GENERALIZED WEYL-VON NEUMANN THEOREMS (II)

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Abstract.

We show that the multiplier algebra $M(A)$ of a $\sigma$-unital $C^*$-algebra with stable rank one and (FU) has real rank zero. We also show that the multiplier algebras $M(A)$ of matroid $C^*$-algebras and many other $C^*$-algebras have (FU). Consequently, if $u$ is a unitary in $M(A)$ and $\varepsilon > 0$, there are projections $\{p_n\} \in A$ such that

$$u = \sum_{n=1}^{\infty} z_n p_n + a$$

$$u = \sum_{n=1}^{\infty} p_n = 1,$$

where $|z_n| = 1$, $a \in A$ and $\|a\| < \varepsilon$.

0. Introduction.

Let $H$ be a separable, infinite dimensional Hilbert space, $K$ be the $C^*$-algebra of compact operators on $H$ and $B(H)$ the $C^*$-algebra of bounded operators on $H$. The Weyl-von Neumann theorem says: if $T$ is a self-adjoint operator in $B(H)$ and $\varepsilon > 0$, then there is a diagonalizable self-adjoint matrix $D$ in $B(H)$ and a compact operator $k \in K$ such that

$$T = D + k$$

with $\|k\| < \varepsilon$. Let $A$ be a $C^*$-algebra and $M(A)$ its multiplier algebra $(M(A) = \{m \in A^{**} : ma, am \in A, \forall a \in A\}$ where $A^{**}$ is the enveloping von-Neumann algebra. So $M(A)$ is the idealizer of $A$ in $A^{**}$.) We say that the Weyl-von Neumann theorem holds for $A$ and $M(A)$ if for any $T \in M(A)_{s.a.}$ and $\varepsilon > 0$, there are projections $p_n$ in $A$ and $a \in A$ such that

$$T = \sum_{i=1}^{\infty} \lambda_n p_n + a,$$

where $\sum_{i=1}^{\infty} p_i = 1$, $\lambda_n$ is a bounded sequence of real numbers and $\|a\| < \varepsilon$. It has been shown ([M] and [Zh 1]) that the Weyl-von Neumann theorem holds for

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A and $M(A)$ if and only if $M(A)$ has real rank zero. (A $C^*$-algebra $A$ has real rank zero if the set of self-adjoint elements with finite spectra is dense in $A_{s.a.}$. If $A$ has real rank zero, we will write $\text{RR}(A) = 0$. See [BP]) When is $\text{RR}(M(A)) = 0$? A necessary condition is $\text{RR}(A) = 0$. $W^*$-algebras and $AW^*$-algebras all have real rank zero. AF-algebras, Bunce-Deniddens algebras and all purely infinite simple $C^*$-algebras have real rank zero (See [BP]). The question whether $\text{RR}(M(A)) = 0$ if $A$ is an AF-algebra was raised formally in [BP]. However, as early as 1974, George A. Elliott raised the same question at Tohoku. It has been shown by L. G. Brown and G. K. Pedersen [BP], S. Zhang [Zh 3, 7, 8] and by N. Higson and M. Rørdam [HR] that the above question has an affirmative answer in the case that $A$ is a matroid $C^*$-algebra. The author shows recently that $\text{RR}(M(A)) = 0$ for every $\sigma$-unital AF-algebra ([Li3]). For more information concerning the generalized Weyl-von Neumann theorem readers are referred to [Zh 1–8] and [Li 3]. One key result we established in [Li 3] is the following:

**Theorem A ([Li 3, 3.2]).** Let $A$ be a $\sigma$-unital $C^*$-algebra. Then $M(A)/A$ has real rank zero if $K_1(B) = 0$ for every hereditary $C^*$-algebra $B$ of $M(M_n(A))$ which contains $M_n(A)$, where $n = 1, 2, \ldots$.

We will show in section 2 that every $\sigma$-unital $C^*$-algebra with real rank zero, stable rank one, zero $K_1$-group and satisfying a certain condition (a) satisfies the conditions in Theorem A. By combining [BP, 3.13 and 3.14] as in [Li 3], we conclude that $\text{RR}(M(A)) = 0$ for these $C^*$-algebras. We also show, in section 3, that every simple $C^*$-algebra with real rank zero, stable rank one and satisfying the condition (a) satisfies conditions in Theorem A. Therefore corona algebras of those $C^*$-algebras have real rank zero. In section 4, we show that the Weyl-von-Neumann theorem for unitaries holds for the multiplier algebras of matroid algebras and other $C^*$-algebras with real rank zero. Applications of these results to the theory of $C^*$-algebra extensions will appear elsewhere.

We would like state the following definitions.

**Definition 1.1.** [Ph 1, 1.2] Let $A$ be a unital $C^*$-algebra and let $U_0(A)$ be the connected component of the unitary group $U(A)$ of $A$. The exponential rank of $A$, written $\text{cer}(A)$, is the largest element of the set of symbols $1, 1 + \varepsilon, 2, 2 + \varepsilon, \ldots, \infty$ (with the obvious order) consistent with the following restrictions:

1. $\text{cer}(A) \leq n$ if every $u \in U_0(A)$, the identity component of the unitary group, is the product $\exp(ih_1)\exp(ih_2)\ldots\exp(ih_n)$ for some $h_1, h_2, \ldots, h_n \in A_{s.a.}$;

2. $\text{cer}(A) \leq n + \varepsilon$ if every $u \in U_0(A)$ is a norm limit of products of $n$ exponentials as in (1).

For nonunital $A$, set $\text{cer}(A) = \text{cer}(\tilde{A})$.

**Definition 1.2.** A unital $C^*$-algebra $A$ is said to have (FU) (weak (FU)) if the set of unitaries with finite spectra is norm dense in $U(A)(U_0(A))$. For onunital $A$,
we say $A$ has (FU) (weak (FU)), if $\tilde{A}$ has (FU). It is known that $W^*$-algebras, $AW^*$-algebras, AF-algebras and many other (see [Ph 1]) have (FU). On the other hand, if $A$ has weak (FU), then $\text{RR}(A) = 0$ and $\text{cer}(A) \leq 1 + \varepsilon$. It is shown in [Ph 1] that the irrational rotation algebras $A_\theta$ have weak (FU) for $\theta$ in a dense $G_\delta$-set of $[0, 1] \setminus \mathbb{Q}$ and that Elliott's $C^*$-algebras $A$ of inductive limits of basic building blocks have weak (FU). It is shown in [Ph 2] that for every purely infinite simple $C^*$-algebra $A$, $\text{cer}(A) \leq 1 + \varepsilon$. Our results in section 3 show that for matroid $C^*$-algebras and purely infinite simple $C^*$-algebras $A$ (and many other $C^*$-algebras), $\text{cer}(M(A)/A) \leq 1 + \varepsilon$.

