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COHOMOLOGICAL PROPERTIES OF MODULES WITH SECONDARY REPRESETATIONS

LEIF MELKERSSON

0. Introduction.

The notion of secondary representation is in some sense dual to that of primary decomposition. A module M over the ring A (always assumed to be a commutative noetherian ring in this paper) is called secondary if for any element x in A, multiplication by x on M is either surjective or nilpotent. The radical of the annihilator of M is then a prime ideal $\mathfrak{p}$ and we say that M is $\mathfrak{p}$-secondary. A secondary representation of a module is the representation of it as a finite sum of secondary submodules. If the module has a secondary representation, we simply call it representable. Basic facts about the theory of secondary representations are found in [2] and [3, appendix to §6]. It is wellknown that any artinian module is representable, and so is any injective module as shown by Sharp, [7], (the ring is noetherian). More generally if E is injective, $\operatorname{Hom}_A(N, E)$ is representable for any finite module, as shown by Melkersson and Schenzel, [5], Toroghy and Sharp, [9] and Zöschinger, [11]. Zöschinger also shows, [11, Folgerung 1.9], that if M is a representable module, which has finite Goldie dimension, then $\operatorname{Hom}_A(N, M)$ is representable for any finite module N, provided that $\operatorname{Ass} M$ is a discrete set. A module M is said to have finite Goldie dimension, if M does not contain an infinite direct sum of nonzero submodules, or equivalently the injective hull $E(M)$ of M decomposes as a finite direct sum of indecomposable (injective) submodules. (The indecomposable injective modules are the modules of the form $E(A/\mathfrak{p})$, where $\mathfrak{p}$ runs through the set of prime ideals of A.) In particular $\operatorname{Ass} M$ must then be a finite set. The condition on a module to have finite Goldie dimension is a natural finiteness condition put on a module. Artinian and finite modules have finite Goldie dimension, and so has any A-module, which has the structure of an artinian module over a localization of A, and a module of that kind is in addition representable, by [4, Proposition 4.1]. In

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section 4 we study the problem, when $\mathrm{Hom}_A(N, M)$ is representable, where N is finite and M is representable of finite Goldie dimension. It is shown that in general this module is not representable. Its representability is connected with the purity of the submodules $\Gamma_{\mathfrak{a}}(M)$ of M. For the concept of purity see e.g. [10]. But in many respects representable modules of finite Goldie dimension behave well. The representability of a module of finite Goldie dimension is in section 1 characterized by vanishing of local cohomology. The local cohomology modules $H^i_{\mathfrak{a}}(M)$, $i = 0, 1, 2, \ldots$ of an A-module M with respect to an ideal $\mathfrak{a}$ of A were introduced by Grothendieck, [1]. They arise as the derived functors of the left exact functor $\Gamma_{\mathfrak{a}}(-)$, where for an A-module M, $\Gamma_{\mathfrak{a}}(M)$ is the submodule of M consisting of all elements annihilated by some power of $\mathfrak{a}$, i.e. $\bigcup_{n=1}^{\infty} \underset{M}{:} \mathfrak{a}^n$. There is a natural isomorphism $H^i_{\mathfrak{a}}(M) \cong \varinjlim \mathrm{Ext}^i_A(A/\mathfrak{a}^n, M)$. The exact sequences connected with local cohomology are exploited to show some results on the representability of modules of finite Goldie dimension, e.g. the local criterion of representability. Connected with $H^i_{\mathfrak{a}}(M)$ are the modules $D^i_{\mathfrak{a}}(M) \cong \varinjlim \mathrm{Ext}^i_A(\mathfrak{a}^n, M)$, $i = 0, 1, 2, \ldots$. There is an exact sequence $0 \to \Gamma_{\mathfrak{a}}(M) \to M \to D^0_{\mathfrak{a}}(M) \to H^1_{\mathfrak{a}}(M) \to 0$, and if $\mathfrak{a}$ is the principal ideal generated by the elements s, $D^0_{\mathfrak{a}}(M)$ is canonically isomorphic to M_s. The structure of the representable modules of finite Goldie dimension is determined in section 3. It is shown that they are close to artinian modules over localizations of the ring. Their minimal injective resolutions are simple; the Bass numbers of the module satisfy strong finiteness properties. The Bass numbers of an A-module M are defined by $\mu_i(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})} \mathrm{Ext}^i_{A_{\mathfrak{p}}}(k(\mathfrak{p}), M_{\mathfrak{p}})$, where $i = 0, 1, 2, \ldots$, and $\mathfrak{p}$ is a prime ideal of A. The cardinal $\mu_i(\mathfrak{p}, M)$ equals the number of times $E(A/\mathfrak{p})$ occurs in the ith module in a minimal injective resolution of M. In the last section we deduce some properties of modules of the form $\mathrm{Hom}_A(F, M)$, where F is flat and M representable, which is also related to the co-localization of an artinian module at a multiplicative set in A, studied by this author and Schenzel in [6].

