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ENTIRE FUNCTIONS WITH ASYMPTOTIC FUNCTIONS

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1. Introduction.

An entire function a is called an asymptotic function for the entire function f if $f(z) - a(z)$ approaches zero as $z \to \infty$ along a path Γ connecting 0 to ∞. Classically if f has n distinct *identically constant* asymptotic functions, then the order of f is at least $n/2$. This leads to the following conjecture [5].

CONJECTURE. *If f is an entire function having n distinct asymptotic functions each with order less than* $1/2$, *then the order of f is at least* $n/2$.

Simple examples show the necessity of the order 1/2 condition in the conjecture. We refer the reader to [6, pp. 575–578] for background material.

In 1983, Fenton [4] proved the conjecture true provided the orders of the asymptotic functions are less than 1/4. In a recent paper of Dudley Ward and Fenton [3, Theorem 4], the bound 1/4 is improved provided the asymptotic paths are rays.

THEOREM A. *Let f be analytic in the sector* $D = \{z: |\arg z| < \eta\}$, *where* $0 < \eta < \pi$, *and continuous up to the boundary. Let $a(z)$ and $b(z)$ be two distinct entire functions of order* $\rho < 1/(2 + 2\eta/\pi)$, *such that*

$$f(te^{i\eta}) - a(te^{i\eta}) \to 0 \quad \textit{and} \quad f(te^{-i\eta}) \to b(te^{-i\eta}) \to 0$$

as $t \to \infty$. *Then*

$$\liminf_{r \to \infty} \frac{\log M(r, D, f)}{r^{\pi/2\eta}} > 0, \tag{1.1}$$

where $M(r, D, f) = \sup\{|f(re^{i\theta})|: re^{i\theta} \in D\}$.

The authors remark that the rays in their theorem can actually be replaced by paths that are *almost* straight in the sense that the length of the part of the path in $\{z: |z| < r\}$ is $O(r)$ as $r \to \infty$.

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It is our purpose in this paper to extend Theorem A. We note that in Theorem A, the intersection of the circle of radius r with the domain D formed by the two rays has angular measure 2η. By this we mean that for all r

$$\operatorname{meas}\{\theta: re^{i\theta} \in D\} = 2\eta.$$

We state our theorem with this property in mind.

Let f have two distinct asymptotic functions a and b on the asymptotic paths Γ_1 and Γ_2 respectively. We may assume without loss of generality that the paths are simple and consist of finitely many line segments in any bounded set. If the paths intersect infinitely often, denote by D_n the bounded region between the nth and $(n-1)$st intersection and let D be the union of the D_n. If the paths intersect finitely often, it is no loss of generality to assume that they only intersect at the origin and in this case we let D be one of the unbounded regions formed by the two curves.

THEOREM 1. *Let a, b, Γ_1, Γ_2, and D be as above and let f be analytic in D and continuous up to the boundary. Suppose that for all sufficiently large r, the angular measure of $D \cap \{z: |z| = r\}$ does not exceed 2η and that a and b have orders strictly less than $1/(2 + 2\eta/\pi)$. Then Γ_1 and Γ_2 do not intersect outside a bounded set and* (1.1) *holds.*

An example in [3] shows that the number $1/(2 + 2\eta/\pi)$ cannot be replaced by a larger one in Theorem 1 when $\eta = \pi$. It is an open question whether our theorem is sharp for any other choices of η.

2. A strong $\cos \pi\rho$ theorem.

We first need a strong version of the $\cos \pi\rho$ theorem due to Eremenko, Shea and Sodin [2].(See also [8].) Recall that a sequence $\{r_k\}$ is a sequence of strong peaks of order λ for a function g subharmonic in the plane if

$$N(r, g) \leqq N(r_k, g)(r/r_k)^\lambda(1 + \eta_k), \quad r \in [\eta_k r_k, r_k/\eta_k], \tag{2.1}$$

holds for some positive $\eta_k \to 0$ and in addition

$$T(r, g) \leqq CN(r_k)(r/r_k)^\lambda, \quad r \in [\eta_k r_k, r_k/\eta_k], \tag{2.2}$$

for some positive constant C. Here

$$N(r, g) = \int_0^r \frac{\mu(\{z: |z| \leqq t)}{t} dt$$

is the usual Nevanlinna integrated "counting" function for g with Riesz measure μ and $T(r, g)$ is the usual characteristic function (we may and will assume that g is harmonic in a neighborhood of the origin). Strong peaks always exist if g has lower order less then one [7].

