REMARKS ON THE EXISTENCE OF NONTRIVIAL SOLUTIONS FOR A CLASS OF VOLterra EQUATIONS WITH SMOOTH KERNELS

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Abstract.

The paper is devoted to the study of the equation \( u = k \ast g(u) \) with a smooth \( k \) and a monotoneous \( g \). Some necessary and sufficient condition for the existence of nontrivial continuous solutions \( u \) of this equation is given.

1. Introduction.

In this paper we study the following equation

\[
(1.1) \quad u(x) = \int_0^x k(x - s)g(u(s)) \, ds \quad (0 \leq x),
\]

where \( k \geq 0 \) is locally integrable and \( g \) is continuous, nondecreasing with \( g(0) = 0 \). We are interested in the problem of the existence of nontrivial nonnegative continuous on \([0, +\infty)\) solutions to (1.1).

Our considerations concern a class of smooth \( k \in C^\infty [0, \infty) \) such that

\[ k(0) = k'(0) = \ldots = k^{(n)}(0) = \ldots = 0 \quad \text{for } n = 1, 2, \ldots. \]

One of the typical examples of such \( k \) is the function \( k(x) = \exp(-x^{-\beta}), \beta > 0 \). In this case it has been shown (see [1], Example 3.1) that under some additional assumptions on \( g \) the equation (1.1) has a nontrivial solution \( u \) if and only if the following condition is satisfied

\[
\int_0^\delta \frac{1}{s \left(-\ln \frac{s}{g(s)}\right)^{\frac{\beta + \Delta}{\beta}}} < \infty.
\]

We are going to give similar sufficient and necessary conditions for the existence of nontrivial solutions for some \( k(x) = \exp(-x^{-\beta}h(x)), \beta > 0 \). The presented

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results generalize those in [1] obtained by more restrictive assumptions on \( g \).

First of all we remove here the assumption that \( \frac{x}{g(x)} \) is nondecreasing, which has been essential for the considerations in [1]. Recently similar problems, in the case of other classes of smooth kernels \( k \), were considered in [3], [5].

2. Statement the results.

Throughout this paper we assume

\[
\begin{align*}
(2.1) \quad & g: [0, \infty) \to [0, \infty) \text{ is a continuous nondecreasing function such that } g(0) = 0 \text{ and } g(x)/x \to \infty, \text{ as } x \to 0 +; \\
(2.2) \quad & h: (0, \infty) \to [0, \infty) \text{ is a continuous monotone function such that } h\left(\frac{x}{2}\right)/h(x) \to 1, \text{ as } x \to 0 +; \\
(2.3) \quad & -x^{-\beta}h(x) \quad (\beta > 0) \text{ is concave on } (0, 1).
\end{align*}
\]

Since the question of uniqueness or nonuniqueness of the trivial solution depends only on the behaviour of \( k \) and \( g \) in a neighbourhood of zero, the assumptions above could be reformulated to take this fact into account.

In this paper \( \varepsilon, \delta > 0 \) always denote sufficiently small constants. We permit them to change their values from paragraph to paragraph.

Typical examples of considered kernels in this paper are the functions \( \beta x^{-\beta-1}\exp(-x^{-\beta}), \exp(-x^{-\beta}(-\ln x)^\gamma) \), where \( -\infty < \gamma < \infty, \beta > 0 \) and \( 0 < x < \delta \).

Set \( K(x) = \int_0^x k(s)\,ds \) and denote by \( K^{-1} \) its inverse. The main result of our paper is stated, as follows

**Theorem 2.1.** Let (2.1)–(2.3) be satisfied and \( k(x) = \exp(-x^{-\beta}h(x)) \) for \( x \geq 0 \). Then the equation (1.1) has a unique continuous solution \( u \) such that \( u(x) > 0 \) for \( x > 0 \) if and only if

\[
(2.4) \quad \int_0^\delta \frac{1}{g(s)}(K^{-1})'(\frac{s}{g(s)})\,ds < \infty.
\]

The proof of this theorem is based on an application of the following facts concerning the existence of nontrivial solutions of (1.1) (see [2], [4]).

