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REMARKS ON THE EXISTENCE OF NONTRIVIAL SOLUTIONS FOR A CLASS OF VOLTERRA EQUATIONS WITH SMOOTH KERNELS

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Abstract.

The paper is devoted to the study of the equation $u = k * g(u)$ with a smooth k and a monotonuous g. Some necessary and sufficient condition for the existence of nontrivial continuous solutions u of this equation is given.

1. Introduction.

In this paper we study the following equation

$$u(x) = \int_0^x k(x-s)g(u(s))\,ds \qquad (0 \leqq x), \tag{1.1}$$

where $k \geqq 0$ is locally integrable and g is continuous, nondecrasing with $g(0) = 0$. We are interested in the problem of the existence of nontrivial nonnegative continuous on $[0, +\infty)$ solutions to (1.1).

Our considerations concern a class of smooth $k \in C^\infty\,[0, \infty)$ such that

$$k(0) = k'(0) = \ldots = k^{(n)}(0) = \ldots = 0 \qquad \text{for } n = 1, 2, \ldots.$$

One of the typical examples of such k is the function $k(x) = \exp(-x^{-\beta})$, $\beta > 0$. In this case it has been shown (see [1], Example 3.1) that under some additional assumptions on g the equation (1.1) has a nontrivial solution u if and only if the following condition is satisfied

$$\int_0^\delta \frac{1}{s\left(-\ln\dfrac{s}{g(s)}\right)^{\frac{\Lambda+\beta}{\beta}}} < \infty.$$

We are going to give similar sufficient and necessary conditions for the existence of nontrivial solutions for some $k(x) = \exp(-x^{-\beta}h(x))$, $\beta > 0$. The presented

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results generalize those in [1] obtained by more restrictive assumptions on g. First of all we remove here the assumption that $\frac{x}{g(x)}$ is nondecreasing, which has been essential for the considerations in [1]. Recently similar problems, in the case of other classes of smooth kernels k, were considered in [3], [5].

2. Statement the results.

Throughout this paper we assume

(2.1) $g\colon [0, \infty) \to [0, \infty)$ *is a continuous nondecreasing function such that* $g(0) = 0$ *and* $g(x)/x \to \infty$, *as* $x \to 0+$;

(2.2) $h\colon (0, \infty) \to [0, \infty)$ *is a continuous monotonuous function such that* $h\left(\frac{x}{2}\right) \Big/ h(x) \to 1$, *as* $x \to 0+$;

(2.3) $-x^{-\beta}h(x) \quad (\beta > 0)$ *is concave on* $(0, 1)$.

Since the question of uniqueness or nonuniqueness of the trivial solution depends only on the behaviour of k and g in a neighbourhood of zero, the assumptions above could be reformulated to take this fact into account.

In this paper $\varepsilon, \delta > 0$ always denote sufficiently small constants. We permit them to change their values from paragraph to paragraph.

Typical examples of considered kernels in this paper are the functions $\beta x^{-\beta-1}\exp(-x^{-\beta})$, $\exp(-x^{-\beta}(-\ln x)^{\gamma})$, where $-\infty < \gamma < \infty$, $\beta > 0$ and $0 < x < \delta$.

Set $K(x) = \int_0^x k(s)\,ds$ and denote by K^{-1} its inverse. The main result of our paper is stated, as follows

THEOREM 2.1. *Let* (2.1)–(2.3) *be satisfied and* $k(x) = \exp(-x^{-\beta}h(x))$ *for* $x \geqq 0$. *Then the equation* (1.1) *has a unique continuous solution* u *such that* $u(x) > 0$ *for* $x > 0$ *if and only if*

$$\int_0^{\delta} \frac{1}{g(s)}(K^{-1})'\left(\frac{s}{g(s)}\right) ds < \infty. \tag{2.4}$$

The proof of this theorem is based on an application of the following facts concerning the existence of nontrivial solutions of (1.1) (see [2], [4]).

THEOREM 2.2. *Let* $\ln k$ *be concave on* $(0, \delta)$ *and* (2.1) *be satisfied. Then the condition* (2.4) *is necessary for the existence of a nontrivial continuous solution of* (1.1).