THEOREM B. *Let g be a function subharmonic in the plane and let $\{r_k\}$ be a sequence of strong peaks of order $\lambda < 1$ of g. Then*

$$\liminf_{n\to\infty} A(s_k, g)/M(s_k, g) \geqq \cos \pi\lambda \tag{2.3}$$

and

$$\liminf_{n\to\infty} A(s_k, g)/N(s_k, g) \geqq \pi\lambda \cot \pi\lambda, \tag{2.4}$$

where $A(r, g)$ and $M(r, g)$ are respectively the infimum and supremum of g on $|z| = r$ and $\{s_k\}$ is also a sequence of strong peaks of order λ such that $s_k/r_k \to 1$ as $k \to \infty$.

We remark that in [7] the result corresponding to Theorem B is stated for entire functions, but the proof in [7] goes through for subharmonic functions as well.

3. Proof of Theorem 1 – infinite intersections.

First of all, without loss of generality we assume that b is identically zero. Denote by $\rho = \rho(a)$ the order of a. If E is a set in C we define E^* to be the circular projection of E onto the negative real axis R_-. Given a measure μ on C, let μ^* be a measure on R which satisfies $\mu^*(\mathsf{C}\setminus\mathsf{R}_-) = 0$ and for each positive r

$$\mu^*(E^* \cap |z| = r) = \mu(E \cap |z| = r).$$

Now let μ be the Riesz measure of $\log^+ |a|$. (If necessary, we replace $\log^+ |a|$ by a slightly modified function without changing notation to make it harmonic in a small neighborhood of the origin. This does not essentially affect the argument that follows.) Since $\log^+ |a|$ has order $\rho < 1$, the function

$$p(z) = \int_0^\infty \log\left|1 - \frac{z}{t}\right| d\mu^*(t)$$

is subharmonic in the plane with order ρ as well. Clearly

$$N(r, \log^+ |a|) = N(r, p). \tag{3.1}$$

Also it is easy to see that

$$A(r, p) \leqq A(r, \log^+ |a|) \leqq M(r, \log^+ |a|) \leqq M(r, p). \tag{3.2}$$

In this section we show that Γ_1 and Γ_2 intersect at most finitely often. Let s_k be as in Theorem B with $g = p$ and $\lambda = \rho$. If the paths intersect infinitely often, then corresponding to infinitely many domains D_n as above, there exists an s_{k_n} such that an arc of the circle $|z| = s_{k_n}$ is contained in D_n and intersects both Γ_1 and Γ_2. In other words we are dealing with two infinite subsequences, one of D_n and one of s_k. For convenience and without loss of generality we assume that s_n corresponds to D_n for all n.

Define U_n to be harmonic in D_n with boundary values $\log(|f|\,|f - a|)$ and set

$$G_n(z) = \int_{D_n} g_n(\zeta, z)\, d\mu(\zeta), \quad z \in D_n, \tag{3.3}$$

where $g_n(z)$ is the Green's function for D_n with pole at ζ and μ is the Riesz measure of $\log^+ |a|$. Then

$$U_n - \log^+ |a| = G_n + h_n, \tag{3.4}$$

where h_n is harmonic in D_n and

$$h_n(z) \leqq K \tag{3.5}$$

for all $z \in D_n$. Here K is a non-negative constant independent of n.

Following an argument in [6, p. 577], let $\gamma(s_n)$ be an arc of $|z| = s_n$ having endpoints z_1 on Γ_1 and z_2 on Γ_2. Now (2.3) holds with $\lambda = \rho < 1/2$ on s_n. This together with (3.2) implies that a is uniformly large on $|z| = s_n$. Then since $f - a$ approaches 0 on Γ_1 and f approaches 0 on Γ_2, we obtain

$$|f(z_1) - a(z_1)| < \tfrac{1}{2}|a(z_1)| \quad \text{and} \quad |f(z_2) - a(z_2)| > \tfrac{1}{2}|a(z_2)|.$$