1. Vanishing of local cohomology.

For any A-module M and an element $s \in A$ the natural homomorphism $M \to M_s$ has kernel $\Gamma_{sA}(M)$ and its cokernel is isomorphic to $H^1_{sA}(M)$. The surjective of this map can therefore be expressed as the vanishing of $H^1_{sA}(M)$. The surjectivity can also be elementwise characterized in the following way: for each $x \in M$, there is $y \in M$ and a number n, such that $s^n(x - sy) = 0$, i.e. M belongs to the class $\mathscr{A}''$ in the terminology of [11].

PROPOSITION 1.1. *Let M be an A-module such that $\mathrm{Ass}_A M$ is finite. If $H^1_{\mathfrak{a}}(M) = 0$ for all principal ideals $\mathfrak{a}$ in A, then $H^i_{\mathfrak{a}}(M) = 0$ for all ideals $\mathfrak{a}$ in A and all $i \geqq 1$.*

PROOF. Let $N = \Gamma_{\mathfrak{a}}(M) \subset M$ and $\overline{M} = M/N$. Then $\mathrm{Ass}\,\overline{M} = \mathrm{Ass}\,M \setminus V(\mathfrak{a})$, where $V(\mathfrak{a})$ is the set of prime ideals $\mathfrak{p}$ such that $\mathfrak{p} \supset \mathfrak{a}$. $\bar{M}$ satisfies the same hypothesis as M. Moreover $H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}}(\bar{M})$ for all $i \geqq 1$. We may therefore assume that $\mathfrak{a} \not\subset \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass} M$. By prime avoidance (Ass M is finite!), there is $s \in \mathfrak{a}$ which is a non-zerodivisor on M. By the hypothesis $H^1_{sA}(M) = 0$, which means that the natural map $M \to M_s$ is an isomorphism. Hence $H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}}(M_s) \cong H^i_{\mathfrak{a}A_s}(M_s) = 0$ for all i.

Proposition 1.2. *If* $\operatorname{Ass} M \subset \operatorname{Max} A$ *then* $H^i_{\mathfrak{a}}(M) = 0$ *for any ideal* $\mathfrak{a}$ *and every* $i \geqq 1$.

Proof. By localization we may assume that A is local with maximal ideal $\mathfrak{m}$. If $\mathfrak{a} \subset \mathfrak{m}$ then $H^i_{\mathfrak{a}}(M) = 0$ for all $i \geqq 1$, since $\operatorname{Supp} M \subset V(\mathfrak{a})$. If $\mathfrak{a}$ is the unit ideal, of course $H^i_{\mathfrak{a}}(M) = 0$ for all i.

Corollary 1.3. *Let* M *be an* A*-module such that* $\operatorname{Ass} M \backslash \operatorname{Max} A$ *is finite. If* $H^1_{\mathfrak{a}}(M) = 0$ *for all principal ideals* $\mathfrak{a}$ *in* A, *then* $H^i_{\mathfrak{a}}(M) = 0$ *for all ideals* $\mathfrak{a}$ *and all* $i \geqq 1$.

Proof. Let $N = \sum_{m \subset \operatorname{Max} A} \Gamma_m(M) \subset M$ and $\bar{M} = M/N$. Then $\operatorname{Ass} \bar{M} = \operatorname{Ass} M \backslash \operatorname{Max} A$ and $\operatorname{Ass} N = \operatorname{Ass} M \cap \operatorname{Max} A$. Of course $\bar{M} \to \bar{M}_s$ is surjective for any $s \in A$. Proposition 1.1 applied to $\bar{M}$ and Proposition 1.2 applied to N together with the exactness of the sequence $H^i_{\mathfrak{a}}(N) \to H^i_{\mathfrak{a}}(M) \to H^i_{\mathfrak{a}}(\bar{M})$ give the conclusion.

The result in Corollary 1.3 is a reformulation of [11, Lemma 2.2 (b)] in terms of local cohomology. Whenever M is a representable module, the map $M \to M_s$ is surjective, or equivalently $H^1_{sA}(M) = 0$. This is immediately reduced to the case that M is a $\mathfrak{p}$-secondary module, and then it is trivial, since multiplication with s on M is surjective, if $s \notin \mathfrak{p}$ and nilpotent, if $s \in \mathfrak{p}$. Note also that if s is an non-zerodivisor on M, then multiplication with s on the representable module M is bijective. However if M is an arbitrary representable module, there might exist a non principal ideal $\mathfrak{a}$, such that $H^1_{\mathfrak{a}}(M) \neq 0$.

Example. Let A be a ring with an ideal $\mathfrak{a}$, such that for some finite A-module N one has $H^2_{\mathfrak{a}}(N) \neq 0$. Since a quotient of an injective module is representable, if E is the injective hull of N, the module $M = E/N$ is representable, but $H^1_{\mathfrak{a}}(M) \cong H^2_{\mathfrak{a}}(N) \neq 0$.