We will use the following notations throughout this paper. $K$ is the $C^*$-algebra of compact operators on a separable infinite-dimensional Hilbert space. $M_n(A)$ is the $n \times n$ matrices over $A$. Her$(a)$ denotes the hereditary $C^*$-subalgebra generated by element $a$ and $C(A)$ denotes the corona algebra $M(A)/A$.

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The main result in this section is Theorem 2.9 which improves our earlier results in [Li 3]. We start with the following lemma.

**Lemma 2.1** ([Zh 10]). *Let $A$ be a $C^*$-algebra with real rank zero and stable rank one, $n$ be a positive integer. Suppose that*

$$p = \sum_{i=1}^{n} p_i \otimes e_{ii},$$

$$p_1 \leq p_2 \leq \cdots \leq p_n,$$

*where the $p_i$'s are projections in $A$ and $\{e_{ij}\}$ is a matrix unit for $M_n$, the $n \times n$ matrices. Then*

$$\text{cer}(pM_n(A)p) \leq d(n) + \text{cer}(p_nAp_n) + \varepsilon,$$

*if $n = 2^{d(n)}$;*

$$\text{cer}(pM_n(A)p) \leq d(n) + 1 + \text{cer}(p_nAp_n) + \varepsilon,$$
if \( 2^{d(n)} < n < 2^{(d(n) + 1)} \), where \( d(n) = \lceil \ln(n)/\ln 2 \rceil \) and \( \{k\} \) is the largest integer smaller or equal to \( k \). Moreover, if the unitary group of \( p_n A p_n \) is connected, \( U(p M_n(A) p) \) is also connected.

**Remark 2.2.** Our earlier estimate is

\[
\operatorname{cer}(p M_n(A) p) \leq 3(n - 1) + \operatorname{cer}(p_n A p_n),
\]

which is enough for our purpose in this paper. But since 2.1 is much better, with S. Zhang’s permission, we quote it from [Zh 10].

**Added in proof:** It has been shown by the author (Exponential rank of \( C^* \)-algebras with real rank zero and the Brown-Pedersen Conjectures, J. Funct. Anal. 114 (1993), 1–11) that \( \operatorname{cer}(B) \leq 1 + \varepsilon \) for every \( C^* \)-algebra of real rank zero.

**Lemma 2.3.** Let \( A \) be a \( C^* \)-algebra with real rank zero. Then the map \( K_1(I) \to K_1(A) \) is injective for any ideal \( I \) of \( A \).

**Proof.** For each \( n \), by [BP, 2.10],

\[
\operatorname{RR}(M_n(A)) = \operatorname{RR}(M_n(I)) = \operatorname{RR}(M_n(A/I)) = 0.
\]

It follows from [Zh 3, 3.2] that every projection in \( M_n(A/I) \) lifts to a projection in \( M_n(A) \).

From the six-term exact sequence in \( K \)-theory

\[
\begin{align*}
K_0(I) & \to K_0(A) \to K_0(A/I) \\
& \uparrow \downarrow \\
K_1(A/I) & \leftarrow K_1(A) \leftarrow K_1(I),
\end{align*}
\]

we see that the map \( K_0(A) \to K_0(A/I) \) is surjective. Hence \( K_1(I) \to K_1(A) \) is injective.

**Lemma 2.4.** Let \( A \) be a \( C^* \)-algebra with real rank zero and \( K_1(A) = 0 \). If \( B \) is a hereditary \( C^* \)-algebra of \( A \), then \( K_1(B) = 0 \).

**Proof.** We may assume that \( A \) is unital. We first consider the case \( B = p A p \) for some projection \( p \) in \( A \). Let \( B_1 = (A \otimes K)^\sim \) and \( \{1 \otimes e_{ij}\} \) be a matrix unit for \( C \cdot 1 \otimes K \). For an integer \( n \), let \( w \) be a unitary in \( \bigoplus_{i=1}^n p \otimes e_{ii} B \bigoplus_{i=1}^n p \otimes e_{ii} \) and \( u = 1 - \sum_{i=1}^n p \otimes e_{ii} + w \). It is enough to show that \( u \) is connected to the identity of \( (p A p \otimes K) \).

Let \( A_1 \) be the \( C^* \)-subalgebra of \( B_1 \) generated by \( \{1, 1 \otimes e_{ij}, i, j = 1, 2, \ldots\} \) and \( w \). Suppose that separable \( C^* \)-algebra \( A_n \) is constructed. Since \( B_1 \) has real rank zero (see [BP]), there is a sequence of projections \( \{p_k\} \) such that every self-adjoint element in \( A_n \) can be approximated by elements with the form \( \sum_{i=1}^n \lambda_i p_k i \), where \( \{\lambda_i\} \) are real numbers and \( \{p_k\} \) are mutually orthogonal. Suppose that \( \{u_k\} \) is
a dense sequence of unitaries of $A_n$. Since $K_1(A) = 0$, each $u_k$ is connected to the identity in $B_1$. Let $u_k^{(1)}, u_k^{(2)}, \ldots, u_k^{(m)}$ be the unitaries along the path which connects $u_k$ to 1 such that

$$\|u_k - u_k^{(1)}\| < 1, \|u_k^{(m)} - 1\| < 1$$

and

$$\|u_k^{(i)} - u_k^{(i+1)}\| < 1, \quad i = 1, 2, \ldots, m - 1.$$  

Let $A_{n+1}$ be the $C^*$-subalgebra of $B_1$ generated by $A_n, p_k, \{u_k, u_k^{(1)}, \ldots, u_k^{(m)}\}$. Set

$$A_\infty = \left( \bigcup_{n=1}^{\infty} A_n \right)^{-}.$$  

By the construction, $A_\infty$ has real rank zero and the unitary group of $A_\infty$ is connected. Let $A_0$ be the norm closure of

$$\bigcup_{n=1}^{\infty} \left( \sum_{i=1}^{n} 1 \otimes e_{ij} \right) A_\infty \left( \sum_{i=1}^{n} 1 \otimes e_{ij} \right).$$  

Then $\tilde{A}_0 \cong A_\infty$. Moreover,

$$(1 \otimes e_{ii})A_0(1 \otimes e_{ii})^* \otimes K \cong A_0.$$  

Thus $K_1(A_0) = 0$. Let $I$ be the ideal generated by

$$(p \otimes e_{11} A_0 p \otimes e_{11}) \otimes K.$$  

By Lemma 2.3, $K_1(I) = 0$. Since, by [Bn 1],

$$I \otimes K \cong (p \otimes e_{11}) A_0 (p \otimes e_{11}) \otimes K,$$

$K_1((p \otimes e_{11}) A_0 (p \otimes e_{11})) = 0$. Hence $u$ is connected to the identity in $(pA_0 \otimes K)^{-}$.  