Hence there exists a point z_n on $\gamma(s_n)$ such that

$$|f(z_n) - a(z_n)| = \tfrac{1}{2}|a(z_n)|.$$

By (2.3) and (3.2), $\log|a(z_n)| = \log^+ |a(z_n)|$ and so, since $\log(|f|\,|f - a|)$ is subharmonic and therefore majorized by U_n in D_n,

$$U(z_n) \geqq \log|f(z_n)| + \log|f(z_n) - a(z_n)| \geqq 2\log^+ |a(z_n)| - C,$$

where C is a non-negative constant not depending on n. Thus (3.4) and (3.5) imply that

$$G_n(z_n) \geqq \log^+ |a(z_n)| - (C + K) \tag{3.6}$$

Our goal is to show that (3.6) is possible for arbitrarily large n only if

$$\rho \geqq 1/(2 + 2\eta/\pi). \tag{3.7}$$

This leads to the required contradiction. To do this, let D_Γ be an unbounded domain containing $D = \cup_{n=1}^{\infty} D_n$, such that for all sufficiently large r, the angular measure of $\{\theta: re^{i\theta} \in D\}$ does not exceed 2η. This is possible by the hypothesis on the angular measure of D.

Define for $z \in D_\Gamma$,

$$G_\Gamma(z) = \int_{D_I} g_\Gamma(\zeta, z)\, d\mu(\zeta),$$

where $g_\Gamma(\zeta, z)$ is the Green's function for D_Γ with pole at ζ. By the monotonicity of the Green's function, we have that

$$G_\Gamma(z) \geqq G_n(z) \tag{3.8}$$

for all z in D_n, $n = 1, 2, \ldots$.

Let $\Omega = \{z: |\arg z| < \eta\}$ and let $g_\Omega(\zeta, z)$ be the Green's function of Ω with pole at ζ. Clearly

$$\sup\{g_\Omega(|\zeta|, z): |z| = r, z \in \Omega\} = g_\Omega(|\zeta|, r).$$

Then by a result of Baernstein [1, Theorem 5] (see also [6, pp. 658–661])

$$g_\Omega(|\zeta|, r) \geqq \sup\{g_\Gamma(\zeta, z): |z| = r,\ z \in D_\Gamma\}.$$

Defining

$$G_\Omega(z) = \int_0^\infty g_\Omega(t, |z|)\, d\mu^*(t),$$

we obtain

$$G_\Omega(r) \geqq \sup\{G_\Gamma(z): |z| = r, z \in D_\Gamma\}. \tag{3.9}$$

By (3.6), (3.1), (3.2) and (2.4) we obtain

$$\begin{aligned} G_\Omega(s_n) &\geqq (1 - o(1)) \log^+ |a(z_n)| \\ &\geqq (1 - o(1)) A(s_n, \log^+ |a|) \\ &\geqq (1 - o(1)) A(s_n, p) \\ &\geqq (1 - o(1)) \pi\rho \frac{\cos \pi\rho}{\sin \pi\rho} N(s_n, p), \end{aligned} \tag{3.10}$$

as $n \to \infty$.

On the other hand,

$$G_\Omega(s_n) = \int_0^\infty \log \left| \frac{t^\beta + s_n^\beta}{t^\beta - s_n^\beta} \right| d\mu^*(t) \tag{3.11}$$

where $\beta = \pi/2\eta$. For the moment let us disregard the logarithmic singularities in (3.11) and formally integrate it by parts twice. We obtain

$$G_\Omega(s_n) = \int_0^\infty K(t, s_n, \beta) N(t, p) \frac{dt}{t} \tag{3.12}$$

where

$$K(t, s_n, \beta) = \frac{2s_n^\beta \beta^2 t^\beta (t^{2\beta} + s_n^{2\beta})}{(t^{2\beta} - s_n^{2\beta})^2} \geqq 0$$

for all $t > 0$ with $t \neq s_n$. The non-negativity of the kernel $K(t, s_n, \beta)$ allows us to use (2.1) and a routine argument (see, for example, [7, p. 177]) to deduce that

$$\begin{aligned} G_\Omega(s_n) &= (1 + o(1)) N(s_n, p) \int_{s_n \eta_n}^{s_n/\eta_n} K(t, s_n, \beta) \left(\frac{t}{s_n}\right)^\rho \frac{dt}{t} \\ &\leqq (1 + o(1)) N(s_n, p) \int_0^\infty K(t, s_n, \beta) \left(\frac{t}{s_n}\right)^\rho \frac{dt}{t}. \end{aligned}$$

