type in the Cartwright class.

1. Introduction.

Let $\mathcal{M}^*$ denote the set of positive Borel measures on the real line having moments of every order and infinite support. We are interested in finding conditions on $\mu \in \mathcal{M}^*$ such that $L^2(\mu)$ consists of entire functions in the following sense: (i) There exists a continuous linear injection $E : L^2(\mu) \rightarrow \mathcal{H}(\mathbb{C})$, where $\mathcal{H}(\mathbb{C})$ denotes the set of entire functions with the topology of compact convergence. (ii) For all $f \in L^2(\mu)$ we have $E(f) = f \mu$-a.e.. We say that $E$ is a realization of $L^2(\mu)$ as entire functions. In the discussion of this problem we need for $\mu \in \mathcal{M}^*$ the corresponding sequence of orthonormal polynomials $(p_n)$. It is uniquely determined by the orthonormality condition and the requirement that $p_n$ is a polynomial of degree $n$ with positive leading coefficient. The sequence of orthonormal polynomials depends only on the moments of $\mu$, so if $\mu$ is indeterminate, i.e. there are other measures having the same moments as $\mu$, all these measures lead to the same sequence $(p_n)$.

When the measure $\mu$ is indeterminate, the Fourier expansion of $f \in L^2(\mu)$

\begin{equation}
\sum_{n=0}^{\infty} \left( \int f(t)p_n(t)d\mu(t) \right) p_n(z)
\end{equation}

converges in $L^2(\mu)$ and uniformly on compact subsets of $\mathbb{C}$ to an entire function $F(f)(z)$, which is the orthogonal projection of $f$ onto the closure in $L^2(\mu)$ of the set $\mathbb{C}[t]$ of polynomials. We recall that $z \mapsto (p_n(z))_n$ is an entire

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function with values in the Hilbert space $\ell^2$, so in particular $(p_n^{(m)}(z))_n \in \ell^2$ for all $z \in \mathbb{C}$, $m \in \mathbb{N}$, cf. [BD1]. By a theorem of M. Riesz ([R], [A]) $F(f)$ is of minimal exponential type. If the indeterminate measure $\mu$ is Nevanlinna extremal (N-extremal in short), which means that $\mathbb{C}[t]$ is dense in $L^2(\mu)$, then $\mu$ is discrete and $F(f)(x) = f(x)$ for $x \in \text{supp}(\mu)$. This means that $F(f)$ is an extension of $f$ to an entire function of minimal exponential type.

Furthermore $f \mapsto F(f)$ is a continuous injection of $L^2(\mu)$ into $\mathcal{H}(\mathbb{C})$. In fact, for any compact set $K \subseteq \mathbb{C}$ we find by (1.1) and Parsevals formula

$$\sup_{z \in K} |F(f)(z)| \leq \|f\|_2 \sup_{z \in K} \rho(z),$$

where

$$\rho(z) = \left( \sum_{k=0}^{\infty} |p_k(z)|^2 \right)^{\frac{1}{2}}$$

is continuous. Riesz ([R]) also showed that

$$\int_{-\infty}^{\infty} \frac{\log \rho(t)}{1 + t^2} \, dt < \infty,$$

and it follows that

$$\int_{-\infty}^{\infty} \frac{\log^+ |F(f)(t)|}{1 + t^2} \, dt < \infty.$$

For a survey of the interplay between entire functions and indeterminate moment problems see [B].

In the following we denote by $\mathcal{C}_0$ the class of entire functions $f$ of minimal exponential type satisfying

$$\int_{-\infty}^{\infty} \frac{\log^+ |f(t)|}{1 + t^2} \, dt < \infty.$$

It is the functions in the Cartwright class which are of minimal exponential type.

In the case of an N-extremal measure $\mu$ we have thus seen that $L^2(\mu)$ consists of entire functions of class $\mathcal{C}_0$. The function $F(f)$ given by (1.1) will be called the \textit{canonical extension} of $f$.

The purpose of the present paper is to establish that also for certain determinate measures $\mu \in \mathcal{M}^*$ the space $L^2(\mu)$ consists of entire functions. A determinate measure $\mu$ with this property must necessarily be discrete, as we shall see below. It turns out that $L^2(\mu)$ consists of entire functions of class $\mathcal{C}_0$, if $\mu$ is a determinate measure of finite index, meaning that there exists a
polynomial \( p \) such that the measure \( |p|^2 \mu \) is indeterminate. If \( k \) is the smallest possible degree of a polynomial \( p \) such that \( |p|^2 \mu \) is indeterminate, then \( k - 1 \) is the index of \( \mu \). This concept was studied in previous papers of the authors, cf. [BD1], [BD2].