THEOREM 2.3. *Let $k \geqq 0$ be locally integrable and let* (2.1) *be satisfied. If there exists a continuous nonnegative function $F \not\equiv 0$ on $[0, \delta)$ such that*

$$F(x) \leqq \int_0^x k(x-s)g(F(s))\,ds \qquad (0 \leqq x \leqq \delta),$$

then the equation (1.1) *has a unique continuous solution u such that $u(x) > 0$ for $0 < x \leqq \delta$. Moreover, u is a nondecreasing function.*

3. Auxiliary lemmas and notations.

For any continuous $w\colon [0, \delta) \to [0, \infty)$ we define

$$T(w)(x) = \int_0^x k(x-s)g(w(s))\,ds, \qquad 0 \leqq x < \delta$$

and denote $\chi(x) = x^{-\beta}h(x)$ for $0 \leqq x$.

For the reader convenience we recall some properties of considered kernels k in the following lemma (see [1]).

LEMMA 3.1. *Let* (2.2), (2.3) *be satisfied. Then*

(i) *for any α, $a > 0$ $h(ax)/h(x) \to 1$ and $x^{-\alpha}h(x) \to \infty$, as $x \to 0+$;*

(ii) *$\chi(x)$ is nonincreasing and absolutely continuous on (δ_1, δ) for every sufficiently small $\delta_1, \delta > 0$.*

(iii) *for any $n, \varepsilon > 0$* $K\left(\dfrac{x}{n^{\frac{1}{\beta}} + \varepsilon}\right) \leqq K^n(x) \leqq K\left(\dfrac{x}{n^{\frac{1}{\beta}} - \varepsilon}\right)$, $0 < x < \delta_\varepsilon$.

In the sequel, we will need some other facts collected in the following three lemmas.

LEMMA 3.2. *Let* (2.2), (2.3) *be satisfied. Then there exist constants $c_1, c_2 > 0$ such that*

$$c_1 \leqq -\frac{x\chi'(x)}{\chi(x)} \leqq c_2 \qquad \textit{for } x \in (0, \delta).$$

PROOF. First we note that in view of Lemma 3.1, (i) there exist $\hat{c}_1, \hat{c}_2 > 0$ such that $\hat{c}_1 \leqq \dfrac{\chi\left(\frac{x}{2}\right) - \chi(x)}{\chi(x)} \leqq \hat{c}_2$ for $x \in (0, \delta)$. Then we apply the inequality $-\dfrac{x}{2}\chi'(x) \leqq \chi\left(\dfrac{x}{2}\right) - \chi(x) \leqq -\dfrac{x}{2}\chi'\left(\dfrac{x}{2}\right)$ a.e. obtained by mean value theorem to derive the required assertion.

LEMMA 3.3. *Let* (2.2), (2.3) *be satisfied. Then*

(i) *there exist constants* $c_1, c_2 > 0$ *such that*

$$c_1 \frac{x}{\chi(x)} \exp(-\chi(x)) \leqq K(x) \leqq c_2 \frac{x}{\chi(x)} \exp(-\chi(x)) \qquad \textit{for } x \in (0, \delta);$$

(ii) $\lim\limits_{z \to 0+} \dfrac{K^{-1}(z^2)}{K^{-1}(z)} = 2^{-\frac{1}{\beta}};$

(iii) *there exists a constant* $c > 0$ *such that* $z(K^{-1})'(z) \leqq cz^2(K^{-1})'(z^2)$ *for* $z \in (0, \delta)$;

Proof. To prove (i) we first notice that $\left(\dfrac{x}{\chi(x)} \exp(-\chi(x))\right)' = -\dfrac{x\chi'(x)}{\chi(x)} \left(1 + \dfrac{1}{\chi(x)}\right) \exp(-\chi(x)) + \dfrac{1}{\chi(x)} \exp(-\chi(x))$. Since in view of Lemma 3.1, (i) $\dfrac{1}{\chi(x)} \to 0$, as $x \to 0+$, our assertion follows from Lemma 3.2.