Remark. The condition $H^1_{\mathfrak{a}}(M) = 0$ for all ideals $\mathfrak{a}$ in A is equivalent to the flasqueness of the quasi-coherent sheaf on Spec A defined by M. However not every A-module M satisfying this condition is representable. For an example take $M = \coprod_{n=1}^{\infty} A/\mathfrak{m}^n$, where $(A, \mathfrak{m})$ is a local ring of dimension $\geqq 1$. By Proposition 1.2, $H^1_{\mathfrak{a}}(M) = 0$ for all ideals $\mathfrak{a}$, but there is no number n, such that $\mathfrak{m}^n M = \mathfrak{m}^{n+1} M$, which there is in case M is representable.

The modules in these two examples did not have finite Goldie dimension. For modules of finite Goldie dimension we have the following criterion of representability, which is [11, Satz 2.4] expressed in terms of local cohomology.

THEOREM 1.4. *Let M be an A-module of finite Goldie dimension. Then the following conditions are equivalent:*

(i) *M is representable*

(ii) *The map $M \to M_s$ is surjective for all $s \in A$*

(ii)′ *$H^1_{\mathfrak{a}}(M) = 0$ for all principal ideals $\mathfrak{a}$ in A*

(iii) *$H^i_{\mathfrak{a}}(M) = 0$ for all ideals $\mathfrak{a}$ and A and all $i \geqq 1$*

(iv) *The sheaf $\tilde{M}$ on $\operatorname{Spec} A$ defined by M is flasque.*

PROOF. The implication (ii)′ $\Rightarrow$ (iii) follows from Proposition 1.1, since $\operatorname{Ass} M$ is finite. The only nontrivial implication is (ii) $\Rightarrow$ (i), so let us assume that (ii) holds. We prove that (i) holds by induction on the number of the prime ideals associated to M. If we put $S = A \setminus \bigcup_{\mathfrak{a} \in \operatorname{Ass} M} \mathfrak{p}$, each element in this multiplicative set is a non-zerodivisor on M, hence acts bijectively on M. Thus M has a natural structure as a module over $S^{-1}A$ and if it is representable as such a module, it is representable also as a module over A, [4, Proposition 4.1]. We may therefore assume that A is a semilocal ring and that $\operatorname{Max} A \subset \operatorname{Ass} M$. Let $L = \sum_{m \in \operatorname{Max} A} L_{\mathfrak{m}}(M)$ and $N = M/L$. Since L as a submodule of M also has finite Goldie dimension, it must be an artinian module. Moreover $\operatorname{Ass} N = \operatorname{Ass} M \setminus \operatorname{Max} A$ and N has finite Goldie dimension, since $N_{\mathfrak{p}} \cong M_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec} A \setminus \operatorname{Max} A$. Clearly N also satisfies (ii). By the induction hypothesis N is representable. Since L is artinian and M/L is representable, it follows from [11, Lemma 2.3], that M is representable.

COROLLARY 1.5. *Let M be a an A-module of finite Goldie dimension. Then M is representable if and only if $M_{\mathfrak{m}}$ is a representable $A_{\mathfrak{m}}$-module for each maximal ideal $\mathfrak{m}$ in A.*

PROOF. Let $\mathfrak{a}$ be a principal ideal in A. Then for any maximal ideal $\mathfrak{m}$ in A, $H^1_{\mathfrak{a}}(M)_{\mathfrak{m}} \cong H^1_{\mathfrak{a}A_{\mathfrak{m}}}(M_{\mathfrak{m}}) = 0$, when $M_{\mathfrak{m}}$ is a representable $A_{\mathfrak{m}}$-module.

COROLLARY 1.6. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. If M' and M'' are representable and M has finite Goldie dimension, then M is representable.*

PROOF. This follows from Theorem 1.4 and the exactness of the sequence $H^1_{\mathfrak{a}}(M') \to H^1_{\mathfrak{a}}(M'')$ for each principal ideal $\mathfrak{a}$. Both modules at the ends vanish here, so $H^1_{\mathfrak{a}}(M) = 0$.

COROLLARY 1.7. *Let M be a module of finite Goldie dimension. Then the sum of any family of representable submodules of M is again representable.*

PROOF. Use condition (ii).

COROLLARY 1.8. *Any module of finite Goldie dimension has a largest representable submodule.*

PROOF. It is the sum of all the representable submodules.

2. Representability of $\Gamma_{\mathfrak{a}}(M)$.

THEOREM 2.1. *Let M be a representable module of finite Goldie dimension. Then for any ideal $\mathfrak{a}$, the submodule $\Gamma_{\mathfrak{a}}(M)$ is representable.*

PROOF. Let E be the injective hull of M. Now by Theorem 1.4, $H^1_{\mathfrak{a}}(M) = 0$, so we get the exact sequence

$$0 \to \Gamma_{\mathfrak{a}}(M) \to \Gamma_{\mathfrak{a}}(E) \to \Gamma_{\mathfrak{a}}(E/M) \to 0$$

For any ideal $\mathfrak{b}$ there is induced exact sequence

$$0 \to \Gamma_{\mathfrak{b}}(\Gamma_{\mathfrak{a}}(M)) \to \Gamma_{\mathfrak{b}}(\Gamma_{\mathfrak{a}}(E)) \to \Gamma_{\mathfrak{b}}(\Gamma_{\mathfrak{a}}(E/M)) \to H^1_{\mathfrak{b}}(\Gamma_{\mathfrak{a}}(M)) \to H^1_{\mathfrak{b}}(\Gamma_{\mathfrak{a}}(E))$$