In the case of an \( N \)-extremal measure \( \mu \) the canonical extension \( F(f) \) of \( f \in L^2(\mu) \) has the additional property that \( F(p)(z) = p(z) \) for all \( z \in \mathbb{C} \), when \( p \) is a polynomial. We shall see that this property cannot subsist in the determinate case. It will be replaced by a condition which involves discrete differential operators of the form

\[
T = \sum_{l=1}^{N} \sum_{j=0}^{k_l} a_{l,j} \delta_{z_l}^{(j)}, \quad a_{l,j} \in \mathbb{C}
\]

associated to a system \( (z_i, k_i), i = 1, \ldots, N \) of mutually different points \( z_i \in \mathbb{C} \) and multiplicities \( k_i \in \mathbb{N} \). These operators act on entire functions \( F \) via the formula

\[
T(F) = \sum_{l=1}^{N} \sum_{j=0}^{k_l} a_{l,j} F^{(j)}(z_l).
\]

It is well-known that any \( T \) of the form (1.2) has a unique continuous extension from \( C[\ell] \) to \( L^2(\mu) \) if \( \mu \) is \( N \)-extremal. This extension \( \tilde{T} \) satisfies

\[
\tilde{T}(f) = T(F(f)), \quad f \in L^2(\mu),
\]

where \( F(f) \) is the canonical extension of \( f \in L^2(\mu) \). In fact, if \( (q_n) \in C[\ell] \) converges in \( L^2(\mu) \) to \( f \in L^2(\mu) \) then \( q_n = F(q_n) \) converges in \( \mathcal{H}(\mathbb{C}) \) to \( F(f) \) and hence \( \lim_{n \to \infty} T(q_n) = T(F(f)) \). We notice that \( (T(p_n)) \in \ell^2 \), and if \( f \in L^2(\mu) \) has the Fourier expansion \( \sum c_n \ell_n \) then

\[
\tilde{T}(f) = \sum_{n=0}^{\infty} c_n T(p_n).
\]

If \( \mu \) is determinate then \( T \) given by (1.2) has a (unique) continuous extension from \( C[\ell] \) to \( L^2(\mu) \) if and only if \( (T(p_n)) \in \ell^2 \). Although \( (p_n(z)) \notin \ell^2 \) for \( z \notin \text{supp}(\mu) \), it is possible to characterize the differential operators \( T \) for which \( (T(p_n)) \in \ell^2 \). This was done in [BD2]. For determinate measures \( \mu \) of finite index there are “many” of these operators, see below, and we shall prove the following:

**Theorem 1.1.** Let \( \mu \) be a determinate measure of finite index. Then \( L^2(\mu) \) consists of entire functions of class \( \mathcal{C}_0 \) via a continuous linear injection \( E : L^2(\mu) \to \mathcal{H}(\mathbb{C}) \) with the additional property that
\( \tilde{T}(f) = T(E(f)) \)

for all \( f \in L^2(\mu) \) and all operators \( T \) of the form (1.2) for which \( (T(p_n)) \in \ell^2 \).

A realization \( f \mapsto E(f) \) satisfying (1.5) is not uniquely determined. We give several different realizations, and to complete the paper, we characterize for given \( f \in L^2(\mu) \) the set of entire functions \( F \) satisfying

\[ \tilde{T}(f) = T(F) \]

for all operators \( T \) such that \( (T(p_n)) \in \ell^2 \). All these functions \( F \) turn out to be of class \( \mathcal{C}_0 \).

2. Preliminary results.

As claimed in the introduction it imposes severe restrictions on a determinate measure \( \mu \), if \( L^2(\mu) \) consists of entire functions.

**Proposition 2.1.** Let \( \mu \in \mathcal{M}^* \) be determinate and assume that \( E : L^2(\mu) \to \mathcal{H}(\mathbb{C}) \) is a realization of \( L^2(\mu) \) as entire functions. Then \( \mu \) is a discrete measure, and for each \( z \in \mathbb{C} \setminus \text{supp}(\mu) \) there exists \( p \in \mathbb{C}[t] \) such that \( p(z) \neq E(p)(z) \).

**Proof.** If the support \( S \) of \( \mu \) is non-discrete we can choose \( x_0 \in S \) and a compact subset \( F \subseteq S \setminus \{x_0\} \) having accumulation points. Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function with compact support vanishing on \( F \) and such that \( f(x_0) = 1 \). The extension \( E(f) \) of \( f \) to an entire function must necessarily vanish identically, but this is a contradiction.