**Theorem 2.2.** Let \( \ln k \) be concave on \((0, \delta)\) and (2.1) be satisfied. Then the condition (2.4) is necessary for the existence of a nontrivial continuous solution of (1.1).
Theorem 2.3. Let \( k \geq 0 \) be locally integrable and let (2.1) be satisfied. If there exists a continuous nonnegative function \( F \not\equiv 0 \) on \([0, \delta]\) such that

\[
F(x) \leq \int_0^x k(x - s)g(F(s)) \, ds \quad (0 \leq x \leq \delta),
\]

then the equation (1.1) has a unique continuous solution \( u \) such that \( u(x) > 0 \) for \( 0 < x \leq \delta \). Moreover, \( u \) is a nondecreasing function.

3. Auxiliary lemmas and notations.

For any continuous \( w : [0, \delta) \to [0, \infty) \) we define

\[
T(w)(x) = \int_0^x k(x - s)g(w(s)) \, ds, \quad 0 \leq x < \delta
\]

and denote \( \chi(x) = x^{-\beta}h(x) \) for \( 0 \leq x \).

For the reader convenience we recall some properties of considered kernels \( k \) in the following lemma (see [1]).

Lemma 3.1. Let (2.2), (2.3) be satisfied. Then

(i) for any \( \alpha, \beta > 0 \) \( h(ax)/h(x) \to 1 \) and \( x^{-\alpha}h(x) \to \infty \), as \( x \to 0^+ \);

(ii) \( \chi(x) \) is nonincreasing and absolutely continuous on \((\delta_1, \delta)\) for every sufficiently small \( \delta_1, \delta > 0 \).

(iii) for any \( n, \varepsilon > 0 \)

\[
K\left(\frac{x}{\frac{1}{n^\beta} + \varepsilon}\right) \leq K^\alpha(x) \leq K\left(\frac{x}{\frac{1}{n^\beta} - \varepsilon}\right), \quad 0 < x < \delta \varepsilon.
\]

In the sequel, we will need some other facts collected in the following three lemmas.

Lemma 3.2. Let (2.2), (2.3) be satisfied. Then there exist constants \( c_1, c_2 > 0 \) such that

\[
c_1 \leq -\frac{x\chi'(x)}{\chi(x)} \leq c_2 \quad \text{for} \quad x \in (0, \delta).
\]

Proof. First we note that in view of Lemma 3.1, (i) there exist \( \hat{c}_1, \hat{c}_2 > 0 \) such that

\[
\frac{\chi\left(\frac{x}{2}\right)}{\chi(x)} - \chi(x) \leq \hat{c}_2 \quad \text{for} \quad x \in (0, \delta).
\]

Then we apply the inequality

\[
-\frac{x}{2} \chi(x) \leq \chi\left(\frac{x}{2}\right) - \chi(x) \leq -\frac{x}{2} \chi\left(\frac{x}{2}\right)
\]

e. obtained by mean value theorem to derive the required assertion.

Lemma 3.3. Let (2.2), (2.3) be satisfied. Then
(i) there exist constants $c_1, c_2 > 0$ such that
\[ c_1 \frac{x}{\chi(x)} \exp(-\chi(x)) \leq K(x) \leq c_2 \frac{x}{\chi(x)} \exp(-\chi(x)) \quad \text{for } x \in (0, \delta); \]

(ii) \[ \lim_{z \to 0^+} \frac{K^{-1}(z^2)}{K^{-1}(z)} = 2^{-\frac{1}{\beta}}; \]

(iii) there exists a constant $c > 0$ such that $z(K^{-1})(z) \leq cz^2(K^{-1})(z^2)$ for $z \in (0, \delta);$

**Proof.** To prove (i) we first notice that \[ \left( \frac{x}{\chi(x)} \exp(-\chi(x)) \right)' = -\frac{x\chi'(x)}{\chi(x)} \]
\[ \left( 1 + \frac{1}{\chi(x)} \right) \exp(-\chi(x)) + \frac{1}{\chi(x)} \exp(-\chi(x)). \] Since in view of Lemma 3.1, (i) \( \frac{1}{\chi(x)} \to 0, \) as $x \to 0^+,$ our assertion follows from Lemma 3.2.