However the last module in this exact sequence is zero, since the module $\Gamma_{\mathfrak{a}}(E)$ is injective. Since $H^1_{\mathfrak{a}+\mathfrak{b}}(M) = 0$, by Theorem 1.4, we have also the exact sequence

$$0 \to \Gamma_{\mathfrak{a}+\mathfrak{b}}(M) \to \Gamma_{\mathfrak{a}+\mathfrak{b}}(E) \to \Gamma_{\mathfrak{a}+\mathfrak{b}}(E/M) \to 0.$$

Now $\Gamma_{\mathfrak{b}}(\Gamma_{\mathfrak{a}}(N) = \Gamma_{\mathfrak{a}+\mathfrak{b}}(N)$ for any A-module N, so comparison of the above exact sequences yields $H^1_{\mathfrak{a}}(\Gamma_{\mathfrak{a}}(M)) = 0$. By Theorem 1.4 it follows that $\Gamma_{\mathfrak{a}}(M)$ is a representable module.

COROLLARY 2.2. *Let M be an A-module of finite Goldie dimension. If $\mathfrak{a}$ is an ideal of A, such that $\Gamma_{\mathfrak{a}}(M)$ is a representable submodule of M, then for any ideal $\mathfrak{c} \supset \mathfrak{a}$, $\Gamma_{\mathfrak{c}}(M)$ is also a representable submodule of M.*

PROOF. $\Gamma_{\mathfrak{c}}(\Gamma_{\mathfrak{a}}(M)) = \Gamma_{\mathfrak{c}}(M)$, whenever the ideal $\mathfrak{c}$ contains $\mathfrak{a}$.

PROPOSITION 2.3. *Let M be an A-module of finite Goldie dimension and let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of A. If the submodules $\Gamma_{\mathfrak{a}}(M)$ and $\Gamma_{\mathfrak{b}}(M)$ both are representable, then the submodule $\Gamma_{\mathfrak{a}\mathfrak{b}}(M)$ is representable, too.*

PROOF. The exactness of

$$0 \to \Gamma_{\mathfrak{b}}(M) \to M \to M/\Gamma_{\mathfrak{b}}(M) \to 0$$

yields the exactness of

$$0 \to \Gamma_{\mathfrak{a}}(\Gamma_{\mathfrak{b}}(M)) \to \Gamma_{\mathfrak{a}}(M) \to \Gamma_{\mathfrak{a}}(M/\Gamma_{\mathfrak{b}}(M)) \to 0,$$

since $H^1_{\mathfrak{a}}(\Gamma_{\mathfrak{b}}(M)) = 0$, by Theorem 1.4. Therefore $\Gamma_{\mathfrak{a}}(M/\Gamma_{\mathfrak{b}}(M))$, as a homomorphic image of the representable module $\Gamma_{\mathfrak{a}}(M)$, is representable and it is the same module as $\Gamma_{\mathfrak{ab}}(M)/\Gamma_{\mathfrak{b}}(M)$. Now apply Corollary 1.6 to the exact sequence

$$0 \to \Gamma_{\mathfrak{b}}(M) \to \Gamma_{\mathfrak{ab}}(M) \to \Gamma_{\mathfrak{ab}}(M)/\Gamma_{\mathfrak{b}}(M) \to 0$$

COROLLARY 2.4. *An A-module M of finite Goldie dimension, such that the submodule $\Gamma_{\mathfrak{p}}(M)$ is representable for any minimal element $\mathfrak{p}$ of* Ass M, *representable.*

PROOF. If $\mathfrak{a}$ is the product of the minimal elements of Ass M, then $\Gamma_{\mathfrak{a}}(M) = M$.

EXAMPLE. When M is representable, but doesn't have finite Goldie dimension, there might exist an ideal $\mathfrak{a}$, such that $\Gamma_{\mathfrak{a}}(M)$ is not representable. Let A be a domain with quotient field K and suppose there is a regular sequence x, y in A. From the exactness of $0 \to A \xrightarrow{x} A \to A/xA \to 0$, we get an exact sequence

$$\Gamma_{yA}(A/xA) \to H^1_{yA}(A) \xrightarrow{x} H^1_{yA}(A) \to H^1_{yA}(A/xA) \to H^2_{yA}(A)$$

Now $\Gamma_{yA}(A/xA) = 0$, since y is regular on A/xA. This means that multiplication with x on $H^1_{yA}(A)$ is injective. However it is not surjective, since $H^1_{yA}(A/xA) \neq 0$ and $H^2_{yA}(A) = 0$. Therefore the module $H^1_{yA}(A)$ is not representable. The A-modulus K and K/A are representable, so the exactness of $0 \to A \to K \to K/A \to 0$ yields exactness of $\Gamma_{yA}(K/A) \to H^1_{yA}(A) \to 0$. Therefore $\Gamma_{yA}(K/A)$ is not representable, because as we just have shown that its homomorphic image $H^1_{yA}(A)$ is not representable.