Now we consider the case that $B$ is not unital. Let $u$ be a unitary in $B' = (\tilde{B} \otimes K)$. Again, it is enough to show that $u$ is connected to the identity of $(\tilde{B} \otimes K)^{-}$. It is easy to see that $u$ is close to a unitary of the form

$$(1 - \sum_{i=1}^{k} 1 \otimes e_{ii}) + w,$$

where $w$ is a unitary in $(\sum_{i=1}^{k} 1 \times e_{ii})B'(\sum_{i=1}^{k} 1 \otimes e_{ii})$. Since $B$ has real rank zero, by [BP, 2.6], $B$ has an approximate identity $\{d_\varepsilon\}$ consisting of projections. This implies that $w$ is close to a unitary of the form

$$(\sum_{i=1}^{k} 1 \otimes e_{ii} - p) + w',$$

where $p \preceq \sum_{i=1}^{k} 1 \otimes e_{ii}$ is a projection and $w'$ is a unitary in $pB'p$. So $u$ is connected to the unitary $(1 - p) + w'$. Since $B$ is a hereditary $C^*$-subalgebra of $A$, we have that

$$pB'p \cong pM_k(A)p.$$
Since $K_1(M_k(A)) = 0$, from what we have shown, $K_1(pM_k(A)p) = 0$. We also have that $(pB'p \otimes K) \cong B'$. Therefore $(1 - p) + w'$ is connected to the identity of $B'$. This implies that $u$ is connected to the identity of $B'$. This completes the proof.

**Definition 2.5.** A C*-algebra $A$ of real rank zero is said to satisfy condition (a) if there is an integer $k$ such that for every projection $p \in A$

$$\text{cer}(pAp) \leq k.$$  

From [Li 3, 1.3], every C*-algebra with (FU) satisfies the condition (a). It follows from 2.1 and [Zh 5, 3.3] that if $A$ has stable rank one and satisfies condition (a), then $M_n(A)$ satisfies condition (a) (with different $k$ though).

**Lemma 2.6.** Let $A$ be a σ-unital C*-algebra with real rank zero, stable rank one and $K_1(A) = 0$. Suppose that $A$ satisfies condition (a) with the integer $k$. Then for any integer $m$, if $B$ is a unital hereditary C*-subalgebra of $M_m(M(A))$ ($= M(M_m(A))$), then

$$\text{cer}(B) \leq 2(\lfloor \ln(m)/\ln 2 \rfloor + 1 + k) + \varepsilon$$

and the unitary group of $B$ is connected.

**Proof.** Fix an integer $m$. Let $B = pM(M_m(A))p$ for some projection $p$ in $M(M_m(A))$. If $p \in M_m(A)$, by [Zh 5, 3.3], we may assume that $p = \sum_{i=1} p_i \otimes e_i$, where $p_i$ is a projection in $A$ and $\{e_i\}$ is a matrix unit for $M_m$. Moreover, we may assume that

$$p'_1 \leq p'_2 \leq \ldots \leq p'_m.$$  

Therefore the estimate of cer$(B)$ follows from 2.1. Since $K_1(M_m(A)) = 0$, by 2.4, $K_1(B) = 0$. It follows from [Rff, 2.10] that the unitary group of $B$ is connected.

Now we assume that $p \in M(M_m(A)) \setminus M_m(A)$. For any unitary $u$ in $B$, by an Elliott's trick (see [Ell 1, 2.4], [Zh 6, 1.6] or [Li 3, 2.1]), for $\varepsilon > 0$ there are projections $\{e_n\}$ in $M_m(A)$ and unitaries $u_1, u_2$ in $pM_m(A)p$ such that

$$\|u - u_1u_2\| < \varepsilon/2$$

and

$$u_1 = \sum_{i=1}^\infty (e_{2n} - e_{2n-2})u_1(e_{2n} - e_{2n-2}),$$  

$$u_2 = \sum_{i=1}^\infty (e_{2n+1} - e_{2n-1})u_2(e_{2n+1} - e_{2n-1}).$$

By 2.1 and our condition (a), there are

$$b^{(i)} \in (e_{2n} - e_{2n-2})M_m(A)(e_{2n} - e_{2n-2})_{\text{sa}},$$

$i = 2, \ldots, l$ such that
\[
\left\| (e_{2n} - e_{2n-2})u_1(e_{2n} - e_{2n-2}) - \prod_{k=1}^{l} \exp(i b_n^{(k)}) \right\| < \varepsilon/2^{n+2},
\]

where \( l = \{ \ln(n)/\ln 2 \} + k + 1 \). Furthermore, since \( A \) has real rank zero, we may assume that \( \| b_n^{(i)} \| \leq 2\pi \). Clearly,

\[
h_1^{(k)} = \sum_{n=1}^{\infty} b_n^{(k)} \in pM(M_m(A))p_{s.a.}.
\]

Hence

\[
\left\| u_1 - \prod_{k=1}^{l} \exp(i h_1^{(k)}) \right\| < \sum_{n=1}^{\infty} \varepsilon/2^{n+2} = \varepsilon/4.
\]

Similarly, there are \( h_2^{(k)} \in pM(M_m(A))p \), \( k = 1, 2, \ldots, l \) such that

\[
\left\| u_2 - \prod_{k=1}^{l} \exp(i h_2^{(k)}) \right\| < \varepsilon/4.
\]

Therefore

\[
\left\| u - \sum_{k=1}^{l} \exp(i h_1^{(k)}) \exp(i h_2^{(k)}) \right\| < \varepsilon.
\]

This completes the proof.

**Theorem 2.7.** If \( A \) is a \( \sigma \)-unital \( C^* \)-algebra with real rank zero, stable rank one and \( K_1(A) = 0 \), and satisfies condition (a), then for any hereditary \( C^* \)-subalgebra \( B \) of \( M_m(M(A)) \), \( K_1(B) = 0 \).

**Proof.** The proof is similar to that of [Zh 6, 2.17].

If \( B \) is unital, Lemma 2.6 applies. So we may assume that \( B \) is not unital. Moreover, by 2.4, we may assume that \( B \nsubseteq A \). Set \( B' = (\tilde{B} \otimes \subseteq) \). It is enough to show that a unitary \( u \in B' \) can be connected to the identity of \( B' \) by a path of unitaries in \( B' \). As in the second part of the proof of Lemma 2.4, without loss of generality, we may assume that \( u = (1 - F_n) + w \), where \( F_n = \sum_{i=1}^{n} 1 \otimes \varepsilon_{ii} \) and \( w \) is a unitary in \( F_nB'F_n \).