To consider (ii) and (iii) it is convenient to take $z = K(y)$ and $K^2(y) = K(\hat{y}).$ It follows from Lemma 3.1, (iii) that $\frac{\hat{y}}{y} \to 2^{-\frac{1}{\beta}},$ as $y \to 0^+.$

Now, to prove (ii), it suffices to note that \[ \frac{K^{-1}(z^2)}{K^{-1}(z)} = \frac{\hat{y}}{y}. \]

To prove (iii), we first note that $z(K^{-1})(z) = \frac{K(y)}{k(y)}$ and $z^2(K^{-1})(z^2) = \frac{K(\hat{y})}{k(\hat{y})}.$ Therefore, in view of just proved estimates in (i), it suffices to consider $\frac{y\chi(\hat{y})}{\hat{y}\chi(y)}.$ By using Lemma 3.1, (i) we obtain
\[ \lim_{y \to 0^+} \frac{y\chi(\hat{y})}{\hat{y}\chi(y)} = 2^{1+\frac{1}{\beta}}, \]
which gives the required assertion.

**Lemma 3.4.** Let (2.1)–(2.3) be satisfied. Then
\[ \int_0^x \frac{1}{g(s)} \left( \frac{s}{g(s)} \right)' \frac{dg(s)}{g(s)} \leq \int_0^x \frac{1}{g(s)} (K^{-1})(s) \left( \frac{s}{g(s)} \right) ds \]
for any $x \in (0, \delta).$

**Proof.** Let $\Psi(s) = K^{-1} \left( \frac{s}{g(s)} \right)$ for $s \in (0, \delta).$ Then $\Psi$ is locally of bounded variation on $(0, \delta)$ and $\Psi(0) = 0.$ Moreover $d\Psi(s) = \frac{1}{g(s)} (K^{-1})(s) \left( \frac{s}{g(s)} \right) ds.$
\[
\frac{s}{g(s)} (K^{-1})' \left( \frac{s}{g(s)} \right) \frac{dg(s)}{g(s)} \quad \text{for } s > 0. \text{ Since } \Psi(x) - \Psi(\delta_1) = \int_{\delta_1}^{x} d\Psi(s), \text{ we get our assertion, when } \delta_1 \to 0^+. \\

4. Proof of the main result.

Since \( \ln k \) is concave the necessity part of Theorem 2.1 follows from Theorem 2.2 immediately.

To prove the sufficiency part we construct some function \( F \), so that Theorem 2.3 could be applied.

This construction we begin with the following observation. Let \( \psi(y) = \sqrt{yg(y)} \) for \( y \geq 0 \) and \( \phi = \psi^{-1} \). Then we have

**Lemma 4.1.** Let (2.1)–(2.3) be satisfied. Then

(i) \( \phi(x) \leq x \) for \( 0 < x < \delta \);

(ii) if (2.4) is satisfied, then

\[
(4.1) \quad \int_{0}^{\delta} \frac{1}{g(\phi(s))} (K^{-1})' \left( \frac{s}{g(\phi(s))} \right) ds < \infty.
\]

**Proof.** The inequality (i) is an immediate consequence of the relation \( x^2 = g(\phi(x))\phi(x) \) and the inequality \( x < g(x) \) valid for \( 0 < x < \delta \) obtained from (2.1).