3. Bass numbers of representable modules with finite Goldie dimension.

We give a structure theorem for representable modules of finite Goldie dimension.

THEOREM 3.1. *Let M be a representable module of finite Goldie dimension and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be its distinct associated prime ideals, ordered in such a way that $\mathfrak{p}_i$ is maximal in $\{\mathfrak{p}_i, \ldots, \mathfrak{p}_n\}$, for each i, $i = 1, \ldots, n$. Then there is a chain $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ of submodules of M, such that for $i = 1, \ldots, n$, the quotient module M_i/M_{i-1} has the structure of an artinian module over the localization $A_{\mathfrak{p}_i}$.*

PROOF. We prove it by induction on n. The prime ideal $\mathfrak{p}_1$ is maximal in Ass M. If we put $M_1 = \Gamma_{\mathfrak{p}_1}(M)$, the module M_1 is representable by Theorem 2.1, and has finite Goldie dimension with $\mathfrak{p}_1$ as its only associated prime ideal. Therefore it has the structure of an artinian module over $A_{\mathfrak{p}_1}$. Consider the quotient module $N = M/M_1$. Since Ass N = Ass $M \setminus \{\mathfrak{p}_1\}$ and $N_{\mathfrak{p}_i} \cong M_{\mathfrak{p}_i}$ for $i = 2, \ldots, n$, it has finite Goldie dimension.

Now just apply the induction hypothesis to N and use the correspondence between the submodules of N and those submodules of M, which include M_1.

THEOREM 3.2. *Let M be a representable module of finite Goldie dimension. Then all its Bass numbers $\mu_i(\mathfrak{p}, M)$ are finite and moreover for each i there are only finitely many prime ideals $\mathfrak{p}$ such that $\mu_i(\mathfrak{p}, M) \neq 0$. This means that in a minimal injective resolution $0 \to M \to E^0 \to E^1 \to \dots$ of M, for each i, E^i is a direct sum of finitely many indecomposable injective modules.*

PROOF. First note that the assertions hold when M is artinian, and more generally when M is artinian over a localization of M. Next if $0 \to M' \to M \to M'' \to 0$ is an exact sequence and the assertions hold for M' and M'', then they hold for M, since for each $\mathfrak{p}$ and each i, there is an exact sequence $\mathrm{Ext}^i_{A_\mathfrak{p}}(k(\mathfrak{p}, M'_\mathfrak{p}) \to \mathrm{Ext}^i_{A_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) \to \mathrm{Ext}^i_{A_\mathfrak{p}}(k(\mathfrak{p}), M''_\mathfrak{p})$.

COROLLARY 3.3. *Let $0 \to M' \to M'' \to 0$ be exact, where M' and M are representable. If M has finite Goldie dimension, then so has the (representable) module M''.*

PROOF. For any prime ideal $\mathfrak{p}$ there is an induced exact sequence $\mathrm{Hom}_{A_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) \to \mathrm{Hom}_{A_\mathfrak{p}}(k(\mathfrak{p}), M''_\mathfrak{p}) \to \mathrm{Ext}^1_{A_\mathfrak{p}}(k(\mathfrak{p}), M'_\mathfrak{p})$. Now M' and M are representable with finite Goldie dimension, so the endterms are by Theorem 3.2 finite dimensional $k(\mathfrak{p})$-vector spaces, and there are only finitely many $\mathfrak{p}$ for which they are non-zero. Hence $\mathrm{Hom}_{A_\mathfrak{p}}(k(\mathfrak{p}), M''_\mathfrak{p})$ is for each $\mathfrak{p}$ a finite dimensional $k(\mathfrak{p})$-vector space, and there are only a finite number of prime ideals $\mathfrak{p}$ for which it is non-zero, which is the condition for a module to have finite Goldie dimension expressed in terms of (the zeroth Bass numbers). It now follows that M'' has finite Goldie dimension.

As another application of this technique we have:

THEOREM 3.4. *The set of representable submodules with finite Goldie dimension of an A-module M satisfies the descending chain condition.*

PROOF. Let $M_1 \supset M_2 \supset \dots$ be a descending sequence of submodules of M such that M_n is representable of finite Goldie dimension for each n. We shall show that it is stationary. By first localizing at the multiplicative set consisting of the non-zerodivisors on M_1, we can assume that A is a semilocal ring. Next we localize at the maximal ideals and we are reduced to the case that A is a local ring with maximal ideal $\mathfrak{m}$, and we use induction on $\dim A$. For each n, $\Gamma_\mathfrak{m}(M_{n+1}) = \Gamma_\mathfrak{m}(M_n) \cap M_{n+1}$, so the sequence $M_n/\Gamma_\mathfrak{m}(M_n)$, $n = 1, 2, \dots$ embeds as a descending sequence of submodules of $M/\Gamma_\mathfrak{m}(M)$. The above localization process applied to this module and the induction hypothesis yields that $M_n/\Gamma_\mathfrak{m}(M_n)$, $n = 1, 2, \dots$ is stationary. Since in addition the descending sequence $\Gamma_\mathfrak{m}(M_n)$, $n = 1, 2, \dots$ of artinian modules is stationary, the original sequence must eventually terminate.