Now consider the map

\[
\tau: F_nB'F_n \cong M_n(\tilde{B}) \to M_n(\tilde{B}/B) \cong M_n(C).
\]

Set \( v = \tau(w) \). If we use the same notation \( v \) for the corresponding scalar matrix in \( M_n(\tilde{B}) \), then we may write \( w = v + b \) for some \( b \in M_n(B) \). Clearly \( w \) is connected to a unitary with form \( F_n + b' \) in the unitary group of \( M_n(\tilde{B}) \) for some \( b' \in M_n(B) \). Therefore we may assume that \( u = 1 + b' = (1 - F_n) + (F_n + b') \).

Notice that for any integer \( m \geq 1 \), \( M_m(B) \) is a hereditary \( C^* \)-subalgebra of
$M_m(M(A)) \cong M(M_m(A))$. Since $M_m(A)$ has real rank zero (see [BP, 2.10]), it follows from [Zh 2, 1.1] that $M_m(B)$ has LP for any $m \geq 1$ ($A$ C*-algebra $A$ is said to have LP if $A$ is the closed linear span of its projections). It is then routine to show that $B \otimes K$ has LP. By [Zh 6, 1.1], $B \otimes K$ has an approximate identity consisting of projections. Then, as in the second part of the proof of Lemma 2.4, it is easy to see that $u = 1 + b'$ is close to a unitary with the form $(1 - p) + v'$ where $p$ is a projection in $B \otimes K$ and $v'$ is a unitary in $p(B \otimes K)p$. So we may assume that $u = (1 - p) + v'$. It easy to see that $p$ is close to a projection which is in $F_k(B \otimes K)F_k$ for some $k \geq n$, without loss generality, we may further assume that $p \leq F_k$.

We notice that $p(B \otimes K)p = pM_k(B)p = pM_k(M(A))p$. It follows from 2.6 that $v'$ is connected to the identity of $F_k(B \otimes K)F_k$ by a path of unitaries in $F_k(B \otimes K)F_k$. This proves that $u$ is connected to the identity of $B'$ by a path of unitaries in $B'$.

**Corollary 2.8.** Let $A$ be a σ-unital C*-algebra with FU) and stable rank one, then for any hereditary C*-algebra $B$ of $M_m(M(A))$ for any $m$, $K_1(B) = 0$.

**Proof.** It follows from [Li 3, 1.3], $pA'$ has (FU) for all projections in $A$. Hence $A$ satisfies condition (a). Moreover, by 2.1, $K_1(A) = 0$.

**Theorem 2.9.** Let $A$ be a σ-unital C*-algebra with real rank zero, stable rank one and $K_1(A) = 0$. If $A$ satisfies condition (a), then $M(A)$ has real rank zero. Equivalently, for any $T \in M(A)_{s.a.}$ and $\varepsilon > 0$, there are an approximate identity $\{e_n\}$ of $A$ consisting of projections and an element $a \in A_{s.a.}$ such that

$$T = \sum_{i=1}^{\infty} \lambda_i (e_i - e_{i-1}) + a,$$

where $\|a\| < \varepsilon$ and $\{\lambda_i\}$ is a bounded sequence of real numbers.

**Proof.** It is an immediate consequence of 2.7, Theorem A and [BP, 3.14].

**Corollary 2.10.** Suppose that $A$ is a σ-unital C*-algebra with (FU) and stable rank one. Then $M(A)$ has real rank zero.

**Corollary 2.11 ([Li 3]).** If $A$ is a σ-unital AF-algebra, then $M(A)$ has real rank zero.

2. Corona algebras $M(A)/A$ with real rank zero.

If $M(A)$ has real rank zero, then the corona algebra $C(A) = M(A)/A$ has real rank zero. However, there are examples of C*-algebras with $RR(A) = RR(C(A)) = 0$ but $RR(M(A)) \neq 0$. It is shown in [Zh 3] that if $A = B \otimes K$, where $B$ is the Bunce-Deddens algebra, then $RR(A) = RR(C(A)) = 0$ but $RR(M(A)) \neq 0$. We se
from Theorem A that if \( A \) is a \( \sigma \)-unital \( C^* \)-algebra such that for any \( n \), \( K_1(B) = 0 \) for every hereditary \( C^* \)-subalgebra \( B \) of \( M_n(M(A)) \) which contains \( M_n(A) \) but is not \( M_n(A) \), then the corona algebras \( C(A) \) has real rank zero. It was shown by Larry Brown that \( K_1(M(B)) = 0 \), where \( B \) is stably isomorphic to a Bunce-Deddens algebra. Notice that \( K_1(B) \neq 0 \). We show in this section that many simple \( C^* \)-algebras \( A \) with real rank zero have this phenomenon. Hence corona algebras of these algebras have real rank zero.

**Lemma 3.1.** Let \( A \) be a non-elemetary simple \( C^* \)-algebra with real rank zero and \( p \) a non-zero projection in \( A \), then for any positive integer \( k \), there are \( k \) non-zero mutually equivalent and mutually orthogonal projections \( q_i \leq p \) \( (i = 1, 2, \ldots, k) \).

**Proof.** Since \( A \) is a non-elemetary, \( p \not\sim p \) is also non-elemetary. Moreover \( p \not\sim p \) has real rank zero (See [BP, 2.8]). We may assume that \( p = 1 \). There is a nonzero projection \( q \) in \( A \) such that \( 1 - q \not\sim 0 \). Suppose that \( a \) is a nonzero positive element in \( (1 - q)A(1 - q) \). By \([Cu, 1.8]\), there is a nonzero element \( y \) in \( A \) such that

\[
y^* y \in qAq, \quad y y^* \in (1 - q)A(1 - q).
\]

Let \( y = u|y| \) be the polar decomposition of \( y \) in \( A^{**} \). Then by [Li, 1.2], the map

\[
\phi(x) = u x u^*
\]

is an isomorphism from \( \text{Her}(|y|) \) onto \( \text{Her}(|y^*|) \). Let \( q_1 \) be a nonzero projection in \( \text{Her}(|y|) \). Then \( u q_1 \in A \) (See [Li 2, 1.2]). Moreover,

\[
(u q_1)^*(u q_1) = q_1
\]

and

\[
(u q_1)(u q_1)^* = q_2
\]

is a projection in

\[
\text{Her}(|y^*|) \subset (1 - p)A(1 - p).
\]

The lemma then follows by induction.

