MATH. SCAND. 77 (1995), 119–128

THE LINEAR GROUPS OF INJECTIVE FACTORS AND OF MATROID C^*ALGEBRAS ARE CONTRACTIBLE TO A POINT

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Abstract.

It is proved that the general linear groups of all injective factors and of σ-finite factors of type III are contractible to a point. This is also shown to be true for the linear group of matroid C^*-algebras.

0. Introduction.

Let X be a Banach space and $\mathrm{GLG}(X)$ its general linear group of all linear isomorphisms of X onto itself. The group $\mathrm{GLG}(X)$ with norm topology is said to be contractible to a point if there exists a continuous map $F\colon \mathrm{GLG}(X) \times [0,1] \to \mathrm{GLG}(X)$ such that $F(B,0) = B$, $F(B,1) = \mathrm{Id}_X$ for every $B \in \mathrm{GLG}(X)$. J. W. Milnor [6] established, that $\mathrm{GLG}(X)$ is contractible to a point if and only if $\mathrm{GLG}(X)$ has trivial homotopy groups. N. Kuiper [4] proved that the group $\mathrm{GLG}(H)$ of a real, complex or quaternionic infinite-dimensional separable Hilbert space H is contractible to a point. D. Arlt [3] and G. Neubauer [9] obtained the analogous results for the groups $\mathrm{GLG}(l_p)$, $1 \leqq p \leqq \infty$ and $\mathrm{GLG}(c_o)$. A far-reaching common generalization of these results was given by B. S. Mityagin in [7], where he introduced two interesting properties of Banach spaces: infinitely divisibility (ID) and smallness of operator blocks (SOB). The group $\mathrm{GLG}(X)$ of a Banach space X which has (ID) and (SOB) is contractible to a point. Checking up the properties (ID) and (SOB), B. S. Mityagin [7], B. S. Mityagin and I. S. Edelstein [8] exhibited many examples of Banach spaces X with the linear group $\mathrm{GLG}(X)$ contractible to a point. Among these examples we mention only $L_\infty(0,1)$, $B(H)$, $C(H)$ – the algebras of all bounded measurable functions on $(0,1)$, all bounded linear operators on H and all compact operators on H, respectively. It can be noticed that the first and the second examples are hyperfinite von Neumann algebras and the third example is a matroid C^*-algebra.

Received October 11, 1993; in revised form March 7, 1994.

Here, using the B. S. Mityagin's scheme, we prove that the linear groups of all injective factors and of matroid C^*-algebras are contractible to a point.

1. Preliminaries.

Let X and Y be Banach spaces. A linear bijective mapping π of X onto Y is called an isomorphism if there exist constants $c_1, c_2 > 0$ such that

$$(1) \qquad c_1 \|\pi(x)\|_Y \leqq \|x\|_X \leqq c_2 \|\pi(x)\|_Y, \quad x \in X.$$

In this case we will write $\pi X \xrightarrow{\sim} Y$ or $X \simeq Y$.

In the case when X and Y are C^*-algebras, π is called a $*$-isomorphism if it satisfies the conditions $\pi(xy) = \pi(x)\pi(y)$ and $\pi(x) = \pi(x)^*$ for every $x, y \in X$ instead of the condition (1). In this case we have $\|\pi(x)\|_Y = \|x\|_X$.

For any Banach space X we denote by $L(X)$ the space of all continuous linear operators in X with usual operators norm, and by $l_p(X)$, $1 \leqq p \leqq \infty$ (respectively, $c_0(X)$), the space of all sequences $x = (x_1, x_2, \ldots)$ with $x_j \in X$ and $(\|x_j\|_X) \in l_p$ (respectively, $\|x_j\|_X \to 0$) normed by $\|x\|_{l_p(x)} = \|(\|x_j\|_X)\|_{l_p}$ (respectively by $\|x\|_{c_0(x)} = \sup \|x_j\|_X$).

The definition and the assertion cited below are due to B. S. Mityagin (see [7]).