For a discrete determinate measure \( \mu \) it is known that \( \sum |p_n(z)|^2 = \infty \) for all \( z \notin \text{supp}(\mu) \). Fix \( z \notin \text{supp}(\mu) \) and let us assume that the realization \( E \) has the property \( E(p)(z) = p(z) \) for all \( p \in \mathbb{C}[t] \). We define a sequence \( S_n \) of continuous linear functionals on \( \ell^2 \) by

\[ S_n(c) = \sum_{k=0}^n c_k p_k(z), \quad c = (c_n) \in \ell^2. \]

For any \( c \in \ell^2 \) there exists \( f \in L^2(\mu) \) such that

\[ \sum_{k=0}^n c_k p_k \to f \quad \text{in} \quad L^2(\mu), \]

and hence

\[ S_n(c) = E \left( \sum_{k=0}^n c_k p_k \right) (z) \to E(f)(z). \]
By the Banach-Steinhaus Theorem this implies that
\[ \sup_n \| S_n \| = \left( \sum_0^\infty |p_k(z)|^2 \right)^{\frac{1}{2}} < \infty, \]
which is a contradiction.

The determinate measures of finite index are discrete, and we shall realize \( L^2(\mu) \) as entire functions for this class of measures.

The index of determinacy of a determinate measure \( \mu \) was introduced and studied by the authors in [BD1]. This index checks the determinacy under multiplication by even powers of \( |t - z| \) for \( z \) a complex number, and it is defined as

\[ (2.1) \quad \text{ind}_z(\mu) = \sup \{ k \in \mathbb{N} \mid |t - z|^{2k} \mu \text{ is determinate} \}. \]

Using the index of determinacy, determinate measures can be classified as follows:

If \( \mu \) is constructed from an \( N \)-extremal measure by removing the mass at \( k + 1 \) points in the support, then \( \mu \) is determinate with

\[ (2.2) \quad \text{ind}_z(\mu) = \begin{cases} \ \ k, & \text{for } z \notin \text{ supp}(\mu) \\ \ k + 1, & \text{for } z \in \text{ supp}(\mu). \end{cases} \]

For an arbitrary determinate measure \( \mu \) the index of determinacy is either infinite for every \( z \), or finite for every \( z \). In the latter case the index has the form (2.2), and \( \mu \) is derived from an \( N \)-extremal measure by removing the mass at \( k + 1 \) points. Such an \( N \)-extremal measure is highly non-unique by a perturbation result of Berg and Christensen, cf. [BC, Theorem 8].

Using that the index of determinancy is constant at complex numbers outside of the support of \( \mu \), we define the index of determinacy of \( \mu \) by

\[ (2.3) \quad \text{ind}(\mu) := \text{ind}_z(\mu), \quad z \notin \text{ supp}(\mu). \]

We stress that a measure \( \mu \) of finite index is discrete and \( \text{ind}(\mu) + 1 \) is the smallest degree of a polynomial \( p \) such that \( |p|^2 \mu \) is indeterminate.

To each measure \( \mu \) which is either \( N \)-extremal or determinate of finite index we associate an entire function \( F_\mu \) with simple zeros at the points of \( \text{ supp}(\mu) \). We recall from [BD1] that

\[ (2.4) \quad F_\mu(w) = \exp \left( -w \sum_{n=0}^\infty \frac{1}{x_n} \prod_{n=0}^\infty \left( 1 - \frac{w}{x_n} \right) \exp \left( \frac{w}{x_n} \right) \right), \]

where \( \{x_n : n \in \mathbb{N}\} \) is the support of \( \mu \). This function \( F_\mu \) is the uniquely determined entire function of minimal exponential type having \( \text{ supp}(\mu) \) as its
set of zeros and satisfying \( F_\mu(0) = 1 \). In the above formulation we tacitly assume \( 0 \notin \text{supp}(\mu) \). If however \( 0 \in \text{supp}(\mu) \), the above expression for \( F_\mu \) shall be multiplied with \( w \) and \( \{ x_n : n \in \mathbb{N} \} = \text{supp}(\mu) \setminus \{0\} \).

That \( F_\mu \) is of minimal exponential type follows by a theorem of M. Riesz [R], according to which the entire functions in the Nevanlinna matrix for an indeterminate moment problem are of minimal exponential type. The function \( F_\mu \) is also in the Cartwright class.

**Theorem 2.2.** Let \( \mu \) be \( N \)-extremal. For each \( f \in L^2(\mu) \) we have

\[
F(f)(z) = \sum_{x \in \text{supp}(\mu)} \frac{F_\mu(z)}{F_\mu'(x)(z-x)}f(x), \quad z \in \mathbb{C},
\]

where the series converges uniformly on compact subsets of \( \mathbb{C} \).