**Lemma 3.2.** Let \( A \) be a simple \( C^* \)-algebra with stable rank one and \( p \) and \( q \) two nonzero, mutually orthogonal projections in \( A \). Suppose that \( u \in U(pAp) \), then there is a \( v \in U(qAq) \) such that

\[
u + v \in U_0((p + q)A(p + q)).
\]

**Proof.** We may assume that \( p + q = 1 \). Working in \( A \otimes K \), let

\[
e = \text{diag}(0, 1, 1, \ldots).
\]

By [Bn 1], there is a \( W \in M(A \otimes K) \) such that
\[ W^*W = p \otimes e, \quad WW^* = q \otimes e. \]

By [Rff, 2.10], there is \( v \in U(qAq) \) such that
\[
[\text{diag}(v, q, q, \ldots)] = [q + Wu^* \otimes e_{22} W^* + (q \otimes e - q_1)]
\]
in \( K_1(qAq) \), where \( \{e_{ij}\} \) is a matrix unit for \( K \) and
\[
q_1 = W(u \otimes e_{22})W^* W(u^* \otimes e_{22})W^* = W(p \otimes e_{22})W^*.
\]

Set \( W_0 = W(p \otimes e) + W^*(q \otimes e), \) and \( W_1 = \text{diag}(1, W_0). \) Then
\[
[W_1 \text{diag}(v + p, u + q, 1, 1, \ldots)W_1^*] = [v + W(u \otimes e_{22})W^* + (e - q_1)] = 0
\]
in \( K_1(A) \). So \( [\text{diag}(v + p, u + q, 1, 1, \ldots)] = 0 \) in \( K_1(A) \). Hence, by [Rff, 2.10], \( u + v \in U_0(A) \). This completes the proof.

**Lemma 3.3.** Let \( A \) be a (non-unital) \( \sigma \)-unital simple \( C^* \)-algebra with real rank zero and stable rank one. Suppose that \( A \) satisfies condition (a), then the unitary group of \( M(A) \) is connected.

**Proof.** We may assume that \( A \) is non-elementary. Suppose that \( v \) is a unitary in \( M(A) \). It follows from [Ell 1, 2.4] (see 2.6 also) that there are unitaries \( u_1 \) and \( u_2 \) in \( M(A) \) such that
\[
\|u - u_1u_2\| < 1
\]
and
\[
u_1 = \sum_{n=1}^{\infty} (e_{2n} - e_{2n-2})u_1(e_{2n} - e_{2n-2})
\]
\[
u_2 = \sum_{n=1}^{\infty} (e_{2n+1} - e_{2n-1})u_2(e_{2n+1} - e_{2n-1})
\]
where \( e_n(e_0 = 0) \) is an approximate identity for \( A \). So \( u \) is connected with \( u_1u_2 \). To show that \( u \) is in \( U_0(M(A)) \), it is enough to show that both \( u_1 \) and \( u_2 \) are in \( U_0(M(A)) \). Therefore we may assume that
\[
u = \sum_{n=1}^{\infty} (e_n - e_{n-1})u(e_n - e_{n-1}).
\]
Set \( u_n = (e_n - e_{n-1})u(e_n - e_{n-1}) \). These \( u_n \) are unitaries in
\[
(e_n - e_{n-1})A(e_n - e_{n-1}).
\]
By factoring \( u \) further, we may assume that \( u_{2n} = (e_{2n} - e_{2n-1}), n = 1, 2, \ldots \). By Lemma 3.1, we write
\[ e_{2n} - e_{2n-1} = \sum_{i=1}^{2n+1} p_{2n}^{(i)} \]

where each \( p_{2n}^{(i)} \) is a projection, \( p_{2n}^{(i)} \neq 0 \) if \( i \neq 2n + 1 \), \( p_{2n}^{(i)} \) are mutually orthogonal and \( p_{2n}^{(i)} \sim p_{2n}^{(j)} \) if \( i, j \neq 2n + 1 \). Suppose that \( s_{2n}^{(i)} \) are partial isometries in \((e_{2n} - e_{2n-1})A(e_{2n} - e_{2n-1})\) such that

\[ (s_{2n}^{(i)})(s_{2n}^{(ij)}*) = p_{2n}^{(i)} \quad (s_{2n}^{(ij)})(s_{2n}^{(ij)}*) = p_{2n}^{(i)}, \]

\( i, j = 1, 2, \ldots, 2n \). By Lemma 3.2, there is a unitary \( \nu_{2n}^{(2i-1)} \) in \( p_{2n}^{(2i-1)}A p_{2n}^{(2i-1)} \) such that \( \nu_{2n}^{(2i-1)} + u_{2i-1}^{(2i-1)} \in U_0(A) \), \( i = 1, 2, \ldots, n \). Set

\[ s_{2n}^{(i)}(t) = p_{2n}^{(2i-1)} \cos t - s_{2n}^{(2i-1, 2i)} \sin t + (s_{2n}^{(2i-1, 2i)})* \sin t + p_{2n}^{(2i)} \cos t, \]

\[ y_{2n} = \sum_{i=1}^{n} \nu_{2n}^{(2i-1)} p_{2n}^{(2i-1)} + \sum_{i=1}^{n} p_{2n}^{(2i)} + p_{2n}^{(2n+1)}, \]

\[ z_{2n} = \sum_{i=1}^{n} p_{2n}^{(2i-1)} + \sum_{i=1}^{n} (\nu_{2n}^{(2i-1)})(p_{2n}^{(2i)}) + p_{2n}^{(2n+1)} \]

and

\[ w_{2n}(t) = y_{2n} \left( \sum_{i=1}^{n} s_{2n}^{(i)}(t) + p_{2n}^{(2n+1)} \right) z_{2n} \left( \sum_{i=1}^{n} s_{2n}^{(i)}(t) + p_{2n}^{(2n+1)} \right)^* \]

\[ w_{2n-1}(t) = u_{2n-1}. \]

So \( \{w_n(t)\} \) is equi-continuous on \([0, \pi/2]\). Thus \( w(t) = \sum_{n=1}^{\infty} w_n(t) \) is a norm continuous path in \( U(M(A)) \) with \( w(0) = u \) and

\[ w(\pi/2) = \sum_{n=1}^{\infty} u_{2n-1} + \sum_{n=1}^{\infty} \left( \sum_{i=1}^{n} \nu_{2n}^{(2i-1)} p_{2n}^{(2i-1)} + \sum_{i=1}^{n} (\nu_{2n}^{(2i-1)})(p_{2n}^{(2i)}) + p_{2n}^{(2n+1)} \right) \]