To consider (ii) and (iii) it is convenient to take $z = K(y)$ and $K^2(y) = K(\hat{y})$. It follows from Lemma 3.1, (iii) that $\dfrac{\hat{y}}{y} \to 2^{-\frac{1}{\beta}}$, as $y \to 0+$.

Now, to prove (ii), it suffices to note that $\dfrac{K^{-1}(z^2)}{K^{-1}(z)} = \dfrac{\hat{y}}{y}$.

To prove (iii), we first note that $z(K^{-1})'(z) = \dfrac{K(y)}{k(y)}$ and $z^2(K^{-1})'(z^2) = \dfrac{K(\hat{y})}{k(\hat{y})}$.

Therefore, in view of just proved estimates in (i), it suffices to consider $\dfrac{y\chi(\hat{y})}{\hat{y}\chi(y)}$. By using Lemma 3.1, (i) we obtain

$$\lim_{y \to 0+} \frac{y\chi(\hat{y})}{\hat{y}\chi(y)} = 2^{\frac{1+\beta}{\beta}},$$

which gives the required assertion.

Lemma 3.4. *Let* (2.1)–(2.3) *be satisfied. Then*

$$\int_0^x \frac{s}{g(s)} (K^{-1})' \left(\frac{s}{g(s)}\right) \frac{dg(s)}{g(s)} \leqq \int_0^x \frac{1}{g(s)} (K^{-1})' \left(\frac{s}{g(s)}\right) ds$$

for any $x \in (0, \delta)$.

Proof. Let $\Psi(s) = K^{-1}\left(\dfrac{s}{g(s)}\right)$ for $s \in (0, \delta)$. Then Ψ is locally of bounded variation on $(0, \delta)$ and $\Psi(0) = 0$. Moreover $d\Psi(s) = \dfrac{1}{g(s)} (K^{-1})' \left(\dfrac{s}{g(s)}\right) ds -$

$\dfrac{s}{g(s)}(K^{-1})'\left(\dfrac{s}{g(s)}\right)\dfrac{dg(s)}{g(s)}$ for $s > 0$. Since $\Psi(x) - \Psi(\delta_1) = \displaystyle\int_{\delta_1}^{x} d\Psi(s)$, we get our assertion, when $\delta_1 \to 0+$.

4. Proof of the main result.

Since $\ln k$ is concave the necessity part of Theorem 2.1 follows from Theorem 2.2 immediately.

To prove the sufficiency part we construct some function F, so that Theorem 2.3 could be applied.

This construction we begin with the following observation. Let $\psi(y) = \sqrt{yg(y)}$ for $y \geqq 0$ and $\phi = \psi^{-1}$. Then we have

LEMMA 4.1. *Let* (2.1)–(2.3) *be satisfied. Then*

(i) $$\phi(x) \leqq x \qquad \text{for } 0 < x < \delta;$$

(ii) *if* (2.4) *is satisfied, then*

$$\int_0^{\delta} \frac{1}{g(\phi(s))}(K^{-1})'\left(\frac{s}{g(\phi(s))}\right) ds < \infty. \tag{4.1}$$

PROOF. The inequality (i) is an immediate consequence of the relation $x^2 = g(\phi(x))\phi(x)$ and the inequality $x < g(x)$ valid for $0 < x < \delta$ obtained from (2.1).

To prove (ii) we substitute $s = \psi(y)$ in the integral in (4.1) and note that in view of Lemma 3.3, (iii) it holds

$$\frac{1}{g(y)}(K^{-1})'\left(\frac{\psi(y)}{g(y)}\right) \leqq c\frac{1}{\psi(y)}\frac{y}{g(y)}(K^{-1})'\left(\frac{y}{g(y)}\right) \tag{4.2}$$

for $y \in (0, \delta)$ and some $c > 0$. Since $2d\psi(y) = \dfrac{g(y)}{\psi(y)}dy + \dfrac{y}{\psi(y)}dg(y)$ for $y > 0$, we have