**Proof.** Without loss of generality we may assume that \( 0 \in \text{supp}(\mu) \), so \( F_\mu \) is proportional to the function \( D \) from the Nevanlinna matrix, cf. [A], and it is well known that

\[
\sum_{n=0}^{\infty} p_n(z)p_n(x) = \frac{B(z)D(x) - B(x)D(z)}{z-x},
\]

cf. [BD1], [BuCa], where

\[
B(z) = -1 + z \sum_{n=0}^{\infty} q_n(0)p_n(z).
\]

Here \( (q_n) \) denotes the sequence of polynomials of the second kind given by

\[
q_n(z) = \int \frac{p_n(z) - p_n(x)}{z-x} d\mu(x).
\]

Since \( D \) vanishes on \( \text{supp}(\mu) \) we get

\[
F(f)(z) = \int \left( \sum_{n=0}^{\infty} p_n(z)p_n(x) \right)f(x)d\mu(x) = -D(z) \int \frac{B(x)f(x)}{z-x} d\mu(x),
\]

and

\[
\frac{B(x)f(x)}{z-x} = -\frac{f(x)}{z-x} + \frac{xf(x)}{z-x} \sum_{n=0}^{\infty} q_n(0)p_n(x)
\]

belongs to \( L^1(\mu) \) because \( \sum q_n(0)p_n(x) \in L^2(\mu) \).

The mass at \( x \in \text{supp}(\mu) \) is given by ([A, p. 114])
\[
\mu(\{x\}) = \frac{-1}{B(x)D'(x)}
\]
showing that
\[
F(f)(z) = \sum_{x \in \text{supp}(\mu)} \frac{D(z)}{D'(x)(z-x)} f(x)
\]
and the series converges uniformly on compact subsets of \(\mathbb{C}\). Since \(D\) and \(F_\mu\) are proportional the result follows.

From Theorem 2.2 it is easy to verify that the realization \(F(L^2(\mu))\) is a Hilbert space of entire functions in the sense of de Branges, see [Br, p. 57]. For details see Corollary 3.3 below.

In [BD2] we obtained the following result:

**Theorem 2.3.** Let \(\mu \in \mathcal{M}^*\) be determinate and let \((p_n)\) be the sequence of orthonormal polynomials corresponding to \(\mu\). Let \((z_1,k_1), \ldots, (z_N,k_N)\) be given, where the \(z\)'s are different complex numbers and the \(k\)'s are nonnegative integers. Putting \(M = \sum_{l=1}^{N} (k_l + 1)\) and

\[
\mathcal{T} = \{ T = \sum_{l=1}^{N} \sum_{j=0}^{k_l} a_{lj} \delta^{(j)}_{z_l} \mid a_{lj} \in \mathbb{C} \},
\]

we have

(i) If

\[
\text{ind}(\mu) \geq \left( \sum_{l : \mu(\{z_l\}) > 0} k_l + \sum_{l : \mu(\{z_l\}) = 0} (k_l + 1) \right) - 1,
\]
then the sequence \((T(p_n))\) belongs to \(\ell^2\) only in the trivial cases, i.e., if and only if \(T\) is a linear combination of Dirac deltas evaluated at points \(z_l\) which are mass points of the measure \(\mu\).

(ii) If

\[
0 \leq \text{ind}(\mu) \leq \left( \sum_{l : \mu(\{z_l\}) > 0} k_l + \sum_{l : \mu(\{z_l\}) = 0} (k_l + 1) \right) - 2,
\]
then,

\[
\dim \{ T \in \mathcal{T} \mid (T(p_n)) \in \ell^2 \} = M - \text{ind}(\mu) - 1 \geq 1.
\]

Furthermore, \((T(p_n)) \in \ell^2\) if and only if \(T(z^k F_\mu(z)) = 0\) for
\(k = 0, 1, \ldots, \text{ind}(\mu)\).
Corollary 2.4. Let $\mu \in M^*$ be a determinate measure of finite index. For an operator $T \in \mathcal{F}$ we have $(T(p_n)) \in \ell^2$ if and only if $T(z^k F_\mu(z)) = 0$ for $k = 0, 1, \ldots, \text{ind}(\mu)$.

Proof. It is enough to consider the case (i), and to prove that the equations $T(z^k F_\mu(z)) = 0$ for $k \leq \text{ind}(\mu)$ imply that $T$ is a linear combination of Dirac deltas at mass points of $\mu$. To simplify the notation we assume that the system is ordered such that there exist positive integers $0 \leq N_1 \leq N_2 \leq N$ for which

\[
\begin{cases}
\mu(\{z_l\}) > 0 \text{ and } k_l = 0 \text{ for } l = 1, \ldots, N_1 \\
\mu(\{z_l\}) > 0 \text{ and } k_l > 0 \text{ for } l = N_1 + 1, \ldots, N_2 \\
\mu(\{z_l\}) = 0 \text{ for } l = N_2 + 1, \ldots, N.
\end{cases}
\]