By rearranging terms, we may write

\[ w(\pi/2) = \sum_{n=1}^{\infty} w_n \]

where each \( w_n \) is a unitary in

\[ U_0((e_n' - e_{n-1}')A(e_n' - e_{n-1}')) \]

and \( \{e_n'\} \) is an approximate identity consisting of projections. Therefore, since \( A \) satisfies condition (a), there is an integer \( k \) such that for each \( n \) there are

\[ h_n^{(i)} \in (e_n' - e_{n-1}')A(e_n' - e_{n-1}'), \quad i = 1, 2, \ldots, k \]

such that
\[ w_n - \prod_{j=1}^{k} \exp(ih_n^{(j)}) \leq 1/2^n. \]

Since \( A \) has real rank zero, we may further assume that \( 0 \leq h_n^{(j)} \leq 2\pi \). Thus

\[ \left\| w(\pi/2) - \sum_{n=1}^{\infty} \left( \prod_{j=1}^{k} \exp(ih_n^{(j)}) \right) \right\| < 1 \]

Hence \( w(\pi/2) \) and \( \sum_{n=1}^{\infty} \left( \prod_{j=1}^{k} \exp(ih_n^{(j)}) \right) \) are in the same connected component in \( U(M(A)) \). Notice that \( \{ \prod_{j=1}^{k} \exp(ih_n^{(j)}(1 - t)) \} \) is equi-continuous on \([0, 1]\). Set

\[ v(t) = \sum_{n=1}^{\infty} \left( \prod_{j=1}^{k} \exp(ih_n^{(j)}(1 - t)) \right). \]

Then \( v(t) \) is a norm continuous path in \( U(M(A)) \) and

\[ v(0) = \prod_{j=1}^{k} \exp \left( i \sum_{n=1}^{\infty} h_n^{(j)} \right), v(1) = 1 \]

This completes the proof.

**Corollary 3.4.** Let \( A \) be a \( \sigma \)-unital simple \( C^* \)-algebra with real rank zero and stable rank one. If \( A \) satisfies condition (a), then for any \( n \) and any unital hereditary \( C^* \)-subalgebra \( B \) of \( M(M_n(A)) \) which contains \( M_n(A) \) but not \( M_n(A) \), the unitary group of \( B \) is connected.

**Proof.** Suppose that \( B = pM(M_n(A))p \) for some projection \( p \) in \( M(M_n(A)) \). If \( q \) is a projection in \( pM_n(A)p \subseteq M_n(A) \), then by [Zh 5, 3.3], \( q \) has the form described in 2.1. By 2.1, \( pM_n(A)p \) has the same properties \( A \) has. Since \( B = M(pM_n(A)p) \), 3.4 follows from 3.3.

**Theorem 3.5.** Let \( A \) be a \( \sigma \)-unital simple \( C^* \)-algebra with real rank zero and stable rank one. If \( A \) satisfies condition (a), for any \( n \) and hereditary \( C^* \)-subalgebra \( B \) of \( M(M_n(A)) \) which contains \( M_n(A) \) but is not \( M_n(A) \), \( K_1(B) = 0 \).

**Proof.** It follows from 3.4 as in 2.7.

**Remark 3.6.** A special case of 3.5 was proved by Larry Brown a few years ago. He showed that 3.5 is true for non-unital \( C^* \)-algebras which are stably isomorphic to Bunce-Deddens algebras.

**Theorem 3.7.** Let \( A \) be a \( \sigma \)-unital simple \( C^* \)-algebra with real rank zero and stable rank one. If \( A \) satisfies condition (a), then

\[ \text{RR}(M(A)/A) = 0. \]

**Proof.** This follows from 3.4 and Theorem A immediately.

We have shown that the Weyl-von Neumann theorem for self-adjoint elements holds for AF-algebras and their multiplier algebras. In this section, we show that if $u$ is a unitary in the multiplier algebra of a $\sigma$-unital $C^*$-algebra with stable rank one, (FU) and finitely many ideals in its corona algebra $A$, then for any $\varepsilon > 0$, there is an element $a \in A$ and an approximate identity $\{e_n\}$ consisting of projections such that

$$u = \sum_{n=1}^{\infty} \alpha_n (e_n - e_{n-1}) + a$$

with $|\alpha_n| = 1$ and $\|a\| < \varepsilon$. Consequently, in these cases, $\text{cer}(C(A)) \leq 1 + \varepsilon$.

Our 4.1 is inspired by a result of Mikael Rørdam that if both $I$ and $Q$ have stable (FU), then $A$ has (FU), where $I$, $Q$ and $A$ are as in 4.1. However, we do not need the stable assumption and our proof is different.

THEOREM 4.1. Let

$$0 \rightarrow I \rightarrow A \rightarrow Q \rightarrow 0$$

be a short exact sequence of $C^*$-algebras.

(i) If $I$ has (FU), $Q$ has real rank zero and $\text{cer}(Q) \leq 1 + \varepsilon$; then $A$ has real rank zero and $\text{cer}(A) \leq 1 + \varepsilon$;

(ii) In (i) if $\tilde{A}$ has connected unitary group, $A$ has (FU);

(iii) If $A$ has (FU), then $I$ has (FU) and $Q$ has real rank zero and $\text{cer}(Q) \leq 1 + \varepsilon$;

(iv) If $I$ has (FU), $Q$ has (FU), then $A$ has (FU).

PROOF. By [Ph 1, 1.4], we may assume that $A$ is unital.

(i) Since $I$ has (FU), it follows from [Zh 3, 3.3] and [Ch, 2] that $A$ has real rank zero. Let

$$I^\perp = \{b \in A : bi = ib = 0, \forall i \in A\}$$

Then $I$ is an ideal of $A$. Moreover $I^\perp + I$ is an essential ideal of $A$. By [P 2], we may assume that $A$ is a $C^*$-subalgebra of $M(I^\perp + I) \cong M(I^\perp) \oplus M(I)$. Hence $A$ may be written as $A_1 \oplus A_2$, where $A_1$ is a $C^*$-subalgebra of $M(I^\perp)$ and $A_2$ is a $C^*$-subalgebra of $M(I)$. Since $A_1$ is isomorphic to a hereditary $C^*$-subalgebra of $Q$, by [Li 3, 1.4], $A_1$ has weak (FU). Therefore we may assume that $I$ is an essential ideal of $A$. Next we assume that $I$ is $\sigma$-unital. It follows from [M, Theorem 9 and Introduction] (see [Zh 1, 3.1] also) that for any selfadjoint element $h \in A_{1,a}$ and $\delta > 0$, there are an approximate identity $\{e_n\}$ consisting of projections and an element $c \in I_{a,a}$ such that
\[ h = \sum_{n=1}^{\infty} \lambda_n (e_n - e_{n-1}) + c \]

and \( \|c\| < \delta \), where \( \{\lambda_n\} \) is a bounded sequence of real numbers. Let \( u \in U_0(A) \).
Then \( \pi(u) \in U_0(Q) \), where \( \pi \) is the map: \( A \to Q \). For \( 1 > \varepsilon > 0 \), there is \( h \in Q_{s.a.} \) such that

\[ \| \pi(u) - \exp(ih) \| < \varepsilon/2^6. \]