4. Representability of $\mathrm{Hom}_A(N, M)$.

For an A-module N, let us call an exact sequence $0 \to M' \to M \to M'' \to 0$, N-pure exact, if $0 \to \mathrm{Hom}_A(N, M') \to \mathrm{Hom}_A(N, M) \to \mathrm{Hom}_A(N, M'') \to 0$ is exact. The given exact sequence is thus by [10, Proposition 3] pure (exact) in the sense that it remains exact after tensoring with any A-module, if it is N-pure exact for all finite (finitely presented in case A is not required to be noetherian) A-modules N. A submodule $L$of M is called N-pure (pure) if the natural exact sequence $0 \to L \to M \to M/L \to 0$ is N-pure (resp. pure) exact. An A-module Q is called pure injective in case any pure exact sequence remains exact after applying $\mathrm{Hom}_A(-, Q)$ to it. Then it is a direct summand of any module, in which it is a pure submodule. Since any module can be embedded as a pure submodule of a pure injective module, the converse also holds. Any injective A-module E is trivially pure injective and from the adjoint homomorphism $\mathrm{Hom}_A(M \otimes_A X, E) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_A(X, E))$, it follows that in fact $\mathrm{Hom}_A(X, E)$ is pure injective for any A-module X, when E is injective.

LEMMA 4.1. *Let ϕ: $A \to B$ is a ring homomorphism, then any pure injective B-module Q is also pure injective considered as an A-module.*

PROOF. Let $0 \to M' \to M \to M'' \to 0$ be a pure exact sequence of A-modules. Then it is easily seen that $0 \to M' \otimes_A B \to M \otimes_A B \to M'' \otimes_A B \to 0$ is a pure exact sequence of B-modules. The conclusions follows by considering the natural isomorphisms $\mathrm{Hom}_B(X \otimes_A B, Q) \cong \mathrm{Hom}_A(X, Q)$ valid for all A-modules X.

COROLLARY 4.2. *Any artinian module M over a (noetherian) ring A is pure injective. Moreover any module artinian over a localization $S^{-1}A$ is pure injective as an A-module.*

PROOF. There is a direct sum decomposition $M = \oplus_{i=1}^{r} M_i$, where for each i the module M_i has the structure of an artinian module over the localization $A_{\mathfrak{m}_i}$ of A at some maximal ideal $\mathfrak{m}_i$, hence also over the completion over this ring. By Lemma 4.1, we may therefore assume that A is a complete local ring. Then by Matlis duality there is a finite A-module N, such that $M \cong \mathrm{Hom}_A(N, E)$, where E is the injective hull of the residue field of A. But as noted above a module of this form is pure injective. The last assertion now follows from what has just been proved using Lemma 4.1 again.

REMARK. Corollary 4.2 remains valid without the noetherian hypothesis on A, since Sharp has shown, [8, Theorem 3.2], that there is a ring homomorphism ϕ: $A \to B$, where B is (complete semilocal) noetherian ring, over which M has the structure of an artinian module. Then merely use the result just shown for artinian modules over noetherian rings and apply Lemma 4.1. Also Corollary 4.2

is a consequence of the characterization in [10, Theorem 2, Proposition 9] of pure injectivity by means of solvability of linear equations (algebraic compactness)

LEMMA 4.3. *Let M be a representable module of finite Goldie dimension and let N be a finite module. Then the module* $\mathrm{Hom}_A(N, M)$ *is representable if and only if for each (principal) ideal $\mathfrak{a}$, $\Gamma_{\mathfrak{a}}(M)$ is an N-pure submodule of M.*

PROOF. Since $\mathrm{Hom}_A(N, M)$ is isomorphic to a submodule of M^n for some number n, it has finite Goldie dimension. By Theorem 1.4, the module $\mathrm{Hom}_A(N, M)$ is representable precisely, when $\mathrm{Hom}_A(N, M) \to D_{\mathfrak{a}}(N, M))$ is a surjection for any (principal ideal $\mathfrak{a}$. Now N is finitely presented so the functor $\mathrm{Hom}_A(N, -)$ commutes with direct limits. Therefore $D_{\mathfrak{a}}(\mathrm{Hom}_A(N, M))$ is naturally isomorphic to $\mathrm{Hom}_A(N, D_{\mathfrak{a}}(M))$, and the surjectivity of the above map is equivalent to the surjectivity of $\mathrm{Hom}_A(N, M) \to \mathrm{Hom}_A(N, D_{\mathfrak{a}}(M))$, which just means that $\Gamma_{\mathfrak{a}}(M)$ is an N-pure submodule of M.