$$2\frac{1}{\psi(y)}\frac{y}{g(y)}(K^{-1})'\left(\frac{y}{g(y)}\right)d\psi(y) = \frac{1}{g(y)}(K^{-1})'\left(\frac{y}{g(y)}\right)dy + \frac{y}{g(y)}(K^{-1})'\left(\frac{y}{g(y)}\right)\frac{dg(y)}{g(y)} \tag{4.3}$$

for $y > 0$. Finally, from (4.2) and (4.3) by using Lemma 3.4 we get

$$\int_0^x \frac{1}{g(\phi(s))}(K^{-1})'\left(\frac{s}{g(\phi(s))}\right) ds < c\int_0^x \frac{1}{g(s)}(K^{-1})'\left(\frac{s}{g(s)}\right) ds,$$

where $c > 0$ is a constant and $0 < x < \delta$, which ends the proof.

Thus, we can define

$$F^{-1}(y) = \int_0^y \frac{1}{g(\phi(s))}(K^{-1})'\left(\frac{s}{g(\phi(s))}\right) ds, \qquad 0 < y < \delta.$$

We are going to find $c > 0$ such that the inverse function F_c to cF^{-1} satisfies Theorem 2.3. Since integrating by parts and then taking $s = F_c^{-1}(\tau)$ we get

$$T(F_c)(x) = \int_0^{F_c(x)} K(x - F_c^{-1}(\tau))\, dg(\tau),$$

it remains to show only the following:

LEMMA 4.2. *There exists a constant $c > 0$ such that*

$$y \leqq \int_0^y K(F_c^{-1}(y) - F_c^{-1}(\tau))\, dg(\tau) \qquad \textit{for } y \in (0, \delta).$$

PROOF. Since $\phi(y) \leqq y$, we have

$$\int_0^y K[c(F^{-1}(y) - F^{-1}(y))]\, dg(s) \geqq \int_0^{\phi(y)} K[c(F^{-1}(y) - F^{-1}(s))]\, dg(s) \geqq$$
$$K[c(F^{-1}(y) - F^{-1}(\phi(y)))]g(\phi(y)).$$

for $y \in (0, \delta)$. From the concavity of K^{-1} it follows that the function $z(K^{-1})'(z)$ is nondecreasing. Therefore, we obtain

$$F^{-1}(y) - F^{-1}(\phi(y)) = \int_{\phi(y)}^y \frac{1}{g(\phi(s))}(K^{-1})'\left(\frac{s}{g(\phi(s))}\right) ds \geqq$$
$$\int_{\varphi(y)}^y \frac{1}{g(\phi(y))}(K^{-1})'\left(\frac{s}{g(\phi(y))}\right) ds = K^{-1}\left(\frac{y}{g(\phi(y))}\right) - K^{-1}\left(\frac{\phi(y)}{g(\phi(y))}\right).$$

Noting that $\dfrac{y}{g(\phi(y))} = \sqrt{\dfrac{\phi(y)}{g(\phi(y))}}$ and using Lemma 3.3, (ii) we get

$$F^{-1}(y) - F^{-1}(\phi(y)) \geqq \hat{c}K^{-1}\left(\frac{y}{g(\phi(y))}\right),$$

where $\hat{c} > 0$ is some constant and $y \in (0, \delta)$. Now it suffices to take $c = \dfrac{1}{\hat{c}}$ to obtain our assertion.

REFERENCES

1. W. Mydlarczyk, *The existence of nontrivial solutions for a class of Volterra equations with smooth kernels*, Math. Scand. 71 (1992), 261–266.
2. W. Mydlarczyk, *Remarks on a nonlinear Volterra equation*, Ann. Polon. Math. 53 (1991), 227–232.
3. W. Okrasiński, *Nontrivial solutions for a class of nonlinear Volterra equations with convolution kernels*, J. Integral Equations Appl. 3 (1991), 399–409.
4. W. Okrasiński, *Nontrivial solutions to nonlinear Volterra integral equations*, SIAM J. Math. Anal. 4 (1991), 1007–1015.
5. R. Szwarc, *Nonlinear integral inequalities of the Volterra type*, Math. Proc. Camb. Phil. Soc. 111 (1992), 599–608.

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