Therefore, there is \( h \in A_{s.a.} \) and \( b \in I \) such that

\[ \| u - \exp(ih) - b \| < \varepsilon/2^6. \]

Therefore, by choosing a small \( \delta \), one obtains

\[ \left\| \exp(ih) - \sum_{n=1}^{\infty} e^{i\lambda_n}(e_n - e_{n-1}) \right\| < \varepsilon/2^6. \]

(Notice also that

\[ \exp(ih) - \sum_{n=1}^{\infty} e^{i\lambda_n}(e_n - e_{n-1}) \in I. \])

There is an integer \( N \) such that

\[ \| e_N b e_N - b \| < \varepsilon/2^7. \]

Therefore

\[ \left\| u - \sum_{n=1}^{\infty} e^{i\lambda_n}(e_n - e_{n-1}) - e_N b e_N \right\| < \sum_{k=1}^{3} \varepsilon/2^{k+4}. \]

Let \( x = \sum_{n=1}^{N} e^{i\lambda_n}(e_n - e_{n-1}) + e_N b e_N \). Clearly

\[ \| e_N u - u e_N \| < \sum_{k=1}^{3} \varepsilon/2^{k+2} + \sum_{k=1}^{3} \varepsilon/2^{k+4} = 4/16\varepsilon. \]

Set \( v = x |x|^{-1} \) (the inverse is taken in \( e_N A e_N \)), then

\[ \| v - x \| \leq \| x \| \| |x|^{-1} - e_N \| < (1 + 3/16\varepsilon)/(1 - 3/16\varepsilon)(3/16\varepsilon) < 5/16\varepsilon. \]

Hence

\[ \left\| u - \sum_{n=N+1}^{\infty} e^{i\lambda_n}(e_n - e_{n-1}) - v \right\| < \sum_{k=1}^{3} \varepsilon/2^{k+4} + 5/16\varepsilon = 6/16\varepsilon. \]

By [Li 3, 1.3], \( e_N A e_N = e_N I e_N \) has (FU). So there is an \( h' \in (e_N I e_N)_{s.a.} \) such that

\[ \left\| v - e_N - \sum_{k=1}^{\infty} (ih')^n/n! \right\| < 10/16\varepsilon. \]
We conclude that there is an \( h_0 \in A_{s.a.} \) such that
\[
\|u - \exp(ih_0)\| < \varepsilon.
\]
Hence \( \text{cer}(A) \leq 1 + \varepsilon. \)

Now we reduce the general case to the case that \( I \) is \( \sigma \)-unital. There are \( \{h_n\} \) in \( A_{s.a.} \) and \( \{j_n\} \) in \( I \) such that
\[
\|u - \exp(ih_n) - j_n\| \to 0.
\]
Let \( A_0 \) be the \( C^* \)-subalgebra generated by \( \{h_n, j_n\} \) and \( I_0 \) be the ideal \( A^0 \cap I. \) Since \( I_0 \) is separable and \( I \) has real ranks zero, there is an increasing sequence of projections \( \{p(0, n)\} \) in \( I \) such that
\[
\|a(1 - p(0, n))\| \to 0
\]
for all \( a \in I. \) Let \( A_1 \) be the \( C^* \)-subalgebra generated by \( A_0 \) and \( \{p(0, n)\} \) and \( I_1 \) be ideal \( A_1 \cap I. \) Suppose that \( \{u_k\} \) is a dense sequence of normal partial isometries in \( I_1. \) Since \( I \) has (FU), there are projections \( \{p(1, n)\} \) in \( I \) such that each \( u_k \) can be approximated by linear combinations of finitely many orthogonal projections in \( \{p(1, n)\}. \) Let \( A_2 \) be the \( C^* \)-subalgebra generated by \( A_1 \) and \( \{p(1, n)\} \) and \( I_2 \) be the ideal \( A_2 \cap I. \) If \( A_m \) and \( I_m \) have been constructed, choose an dense sequence of normal partial isometries \( \{v_k\} \) in \( I_m \) and a sequence of projections \( \{p(m, n)\} \) in \( I \) such that each \( v_k \) can be approximated by linear combinations of finitely many orthogonal projections in \( \{p(m, n)\}. \) Then let \( A_{m+1} \) be the \( C^* \)-subalgebra generated by \( A_m \) and \( \{p(m, n)\} \) and \( I \) be the ideal \( A_{m+1} \cap I. \) Set \( A_\infty = \bigcup A_m \) and \( I_\infty = \bigcup I_m \) \( (= I \cap A_\infty). \) It is then easy to check \( A_\infty \) is separable and \( I_\infty \) is separable and has (FU). Now consider the elements \( w_n = \exp(ih_n) + j_n. \) As before, we may assume that \( I_\infty \) is essential in \( A_\infty. \) Then we can apply the above argument to the elements \( w_n \) and (i) follows.

(ii) This is an immediate consequence of (i).

(iii) That \( I \) has (FU) follows from [Li 3, 1.3] and that \( Q \) has real rank zero and \( \text{cer}(Q) \leq 1 + \varepsilon \) follows from [Ph 1, 1.6].

(iv) For every \( u \in U(A), \pi(u) \in U_0(A). \) So (iv) follows from the proof of (i).

**Theorem 4.2.** Let \( A \) be a \( \sigma \)-unital \( C^* \)-algebra with (FU) and with stable rank one. If there is a sequence of ideals
\[
A = I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n = M(A)
\]
such that \( I_k/I_{k-1} \) is simple for each \( k = 1, 2, \ldots, n, \) then \( \text{cer}(M(A)) \leq 1 + \varepsilon \) and \( M(A) \) has real rank zero. Moreover, if \( u \in M(A) \) and \( \varepsilon > 0, \) then there are \( a \in A \) and an approximate identity \( \{e_n\} \) consisting of projections such that
\[ u = \sum_{n=1}^{\infty} \alpha_n (e_n - e_{n-1}) + a \]

with \(|\alpha_n| = 1\) and \(\|a\| < \varepsilon\). Consequently \(\text{cer}(C(A)) = 1 + \varepsilon\).