THEOREM 4.4. *Let M be a representable module of finite Goldie dimension, with $\mathfrak{p}_1, \ldots \mathfrak{p}_n$ as its distinct associated primes. Then* $\mathrm{Hom}_A(N, M)$ *is representable for every finite A-module N if and only if $M \cong \bigoplus_{i=1}^{n} M_i$ where for each i, M_i is an artinian $A_{\mathfrak{p}_i}$-module.*

PROOF. If N is a finite A-module and M is an artinian $A_{\mathfrak{p}}$-module, then $\mathrm{Hom}_A(N, M)$ is an artinian module over $A_{\mathfrak{p}}$ and thus representable also as an A-module. From this follows one of the implications. The converse implication is shown by induction on the number of the associated primes of M. Let $\mathfrak{p}$ be maximal in $\mathrm{Ass}\, M$. Then as in the proof of Theorem 3.1, $\Gamma_{\mathfrak{p}}(M)$ has the structure of an artinian module over $A_{\mathfrak{p}}$. By the hypothesis and Lemma 4.3, the sequence $0 \to \Gamma_{\mathfrak{p}}(M) \to M \to D_{\mathfrak{p}}(M) \to 0$ is pure exact. Now $\Gamma_{\mathfrak{p}}(M)$ is an artinian module over $A_{\mathfrak{p}}$, hence by Corollary 4.2 pure injective as an A-module. Consequenctly the above sequence splits. Thus $M \cong \Gamma_{\mathfrak{p}}(M) \oplus D_{\mathfrak{p}}(M)$; but $D_{\mathfrak{p}}(M)$ has one less associated prime than M, since its set of associated primes is $\mathrm{Ass}\, M \setminus \{\mathfrak{p}\}$.

EXAMPLE. We now give an example of a representable module of finite Goldie dimension, which does not satisfy the condition in Theorem 4.4. Let $(A, \mathfrak{m})$ be a local ring of dimension at least 2, such that there is a regular element x in $\mathfrak{m}$. Take a prime ideal $\mathfrak{p}$ minimal over xA. Consider the exact sequence $0 \to 0 \underset{E}{:} x \to E \xrightarrow{x} E \to 0$, where E is the injective hull of $A/\mathfrak{m}$. We get the exact sequence

$$0 \to \mathrm{Hom}(E(A/\mathfrak{p}), E) \xrightarrow{x} \mathrm{Hom}(E(A/\mathfrak{p}), E) \to \mathrm{Ext}^1(E(A/\mathfrak{p}), 0 \underset{E}{:} x) \to 0$$

Since multiplication by x on $E(A/\mathfrak{p})$, is not injective, (as $x \in \mathfrak{p}$), multiplication by x on $\mathrm{Hom}(E(A/\mathfrak{p}), E)$ is not surjective and we conclude $\mathrm{Ext}^1(E(A/\mathfrak{p}), 0 \underset{E}{:} x) \neq 0$. Next let $M' = 0 \underset{E}{:} x$, which is an artinian A-module and $M'' = E(A/\mathfrak{p})$, which is an artinian $A_{\mathfrak{p}}$-module. Since $\operatorname{Ext}_A^1(M'', M') \neq 0$, there is a nonsplit rxact sequence $0 \to M' \to M \to M'' \to 0$. By Corollary 1.6, M is representable of finite Goldie dimension. It is easily seen that $\Gamma_{\mathfrak{m}}(M)$ is naturally isomorphic to M'. Consequently $\Gamma_{\mathfrak{m}}(M)$ cannot be a pure submodule of M, since it as an artinian module is pure injective and thus would be a direct summand of M, in case it would be a pure submodule of M.

If $\mathfrak{a}$ is an ideal of A, then $\operatorname{Hom}_A(A/\mathfrak{a}, M) \cong 0 \underset{M}{:} \mathfrak{a}$, so one could ask if the finite module N in the above counterexample could be taken to be cyclic. Now the equality $0 \underset{M}{:} (\mathfrak{a} + xA) = 0 \underset{M'}{:} x$, where $M' = 0 \underset{M}{:} \mathfrak{a}$, shows by induction on the number of generators of $\mathfrak{a}$, that if for all representable A-modules of finite Goldie dimension M and any $x \in A$ the submodule $0 \underset{M}{:} x$ of M is representable, then $0 \underset{M}{:} \mathfrak{a}$ is representable for any ideal $\mathfrak{a}$ of A. Note that $0 \underset{M}{:} x$ is the kernel of the endomorphism on M defined to be multiplication by the element x. We therefore ask the following:

QUESTION. Let M be a representable A-module of finite Goldie dimension and $f\colon M \to M$ an A-linear endomorphism. Is then $\operatorname{Ker} f$ always a representable module?

Note that if $f\colon M' \to M$ is an A-linear map, where the A-modules M' and M are representable of finite Goldie dimension, but are not required to be the same module, then $\operatorname{Ker} f$ is not necessarily representable. Consider e.g. the canonical surjection of K onto K/A, where A is a discrete valuation ring and K its quotient field. Both K and K/A are indecomposable injective modules, thus representable of finite Goldie dimension, but A is not representable.