DEFINITION 1. A Banach space X is said to be *infinitely divisible* (ID) if

(a) there exists a sequence $(p_k)_{k=1}^{\infty} \subset L(X)$ of mutually disjoint projections such that the series $\sum_{k=1}^{\infty} p_k$ strongly unconditionally converges to Id_X;

(b) for any $k = 1, 2, \ldots$ there exists an isomorphism τ_k: $P_k X \hookrightarrow X$, where $P_k X$ denotes the image of P_k;

(c) if $i_k: P_k X \to X$ are identity embeddings, then the operators $T = \sum_{k=1}^{\infty} i_{k-1} \tau_{k-1}^{-1} \tau_k P_k$ and $T' = \sum_{k=0}^{\infty} i_{k+1} \tau_{k+1}^{-1} \tau_k P_k$ are well-defined and belong to $L(X)$;

(d) for any $B \in L(X)$ the operator $\tilde{B} = \sum_{=1}^{\infty} i_k \tau_k^{-1} B \tau_k P_k$ is well-defined, $\tilde{B} \in L(X)$ and $\|\tilde{B}\|_{L(X)} \leqq \gamma \|B\|_{L(X)}$ for some $\gamma > 0$.

The following proposition is very useful for verification of (ID) in concrete Banach spaces (see [7]).

PROPOSITION 1.1. *If a Banach space X is isomorphic either to $l_p(X)$ for some $p \in [1, \infty]$, or to $c_0(X)$, then $X \in$ (ID).*

We recall also the decomposition principle [7]:

PROPOSITION 1.2. *Let X and Y be Banach spaces and*

1) $X \in (\mathrm{ID})$;
2) *X is isomorphic to a complemented subspace of Y;*
3) *Y is isomorphic to a complemented subspace of X;*
Then $X \simeq Y$.

Since $l_p(X) \simeq l_p(l_p(X))$ and $c_0(X) \simeq c_0(c_0(X))$ for every Banach space X and every $p \in [1, \infty]$, the following lemma is valid.

LEMMA 1.3. *The spaces $l_p(X)$, $1 \leqq p \leqq \infty$ and $c_0(X)$ are infinitely divisible for every Banach space X.*

It follows immediately from this lemma and from the decomposition principle that, in order to check the property (ID) in any Banach space X, it is sufficient to find a complemented subspace in X isomorphic to $l_p(X)$ for some $p \in [1, \infty]$ (or, to $c_0(X)$).

DEFINITION 2. A Banach space X is said to have the *property of smallness of operator blocks* ($X \in (\mathrm{SOB})$), if for any compactum $\mathscr{B} \subset L(X)$ and any $\varepsilon > 0$ there exist projections Q_1 and Q_2 such that
(a) $Q_1 Q_2 = Q_1 Q_2 = 0$;
(b) $\|Q_1 B Q_2\|_{L(X)} < \varepsilon$ for any $B \in \mathscr{B}$;
(c) there exist isomorphisms $\sigma_k \colon Q_k X \xrightarrow{\ \simeq\ } X$, $k = 1, 2$.

The following result of B. S. Mityagin [7] gives sufficient conditions for contractibility to a point of the general group of a Banach space X.

THEOREM 1.4. *If $X \in (\mathrm{ID})$ and $X \in (\mathrm{SOB})$, then $\mathrm{GLG}(X)$ is contractible to a point.*

Let us fix the following notation (see for example [10]): if M is a von Neumann algebra with the C^*-norm $\| \ \|_M$ then 1_M is the unit of M, M_* is the predual space of M, $M_+ = \{x^*x, x \in M\}$ is the cone of positive elements of M, the symbol "$\leqq$" denotes the order relation on M generated by M_+, and P_M is the lattice of all projections of M.

2. Contractibility to a point of the linear groups of injective factors and factors of type III.

Since contractibility of a linear group is an invariant with respect to an isomorphism of Banach spaces, every factor which is isomorphic to $B(H)$ has a homotopically trivial linear group [8]. Hence contractibility to a point of $\mathrm{GLG}(M)$ for a properly infinite injective factor M follows from the following theorem:

THEOREM 2.1. *The injective factors of the types $\mathrm{I}_\infty, \mathrm{II}_\infty$ and III are mutually linear-topologically isomorphic.*

PROOF. The space $B(H)$ is infinitely divisible, so that we may use the decomposition principle. Since M is an injective factor, there exists a continuous projection from $B(H)$ onto M. It means that M is complemented in $B(H)$. Further, since M is a properly infinite factor, M is isomorphic to $M \bar{\otimes} B(H)$ [5], that implies that $B(H)$ is complemented in M. Hence, by Proposition 1.2, $M \simeq B(H)$.