\text{Proof.} It follows \([Zh\ 2,\ 1.3]\) that \(I_1/A\) is purely infinite. It then follows from \([Ph\ 2]\) that \(\text{cer}(I_1/A) \leq 1 + \varepsilon\). Moreover, \(I_1/A\) has real rank zero. It follows from 4.1 that \(\text{cer}(I_1) \leq 1 + \varepsilon\) and \(I_1\) has real rank zero. By induction and repeated application of 4.1, we conclude that \(\text{cer}(M(A)) \leq 1 + \varepsilon\) and \(M(A)\) has real rank zero. Since \(M(A)\) has connected unitary group, then by 4.1, \(M(A)\) has (FU). To show that every unitary \(u \in M(A)\) has the form

\[ u = \sum_{n=1}^{\infty} \alpha_n (e_n - e_{n-1}) + a \]

as described in the theorem, we use the proof of 3.1 in \([Zh\ 1]\). Since \(M(A)\) has (FU), for any \(\varepsilon > 0\), there is a selfadjoint element \(h \in M(A)\) such that

\[ \|u - \exp(ih)\| < \varepsilon. \]

Since \(M(A)\) has real rank zero, this implies that

\[ \left\| u - \sum_{n=1}^{\infty} e^{\lambda_n} (e_n - e_{n-1}) \right\| < \varepsilon, \]

where \(\{\lambda_n\}\) is a bounded sequence of real numbers and \(\{e_n\}\) is an approximate identity for \(A\) consisting of projections. It follows from \([Zh\ 1,\ 3.9]\) (the equivalence of (b) and (e)) that we may assume that

\[ u = \sum_{n=1}^{\infty} (e_n - e_{n-1})u(e_n - e_{n-1}) + a \]

with \(a \in A\) and \(\|a\| < \varepsilon/4\). Since

\[ \|(1 - e_n)a\| \to 0, \quad \text{as} \quad n \to \infty, \]

by passing a subsequence if necessary, we may assume that

\[ \|(e_n - e_{n-1})u - u(e_n - e_{n-1})\| \to 0 \]

as \(n \to \infty\). As in the proof of 4.1 (i), by a standard argument, we may write

\[ (e_n - e_{n-1})u(e_n - e_{n-1}) = w_n + a_n, \]

where \(w_n\) is a unitary in \((e_n - e_{n-1})A(e_n - e_{n-1}), a_n \in (e_n - e_{n-1})A(e_n - e_{n-1})\) and \(\|a_n\| \leq \varepsilon/2^{n+4}\). For each \(n\), there is a selfadjoint element \(h_n \in (e_n - e_{n-1})A(e_n - e_{n-1})\) with \(\|h_n\| \leq 2\pi\) such that

\[ \|w_n - \exp(ih_n)\| < \varepsilon/2^{n+4}. \]
This implies that there is \( x \in M(A)_{s.a.} \) such that
\[
  u = \exp(ix) + a + b,
\]
where \( b = \sum_{n=1}^{\infty} a_n \in A \) and \( \|b_n\| < \varepsilon/4 \). Again, since \( M(A) \) has real rank zero, we obtain
\[
  u = \sum_{n=1}^{\infty} x_n(e_n - e_{n-1}) + (a + b + c),
\]
where \( \|c\| < \varepsilon/2 \). Finally, we notice that, since \( M(A) \) has real zero, the last conclusion follows immediately from 4.1. This completes the proof.

**Remark 4.3.** If we assume that \( A \) has stable rank one then it follows from section two that \( M(A) \) has connected unitary group.

**Theorem 4.4.** The multiplier algebra \( M(A) \) of any matroid \( C^* \)-algebra \( A \) has (FU). Moreover, for any \( u \in U(M(A)) \) and \( \varepsilon > 0 \), there is an approximate identity \( \{e_n\} \) of \( A \) consisting of projections and an element \( a \in A \) such that
\[
  u = \sum_{i=1}^{\infty} x_i(e_i - e_{i-1}) + a,
\]
where \( |x_i| = 1 \) and \( \|a\| < \varepsilon \). Furthermore, \( \text{cer}(C(A)) \leq 1 + \varepsilon \).

**Proof.** If \( A \) is a finite matroid \( C^* \)-algebra then, by [Ell 1, Theorem 3.1], \( M(A)/A \) is simple. If \( A \) is infinite, then by [Ell 1, 3.2], \( M(A)/A \) has only one nontrivial ideal. So 4.2 applies.

**Examples 4.5.** There are many examples of \( C^* \)-algebras satisfying the conditions in 4.2. It follows from [Li 1, Theorem 2] that every simple AF-algebra with trace space having finitely many extreme points satisfying the condition in 4.2. For simple AF-algebras with infinitely many points in their extreme sets of trace spaces, if they have continuous scales then, by [Li 1, Theorem 1], \( M(A)/A \) are simple. In fact every \( \sigma \)-unital simple \( C^* \)-algebra with continuous scale has a simple corona algebra \( M(A)/A \). We also notice that every \( \sigma \)-unital simple \( C^* \)-algebra with real rank zero has many hereditary \( C^* \)-subalgebras with continuous scales (see [Li 5]).

**Corollary 4.6.** Let \( A \) be a \( \sigma \)-unital simple \( C^* \)-algebra with stable rank one, (FU) and continuous scale, then \( M(A) \) has (FU). Moreover \( \text{cer}(C(A)) \leq 1 + \varepsilon \).

**Theorem 4.7.** Let \( A \) be a \( \sigma \)-unital purely infinite simple \( C^* \)-algebra. If \( K_1(A) = 0 \), then \( M(A) \) has (FU). Moreover, \( \text{cer}(C(A)) = 1 \).

**Proof.** It follows from [Zh 3, 1.2 and 2.6] \( A \) is stable, \( M(A) \) has real rank zero and \( M(A) \) has connected unitary group. By [Ph 2], \( \text{cer}(A) \leq 1 + \varepsilon \) and \( \text{cer}(C(A)) \leq 1 + \varepsilon \). Since \( A \) is stable and \( K_1(A) = 0 \), \( A \) has (FU). By 4.1,
\[
\text{cer}(M(A)) \leq 1 + \varepsilon. \text{ Since } M(A) \text{ has real rank zero and connected unitary group, } M(A) \text{ has (FU).}
\]

4.8. We notice that \( C^* \)-algebras \( O_n \) are purely infinite, simple and have \( K_1(O_n) = 0 \). So, by 4.8, \( M(O_n) \) has (FU).

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