To prove (ii) we substitute \( s = \psi(y) \) in the integral in (4.1) and note that in view of Lemma 3.3, (iii) it holds

\[
(4.2) \quad \frac{1}{g(y)} (K^{-1})' \left( \frac{\psi(y)}{g(y)} \right) \leq c \cdot \frac{1}{\psi(y)} \frac{y}{g(y)} (K^{-1})' \left( \frac{y}{g(y)} \right)
\]

for \( y \in (0, \delta) \) and some \( c > 0 \). Since \( 2d\psi(y) = \frac{g(y)}{\psi(y)} dy + \frac{y}{\psi(y)} dg(y) \) for \( y > 0 \), we have

\[
(4.3) \quad 2 \cdot \frac{1}{\psi(y)} \frac{y}{g(y)} (K^{-1})' \left( \frac{y}{g(y)} \right) d\psi(y) = \frac{1}{g(y)} (K^{-1})' \left( \frac{y}{g(y)} \right) dy + \frac{y}{g(y)} (K^{-1})' \left( \frac{y}{g(y)} \right) \frac{dg(y)}{g(y)}
\]

for \( y > 0 \). Finally, from (4.2) and (4.3) by using Lemma 3.4 we get

\[
\int_{0}^{\delta} \frac{1}{g(\phi(s))} (K^{-1})' \left( \frac{s}{g(\phi(s))} \right) ds < c \int_{0}^{x} \frac{1}{g(s)} (K^{-1})' \left( \frac{s}{g(s)} \right) ds,
\]
where $c > 0$ is a constant and $0 < x < \delta$, which ends the proof.

Thus, we can define

$$F^{-1}(y) = \int_{0}^{y} \frac{1}{g(\phi(s))} (K^{-1})' \left( \frac{s}{g(\phi(s))} \right) ds, \quad 0 < y < \delta.$$  

We are going to find $c > 0$ such that the inverse function $F_c$ to $cF^{-1}$ satisfies Theorem 2.3. Since integrating by parts and then taking $s = F_c^{-1}(\tau)$ we get

$$\mathcal{I}(F_c)(x) = \int_{0}^{F_c(x)} K(x - F_c^{-1}(\tau)) d\tau,$$

it remains to show only the following:

**Lemma 4.2.** There exists a constant $c > 0$ such that

$$y \leq \int_{0}^{y} K(F_c^{-1}(y) - F_c^{-1}(\tau)) d\tau \quad \text{for } y \in (0, \delta).$$

**Proof.** Since $\phi(y) \leq y$, we have

$$\int_{0}^{y} K[c(F^{-1}(y) - F^{-1}(y))] d\tau \geq \int_{0}^{\phi(y)} K[c(F^{-1}(y) - F^{-1}(\phi(y)))] d\tau \geq K[c(F^{-1}(y) - F^{-1}(\phi(y)))].$$

for $y \in (0, \delta)$. From the concavity of $K^{-1}$ it follows that the function $z(K^{-1})'(z)$ is nondecreasing. Therefore, we obtain

$$F^{-1}(y) - F^{-1}(\phi(y)) = \int_{\phi(y)}^{y} \frac{1}{g(\phi(s))} (K^{-1})' \left( \frac{s}{g(\phi(s))} \right) ds \geq$$

$$\int_{\phi(y)}^{y} \frac{1}{g(\phi(y))} (K^{-1})' \left( \frac{s}{g(\phi(y))} \right) ds = K^{-1} \left( \frac{y}{g(\phi(y))} \right) - K^{-1} \left( \frac{\phi(y)}{g(\phi(y))} \right).$$

Noting that $\frac{y}{g(\phi(y))} = \sqrt{\frac{\phi(y)}{g(\phi(y))}}$ and using Lemma 3.3, (ii) we get

$$F^{-1}(y) - F^{-1}(\phi(y)) \geq \hat{c} K^{-1} \left( \frac{y}{g(\phi(y))} \right),$$

where $\hat{c} > 0$ is some constant and $y \in (0, \delta)$. Now it suffices to take $c = \frac{1}{\hat{c}}$ to obtain our assertion.
REFERENCES


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