The following lemmas prove contractibility of the linear group of the hyperfinite factor of type II_1.

LEMMA 2.2. *Let R be the hyperfinite factor of type* II_1. *Then R is infinitely divisible.*

PROOF. It is sufficient to show that $l_\infty(R)$ is isomorphic to some complemented subspace of R. Let $(p_k)_{k=1}^\infty \subset P_R$ be a sequence of mutually orthogonal projections. Put $E = \{x \in R\colon p_k x p_j = 0,\ k \neq j\}$. It is clear that $E = \{x \in R\colon x = (\mathrm{so})\text{-}\sum_{k=1}^\infty p_n x p_n\}$. Since $\Phi\colon x \to (\mathrm{so})\text{-}\sum_{n=1}^\infty p_n x p_n$ is a projection, E is a complemented subspace in R.

We put $R_k = p_k R p_k = \{p_k x p_k,\ x \in R\}$, $k = 1, 2$. Then R_k is also a hyperfinite factor of the type II_1 and, by the uniqueness of this factor, we get that R_k and R are isometric. Let us denote by τ_k the corresponding isometry and consider the mapping $T\colon l_\infty(R) \to R$, $T((x_k)_{k=1}^\infty) = \sum_{k=1}^\infty \tau_k^{-1}(x_k)$. Since $R_k \subset E$ for all k, we have $T\colon l_\infty(R) \to E$. It is clear that T is an isometric and bijective mapping of $l_\infty(R)$ onto E. Thus, $T\colon l_\infty(R) \xrightarrow{\sim} E$ and, therefore, $R \in (\mathrm{ID})$.

For a projection $p \in \mathsf{P}_M$ we will denote by Q_p the operator from $L(M)$ given by the equality $Q_p x = pxp$, $x \in M$, and by $s(x)$ the support projection of a self-adjoint operator $x \in M$.

LEMMA 2.3. *Let M be a non-atomic von Neumann algebra and let $e_1, e_2 \in \mathsf{P}_M \backslash \{0\}$. Then for every $\varepsilon > 0$ and $B \in L(M)$ there exist $p_1, p_2 \in \mathsf{P}_M \backslash \{0\}$ such that $p_i \leqq e_i$, $i = 1, 2$, and $\|Q_{p_2} B Q_{p_1}\|_{L(M)} < \varepsilon$.*

PROOF. If it is not true, then for some $B \in L(M)$ and $\varepsilon_0 > 0$ we have $\|Q_{p_2} B Q_{p_1}\|_{L(M)} \geqq \varepsilon_0$ for every $p_i \in \mathsf{P}_M \backslash \{0\}$, $p_i \leqq e_i$, $i = 1, 2$.

Let us divide the projection e_1 into $N > 17\,\|B\|_{L(M)}/\varepsilon_0$ mutually orthogonal projections $e_i^1, e_1^2, \ldots, e_1^N$ (this is possible because M is non-atomic). By the assumption, there exists $x_1 \in M$ such that $\|x_1\|_M \leqq 1$ and $\|Q_{e_2} B Q_{e_1^1} x_1\|_M \geqq \varepsilon_0/2$. It is clear that either $\|Q_{e_2} \operatorname{Re}(B Q_{e_1^1} x_1)\|_M \geqq \varepsilon_0/4$ or $\|Q_{e_2} \operatorname{Im}(B Q_{e_1^1} x_1)\|_M \geqq \varepsilon_0/4$. Multiplying if necessary x_1 by i we may assume that the first inequality holds.

We put $\tilde{x}_1 = \operatorname{Re}(B Q_{e_1^1} x_1)$. Let $(Q_{e_2}\tilde{x}_1)_+ - (Q_{e_2}\tilde{x}_1)_-$ be the decomposition of the self-adjoint operator $Q_{e_2}\tilde{x}_1$ into positive and negative parts. Then either

$\|(Q_{e_2}\tilde{x}_1)_+\|_M \geqq \varepsilon_0/8$ or $\|(Q_{e_2}\tilde{x}_1)_-\|_M \geqq \varepsilon_0/8$. Hence there exist a non-zero projection $e_2^1 \leqq s(Q_{e_2}\tilde{x}_1) \leqq e_2$ and a number $\delta_1 = \pm 1$ such that the inequality $\delta_1 Q_{e_2^1}\tilde{x}_1 \geqq \varepsilon_0 e_2^1/16$ holds.