Using $F_\mu(z_l) = 0$ for $l = 1, \ldots, N_2$, the equations $T(z^k F_\mu(z)) = 0$ can be written

\[
\sum_{l=N_1+1}^{N_2} \sum_{j=1}^{k_l} a_{l,j} \delta^{(j)}_{z_l}(z^k F_\mu(z)) + \sum_{l=N_2+1}^{N} \sum_{j=0}^{k_l} a_{l,j} \delta^{(j)}_{z_l}(z^k F_\mu(z)) = 0.
\]

This system has

\[
p := \sum_{l=N_1+1}^{N_2} k_l + \sum_{l=N_2+1}^{N} (k_l + 1)
\]

variables $a_{l,j}$ and $\text{ind}(\mu) + 1$ equations, and $p \leq \text{ind}(\mu) + 1$ since we consider the case (i). We claim that the system of equations with $k \leq p - 1$ ($\leq \text{ind}(\mu)$) has a non-singular matrix, and therefore the variables involved are 0, i.e.

\[
T = \sum_{l=1}^{N_2} a_{l,0} \delta_{z_l}.
\]

The columns of the matrix can be put together in blocks

\[
\left\{ \delta^{(j)}_{z_l}(z^k F_\mu(z)) \right\}_{k=0,\ldots,p-1, \ l = N_1 + 1, \ldots, N_2, \ j=1,\ldots,k_l}
\]

and

\[
\left\{ \delta^{(j)}_{z_l}(z^k F_\mu(z)) \right\}_{k=0,\ldots,p-1, \ l = N_2 + 1, \ldots, N, \ j=0,\ldots,k_l}
\]

Since $F_\mu(z_l) = 0$, $F_{\mu}'(z_l) \neq 0$ for $l = N_1 + 1, \ldots, N_2$ and $F_{\mu}(z_l) \neq 0$ for $l = N_2 + 1, \ldots, N$, column operations show that these blocks are equivalent to the blocks
The determinant of the matrix formed by these blocks is a variant of Vandermondes determinant and is non-zero.

3. The determinate case.

For a given measure $\mu \in M^*$ of finite index of determinacy we denote by $\mathcal{D}(\mu)$ the set of operators of the form (1.2) for which $(T(p_n)) \in \ell^2$, allowing the system $(z_l, k_l)$ and $N$ to vary. It is an infinite dimensional vector space. Any $T \in \mathcal{D}(\mu)$ can be extended from $C[t]$ to a continuous linear operator $\tilde{T}$ in the space $L^2(\mu)$ via Fourier expansions:

$$\tilde{T}(f) = \sum_n \left( \int_R f(t)p_n(t)d\mu(t) \right) T(p_n), \quad \text{for} \quad f \in L^2(\mu).$$

We choose different real numbers $x_0, \ldots, x_{\text{ind}(\mu)}$ outside of the support of $\mu$ and consider the measure

$$\sigma = \mu + \sum_{i=0}^{\text{ind}(\mu)} \delta_{x_i}. \quad (3.1)$$

From the above, cf. Theorem 3.9 (1) in [BD1], it follows that the measure $\sigma$ is $N$-extremal.

Given a function $f \in L^2(\mu)$, we extend it to a function $\tilde{f}$ in the space $L^2(\sigma)$ in the following way

$$\tilde{f}(t) = \begin{cases} f(t), & \text{for } t \in \text{supp}(\mu) \\ 0, & \text{for } t = x_i, i = 0, \ldots, \text{ind}(\mu). \end{cases} \quad (3.2)$$

Clearly, $f \mapsto \tilde{f}$ is a linear isometry of $L^2(\mu)$ into $L^2(\sigma)$.

Since $\sigma$ is $N$-extremal, $\tilde{f}$ has a canonical extension to an entire function of class $\mathcal{C}_0$ given by

$$F(\tilde{f})(z) = \sum_n \left( \int_R \tilde{f}(t)q_n(t)d\sigma(t) \right) q_n(z), \quad (3.3)$$

where $(q_n)$ is the sequence of orthonormal polynomials with respect to $\sigma$. We can now formulate:

**Theorem 3.1.** Let $\mu$ be a determinate measure with finite index of determinacy $\text{ind}(\mu)$. The mapping $E(f) := F(\tilde{f})$ given by (3.3) is a realization of $L^2(\mu)$ as entire functions of class $\mathcal{C}_0$ such that for any operator $T \in \mathcal{D}(\mu)$.
\[ (3.4) \quad \tilde{T}(f) = T(E(f)) \, , \quad f \in L^2(\mu). \]

**Proof.** It is clear that \( E(f) = F(\tilde{f}) \) is a realization of \( L^2(\mu) \) as entire functions of class \( \mathcal{C}_0 \).