Formally integrating the right hand side of the above inequality by parts twice and noting by a suitable contour integral that

$$\rho^2 \int_0^\infty \log \left| \frac{t^\beta + r^\beta}{t^\beta - r^\beta} \right| \left(\frac{t}{r}\right)^\rho \frac{dt}{t} = \pi\rho \frac{\sin \rho\eta}{\cos \rho\eta},$$

for any positive r, we obtain

$$G_\Omega(s_n) \leqq (1 + o(1)) \pi\rho \frac{\sin \rho\eta}{\cos \rho\eta} N(s_n, p). \tag{3.13}$$

Hence (3.10) and (3.13) yield

$$1 - \tan(\pi\rho)\tan(\rho\eta) \leqq 0.$$

Since $0 < \eta < \pi$ and $0 \leqq \rho < 1/2$, we also have $0 \leqq \pi\rho < \pi/2$ and $0 \leqq \rho\eta < \pi/2$ so that we further get $\tan(\pi\rho + \rho\eta) \leqq 0$. This gives $\pi\rho + \rho\eta \geqq \pi/2$, which is equivalent to (3.7), as required.

To make (3.4) rigorous, we follow an argument used in [2, pp. 391–395] or [8, pp. 69–71]. As in those articles, we can write for any r sufficiently close to s_n,

$$G_\Omega(r) = \int_0^{R_n} \log \left| \frac{t^\beta + r^\beta}{t^\beta - r^\beta} \right| du^*(t) + o(N(s_n)),$$

where $R_n = s_n/(2\eta_n)$. Also we can choose an appropriate sequence $\tau_n \geqq 1$ with $\tau_n \to 1$ as $n \to \infty$, and write

$$G_\Omega(r) = \left(\int_0^{s_n/\tau_n} + \int_{s_n\tau_n}^{R_n}\right)\log\left|\frac{t^\beta + r^\beta}{t^\beta - r^\beta}\right| d\mu^*(t)$$
$$+ \int_{s_n/\tau_n}^{s_n\tau_n} \log\left|\frac{t^\beta + r^\beta}{t^\beta - r^\beta}\right| d\mu^*(t) + o(N(s_n)).$$

We integrate the first two integrals by parts twice and estimate the boundary terms as well as the third integral in exactly the same fashion as in [2] and [8]. The only change is the use of a *subharmonic version* of the Boutroux-Cartan lemma which can be found in [6, Lemma 6.17, p. 366]. Thus we have proved that there can be only finitely many D_n.

4. Proof of (1.1).

We now may assume that the paths Γ_1 and Γ_2 intersect only at the origin and bound the unbounded region D which has angular measure not exceeding 2η. We assume that (1.1) is false. Then there exists a sequence $R_n \to \infty$ on which it fails. Let D_n be the component of $D \cap \{z: |z| < R_n\}$ containing the origin on its boundary. We define the harmonic function U_n in D_n as we did before with boundary values $\log|f| + \log|f - a|$ and obtain that

$$U_n - \log^+|a| = G_n + h_n, \tag{4.1}$$

where G_n is as before. The function h_n is a harmonic function in D_n, with boundary values not exceeding a positive constant K independent of n on $\partial D_n \cap (\Gamma_1 \cup \Gamma_2)$ and equal to $\log(|f|\,|f - a|) - \log^+|a|$ on $\partial D_n \cap \{z: |z| = R_n\}$. Since (1.1) fails on R_n and since the order of a is $\rho < \pi/2\eta$, we must have

$$\limsup_{n\to\infty} \max\{h_n(z)/r^{\pi/2\eta}: z \in \partial D_n, |z| = R_n\} \leqq 0. \tag{4.2}$$

But (4.2), the fact that the angular measure of D never exceeds η and a trivial application of the Carleman-Tsuji inequality [9, p. 116], show that given $\varepsilon > 0$ and r, there exists n such that

$$\sup\{h_n(z): |z| = r, z \in D_n\} < K + \varepsilon$$

Thus picking a sequence s_k as in Theorem B, we can find a corresponding subsequence R_{n_k} such that

$$\limsup_{k\to\infty} \max(h_{n_k}(z): |z| = s_k, z \in \partial D_{n_k}\} \leqq K. \tag{4.3}$$

Thus by (4.3), the term h_{n_k} in (4.1) is asymptotically at most K and we are in exactly the same situation as in the preceding section. The proof of Theorem 1 is complete.

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