5. Connections with flat modules.

An A-module M is called a cotorsion module if $\operatorname{Ext}_A^1(F, M) = 0$ for any flat A-module F. Let $0 \to K \to L \to F \to 0$ be exact, where F is flat and L is free. By considering the long exact Tor-sequence it follows that K is also flat. If now M is cotorsion we get from the long exact Ext-sequence $\operatorname{Ext}_A^i(F, M) \cong \operatorname{Ext}_A^{i-1}(K, M)$ for any $i \geqq 2$. Hence induction yields that if M is cotorsion, then actually $\operatorname{Ext}_A^i(F, M) = 0$ for every flat A-module F and every $i \geqq 1$. An exact sequence $0 \to \mathrm{K} \to \mathrm{L} \to \mathrm{F} \to 0$ with F flat and L free is pure exact. This follows from the long exact Tor-sequence, since $\operatorname{Tor}_1^A(X, F) = 0$. So if M is a pure injective A-module, then $0 \to \operatorname{Hom}_A(F, M) \to \operatorname{Hom}_A(L, M) \to \operatorname{Hom}_A(K, M) \to 0$ is exact, and again using the long exact Ext-sequence, it follows that $\operatorname{Ext}_A^1(F, M) = 0$, i.e. M is cotorsion. Since any artinian module is pure injective by Corollary 4.2, it follows that any artinian module is cotorsion. Moreover Theorem 3.1 implies that any representable module of finite Goldie dimension is cotorsion, because if

$0 \to M' \to M \to M'' \to 0$ is exact, then M is cotorsion, if both M' and M'' are. Thus we have

THEOREM 5.1. *Let M be a representable module of finite Goldie dimension. Then M is a cotorsion module,* i.e. $\mathrm{Ext}^i_A(F, M) = 0$ *for every flat A-module F and every $i \geqq 1$. If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of representable modules of finite Goldie dimension, then the sequence $0 \to \mathrm{Hom}_A(F, M') \to \mathrm{Hom}_A(F, M) \to \mathrm{Hom}_A(F, M'') \to 0$ is exact for all flat A-modules F.*

COROLLARY 5.2. *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of representable A-modulus of finite Goldie dimension, then for any multiplicative set S in A, the sequence $0 \to \mathrm{Hom}_A(S^{-1}A, M') \to \mathrm{Hom}_A(S^{-1}A, M) \to \mathrm{Hom}_A(S^{-1}A, M'') \to 0$ is exact.*

In [6] it was shown that what there was called the co-localization $\mathrm{Hom}_A(S^{-1}A, -)$ preserves exactness of exact sequences of artinian A-modules. In [6] it was also shown that if M is an artinian A-module then for any multiplicative set S in A, the $S^{-1}A$-module $\mathrm{Hom}_A(S^{-1}A, M)$ is representable.

QUESTION. Let M be a representable module of finite Goldie dimension. Is then the $S^{-1}A$-module $\mathrm{Hom}_A(S^{-1}A)$ representable?

Note that an $S^{-1}A$-module is representable, if it is representable as an A-module, according to [11, Lemma 1.7]. A stronger question is the following one:

QUESTION. Let M be a representable A-module of finite Goldie dimension. Is then, for any flat A-module F, the module $\mathrm{Hom}_A(F, M)$ representable?

Of course these modules do not in general have finite Goldie dimension. There are however some positive results:

PROPOSITION 5.3. *Let M be an artinian A-module. Then for any flat module F, the module $\mathrm{Hom}_A(F, M)$ is representable.*

PROOF. M is a finite direct sum of modules artinian over localizations of A at maximal ideals. We may therefore assume that A is a local ring. Since M is artinian, it can be considered as a module over the completion $\hat{A}$ of A. Since $F \otimes_A \hat{A}$ is a flat module over $\hat{A}$ and $\mathrm{Hom}_A(F, M) \cong \mathrm{Hom}_{\hat{A}}(F \otimes_A \hat{A}, M)$, we may assume according to [4, Proposition 4.1] that A is complete. By Matlis duality, then $M \cong \mathrm{Hom}_A(N, E)$ for some finite A-module N, where E as usual denotes the injective hull of the residue field of A. Then $\mathrm{Hom}_A(F, M) \cong \mathrm{Hom}_A(F, \mathrm{Hom}_A(N, E)) \cong \mathrm{Hom}_A(N, \mathrm{Hom}_A(F, E))$, which is representable by [5, Theorem 1], since $\mathrm{Hom}_A(F, E)$ is an injective module.

Even if we don't know, whether $\mathrm{Hom}_A(F, M)$ for an arbitrary representable module of finite Goldie dimension is representable or not, we can at least prove that it has some property in that direction.

PROPOSITION 5.4. *Let M be a representable module of finite Goldie dimension and F a flat module. Then for any ideal a of A*

$$H^i_{\mathfrak{a}}((\mathrm{Hom}_A(F, M)) = 0, \quad i \geqq 1$$

PROOF. If M is artinian, we use Matlis duality in the same way as in the proof of Proposition 5.3. For a general M, use Theorem 3.1 and 5.1.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LUND
BOX 118
S-22100 LUND
SWEDEN