The set of functions \( f \in L^2(\mu) \) for which (3.4) holds is a closed subspace, and therefore it suffices to prove (3.4) for \( f = \chi_{\{x\}} \), \( x \in \text{supp}(\mu) \), where \( \chi_A \) denotes the indicator function of the set \( A \). This is a consequence of the following result:

**Proposition 3.2.** For \( x \in \text{supp}(\mu) \) we have

\[
E(\chi_{\{x\}})(z) = \frac{F_\mu(z)p(z)}{F'_\mu(x)p(x)(z - x)} \, , \quad z \in \mathbb{C},
\]

where \( p \) is the unique monic polynomial of degree \( \text{ind}(\mu) + 1 \) which vanishes at \( x_0, \ldots, x_{\text{ind}(\mu)} \).

The function

\[
\frac{F_\mu(z)}{F'_\mu(x)(z - x)}
\]

is an entire function of class \( \mathcal{C}_0 \) equal to \( \chi_{\{x\}} \) on \( \text{supp}(\mu) \) and we have

\[
\tilde{T}(\chi_{\{x\}}) = T(E(\chi_{\{x\}})) = T\left(\frac{F_\mu(z)}{F'_\mu(x)(z - x)}\right) \quad \text{for} \quad T \in \mathcal{D}(\mu).
\]

**Proof.** For \( f = \chi_{\{x\}} \) we find

\[
\tilde{f}(t) = \begin{cases} f(t), & \text{if} \ t \in \text{supp}(\mu) \\ 0, & \text{for} \ t = x_i, \ i = 0, \ldots, \text{ind}(\mu) \end{cases}
\]

\[
= \begin{cases} 1, & \text{for} \ t = x, \\ 0, & \text{otherwise} \end{cases}
\]

\[
= \chi_{\{x\}}(t).
\]

For \( T \in \mathcal{D}(\mu) \) we denote by \( \tilde{T} \) and \( \tilde{T}_\sigma \) the continuous extensions of \( T \) from \( \mathbb{C}[t] \) to \( L^2(\mu) \) and \( L^2(\sigma) \) respectively. We then have \( \tilde{T}(f) = \tilde{T}_\sigma(\tilde{f}) \) for \( f \in L^2(\mu) \) because \( \|f - p\|_{L^2(\mu)} \leq \|\tilde{f} - p\|_{L^2(\sigma)} \) when \( p \in \mathbb{C}[t] \), and in particular \( \tilde{T}(\chi_{\{x\}}) = \tilde{T}_\sigma(\chi_{\{x\}}) \) when \( x \in \text{supp}(\mu) \).

By Theorem 2.2 we have

\[
F(\tilde{f})(z) = \frac{F_\sigma(z)}{F'_\sigma(x)(z - x)} = \frac{F_\mu(z)p(z)}{F'_\mu(x)p(x)(z - x)},
\]
because $F_\sigma(z) = \beta p(z) F_\mu(z)$ for a certain constant $\beta$, and hence $F'_\sigma(x) = \beta p'(x) F_\mu(x) + \beta p(x) F'_\mu(x) = \beta p(x) F'_\mu(x)$. This gives by (1.3)

$$\tilde{T}(\chi(x)) = T \left( \frac{F_\mu(z)p(z)}{F'_\mu(x)p(x)(z-x)} \right),$$

but since

$$\frac{F_\mu(z)p(z)}{F'_\mu(x)p(x)(z-x)} = \frac{F_\mu(z)}{F'_\mu(x)(z-x)} + q(z)F_\mu(z),$$

where

$$q(z) = \frac{p(z) - p(x)}{F'_\mu(x)(z-x)p(x)}$$

is a polynomial of degree $\text{ind}(\mu)$, we have $T(qF_\mu) = 0$ by Corollary 2.4, and the second assertion follows.

**Corollary 3.3.** With the notation above we have

$$(3.5) \quad E(f)(z) = \sum_{x \in \text{supp}(\mu)} \frac{F_\mu(z)p(z)}{F'_\mu(x)p(x)(z-x)} f(x) \quad \text{for} \quad f \in L^2(\mu),$$

where the series converges uniformly on compact subsets of $C$.

The realization $E(L^2(\mu)) \subseteq \mathcal{H}(C)$ is a Hilbert space of entire functions in the sense of de Branges.

**Proof.** Formula (3.5) follows immediately from Theorem 2.2 and Proposition 3.2. To see that $E(L^2(\mu))$ is a Hilbert space of entire functions in the sense of de Branges we shall verify the properties (H1)–(H3) from [Br, p. 57]. We shall only comment on (H1): If $w \in \mathbb{C} \setminus \mathbb{R}$ is a zero of $E(f)$ we have

$$\sum_{x \in \text{supp}(\mu)} \frac{f(x)}{F'_\mu(x)p(x)(w-x)} = 0,$$

and hence for $z \neq w$

$$E \left( f(x) \frac{x-w}{x-w} \right)(z) = F_\mu(z)p(z) \sum_{x \in \text{supp}(\mu)} \frac{f(x)}{F'_\mu(x)p(x)(z-x)} \left( 1 + \frac{w-w}{x-w} \right)$$

$$= \left[ E(f)(z) + F_\mu(z)p(z)(w-w)S(z) \right],$$

where
\[ S(z) = \sum_{x \in \text{supp}(\mu)} \frac{f(x)}{F'_\mu(x)p(x)} \left( \frac{1}{(z-x)(x-w)} + \frac{1}{(z-w)(w-x)} \right). \]

Therefore we get
\[ E \left( f(x) \frac{x-w}{x-w} \right)(z) = E(f)(z) \frac{z-w}{z-w}, \]

which shows (H1).

In Theorem 3.1, to get an extension of \( f \in L^2(\mu) \) to an entire function, we add mass points to the measure \( \mu \) until we reach an \( N \)-extremal measure \( \sigma \). We next extend \( f \) by zero to a function in \( L^2(\sigma) \), and use its canonical extension to an entire function. However, there is a different way to obtain \( N \)-extremal measures from a determinate measure \( \mu \) having finite index of determinacy. We prove that this approach can also be used to find entire extensions of functions in \( L^2(\mu) \), such that (3.4) holds.

For a determinate measure \( \mu \) with finite index of determinacy \( \text{ind}(\mu) \), we take a polynomial
\[ R(t) = \prod_{l=1}^{N} (t - z_l)^{k_l+1}, \quad \text{with} \quad \sum_{l=1}^{N} (k_l + 1) = \text{ind}(\mu) + 1, \]

where \( z_l \notin \text{supp}(\mu) \), \( l = 1, \ldots, N \).

It follows from Lemma 2.1 in [BD2] that \( \sigma = |R|^2 \mu \) is an indeterminate measure, but the measure \( |R(t)/(t - z_1)|^2 \mu \) is determinate. According to Lemma A in Section 3 of [BD1], we conclude that the measure \( \sigma = |R|^2 \mu \) is \( N \)-extremal.

Given a function \( f \in L^2(\mu) \), we define \( f^\sigma \in L^2(\sigma) \) by \( f^\sigma = f/R \). Since \( \sigma \) is \( N \)-extremal, \( f^\sigma \) has a canonical extension \( F(f^\sigma) \) and we define
\[ (3.6) \quad E(f)(z) := R(z)F(f^\sigma)(z). \]

**Theorem 3.4.** Let \( \mu \) be a determinate measure of finite index and let \( R \) be as above. Then \( L^2(\mu) \) is realized as entire functions of class \( \mathcal{C}_0 \) via (3.6), and it has the property
\[ (3.7) \quad \tilde{T}(f) = T(E(f)), \quad f \in L^2(\mu) \]

for any discrete differential operator \( T \in D(\mu) \).

**Proof.** The set of functions \( f \in L^2(\mu) \) for which (3.7) holds is a closed subspace, and therefore it suffices to prove (3.7) for \( f = \chi_{\{x\}} \), \( x \in \text{supp}(\mu) \).

In this case \( f^\sigma(t) = (1/R(x))\chi_{\{x\}}(t) \), and since \( F_\mu = F_\sigma \) we get
\[ F(f^\circ)(z) = \frac{F_\mu(z)}{R(x)F'_\mu(x)(z-x)}, \]
hence
\[ R(z)F(f^\circ)(z) = \frac{F_\mu(z)}{F'_\mu(x)(z-x)} + r(z)F_\mu(z), \]
where
\[ r(z) = \frac{1}{R(x)F'_\mu(x)} \frac{R(z) - R(x)}{z-x} \]
is a polynomial of degree \( \text{ind}(\mu) \). Now formula (3.7) follows from Corollary 2.4 and Proposition 3.2.

Like in Corollary 3.3 we have
\[ E(f)(z) = \sum_{x \in \text{supp}(\mu)} \frac{F_\mu(z)R(z)}{F'_\mu(x)R(x)(z-x)}f(x) \quad \text{for } f \in L^2(\mu). \]
The realization \( E(L^2(\mu)) \) is a Hilbert space in the sense of de Branges if \( R \) is a real polynomial.

For given \( f \in L^2(\mu) \) we shall now describe the set of all entire functions \( F \) satisfying
\[ \tilde{T}(f) = T(F) \quad \text{for all } T \in \mathcal{D}(\mu). \]

**Theorem 3.5.** Let \( \mu \) be a determinate measure of finite index and let \( f \in L^2(\mu) \).

(i) Given \( (z_1, k_1), \ldots, (z_N, k_N) \), where \( z_1, \ldots, z_N \) are different points of \( \mathbb{C}, k_1, \ldots, k_N \in \mathbb{N} \), and assume that \( 0 \leq N_2 \leq N \) exists such that \( z_l \in \text{supp}(\mu) \) and \( k_l > 0 \) for \( l = 1, \ldots, N_2 \) and \( z_l \notin \text{supp}(\mu) \) for \( l = N_2 + 1, \ldots, N \) and that
\[ \sum_{l=1}^{N_2} k_l + \sum_{l=N_2+1}^{N} (k_l + 1) = \text{ind}(\mu) + 1, \]
then there exists a unique entire function \( F \) satisfying (3.8) and the interpolation conditions
\[ F^{(i)}(z_l) = \alpha_{l,j} \begin{cases} j = 1, \ldots, k_l, & l = 1, \ldots, N_2 \\ j = 0, \ldots, k_l, & l = N_2 + 1, \ldots, N \end{cases} \]
where \( \alpha_{l,j} \) are arbitrarily given. This entire function \( F \) is of class \( \mathcal{C}_0 \).

(ii) If \( F \) is an entire function satisfying (3.8), then \( F + pF_\mu \), where \( p \) is any polynomial of degree not bigger than \( \text{ind}(\mu) \), are the only entire functions satisfying (3.8). All of them are of class \( \mathcal{C}_0 \).
PROOF. (i) We first prove the existence. Assume that $F$ is an entire function satisfying (3.8). From the hypothesis on the $z_i$'s and since $F_\mu$ has simple zeros, we deduce that $F_\mu'(z_i) \neq 0$ for $l = 1, \ldots, N_2$ and $F_\mu(z_i) \neq 0$ for $l = N_2 + 1, \ldots, N$. Hence, if $p$ denotes a polynomial, the equations

$$\delta_{z_i}^{(j)}(p(z)F_\mu(z)) = F^{(j)}(z_i) - \alpha_{l,j},$$

for $j = 1, \ldots, k_l$, $l = 1, \ldots, N_2$ and for $j = 0, \ldots, k_l$, $l = N_2 + 1, \ldots, N$. The hypothesis (3.9) guarantees that $p$ is uniquely determined as a polynomial of degree $\leq \text{ind}(\mu)$. This means that $F - pF_\mu$ satisfies the interpolation conditions (3.10), and $F - pF_\mu$ still satisfies (3.8) by Corollary 2.4.

To prove uniqueness, assume that $F$ and $G$ are entire functions satisfying (3.8) and (3.10). We shall prove that $F(x) = G(x)$ for all $x \in \mathbb{C} \setminus \left(\text{supp}(\mu) \cup \{z_{N_2+1}, \ldots, z_N\}\right)$. This clearly implies $F = G$. For $x$ as above we consider the linear system

$$\sum_{l=1}^{N_2} \sum_{j=1}^{k_l} a_{l,j} \delta_{z_i}^{(j)}(z^k F_\mu(z)) + \sum_{l=N_2+1}^{N} \sum_{j=0}^{k_l} a_{l,j} \delta_{z_i}^{(j)}(z^k F_\mu(z)) = x^k F_\mu(x),$$

where $0 \leq k \leq \text{ind}(\mu)$. The system is quadratic by (3.9), and it has a unique solution $(a_{l,j})$, cf. the proof of Corollary 2.4. This means that the operator

$$T := \sum_{l=1}^{N_2} \sum_{j=1}^{k_l} a_{l,j} \delta_{z_i}^{(j)} + \sum_{l=N_2+1}^{N} \sum_{j=0}^{k_l} a_{l,j} \delta_{z_i}^{(j)} - \delta_x$$

belongs to $\mathcal{B}(\mu)$, so $T(F) = T(G) = \tilde{T}(f)$ by (3.8), but since $F$ and $G$ both satisfy (3.10) we conclude that $F(x) = G(x)$.

Since (3.8) has a solution $F$ which is of class $\mathcal{C}_0$, the solution $F - pF_\mu$ from the existence part is again of class $\mathcal{C}_0$.

(ii) Let $F, G$ be entire functions satisfying (3.8). The method in (i) shows that it is possible to find a polynomial $p$ of degree $\leq \text{ind}(\mu)$ such that $G - pF_\mu$ satisfies the interpolation conditions

$$\delta_{z_i}^{(j)}(G - pF_\mu) = F^{(j)}(z_i)$$

with $l, j$ as in (3.10). By the uniqueness assertion $G - pF_\mu = F$. 

REFERENCES