Replacing in the above arguments e_1^1 by e_1^2 and e_2 by e_2^1, we get that there exists a $x_2 \in M$, a non-zero projection $e_2^2 \leqq e_2^1$ and a number $\delta_2 = \pm 1$ such that for $\tilde{x}_2 = \mathrm{Re}(BQ_2 x_2)$, the inequality $\delta_e Q_{e_2^2}\tilde{x}_2 \geqq e_2^2$ holds.

After N steps, we obtain N operators $x_1 \in M$, $\|x_i\|_M \leqq 1$, N non-zero projections $e_2^N \leqq e_2^{N-1} \leqq \ldots \leqq e_2^1$ and N numbers $\delta_i = \pm 1$ such that for $\tilde{x}_i = \mathrm{Re}(BQ_{e_1^i}x_i)$ the inequality $\delta_i Q_{e_2^i}\tilde{x}_i \geqq \varepsilon_0 e_2^i/16$ holds, $i = \overline{1,N}$. It follows from $e_2^N \leqq e_2^i$ for each $i = 1, N$, that

$$\delta_i Q_{e_2^N}\tilde{x}_i = \delta_i e_2^N \tilde{x}_i e_2^N = \delta_i e_2^N e_2^i \tilde{x}_i e_2^i e_2^N \geqq e_2^N(\varepsilon_0 e_2^i/16)e_2^N = \varepsilon_0 e_2^N/16,$$

which implies $\left\|\sum_{i=1}^N \delta_i Q_{e_2^N}\tilde{x}_i\right\|_M \geqq \|N\varepsilon_0 e_2^N/16\|_M > \|B\|_{L(M)}$.

On the other hand

$$\left\|\sum_{i=1}^N Q_{e_1^i}\delta_i\tilde{x}_i\right\|_M = \left\|\sum_{i=1}^N e_1^i\delta_i x_i e_1^i\right\|_M \leqq \max_{1\leqq i\leqq N} \|x_i\|_M \leqq 1$$

implies

$$\left\|\sum_{i=1}^N \delta_i Q_{e_2^N}\tilde{x}_i\right\|_M = \left\|Q_{e_2^N}\left(\sum_{i=1}^N \delta_i\tilde{x}_i\right)\right\|_M \leqq \left\|\sum_{i=1}^N \delta_i\tilde{x}_i\right\|_M =$$

$$= \left\|\sum_{i=1}^N \delta_i \,\mathrm{Re}(BQ_{e_1^i}x_i)\right\|_M = \left\|\mathrm{Re}\left(\sum_{i=1}^N BQ_{e_1^i}\delta_i x_i\right)\right\|_M \leqq$$

$$\leqq \left\|B\left(\sum_{i=1}^N Q_{e_1^i}\delta_i x_i\right)\right\|_M \leqq \|B\|_{L(M)}.$$

LEMMA 2.4. *Let M be a non-atomic von Neumann algebra and let $\mathscr{B} \subset L(M)$ be a compactum. Then for any $\varepsilon > 0$ there exists $p_1, p_2 \in \mathrm{P}_M\backslash(0)$ such that $\|Q_{p_2}BQ_{p_1}\|_{L(M)} < \varepsilon$ for all $B \in \mathscr{B}$.*

PROOF. We choose a finite $\varepsilon/2$-net $\{B_1, \ldots, B_m\}$ in $\mathscr{B}$. By Lemma 2.3, there exist non-zero subprojections $e_1^1 \leqq e_1$ and $e_2^1 \leqq e_2$ for arbitrary $e_1, e_2 \in \mathrm{P}_M\backslash\{0\}$ such that $\|Q_{e_2^1}B_1Q_{e_1^1}\|_{L(M)} < \varepsilon/2$. Similarly, we find e_1^2, e_2^2 such that $0 \neq e_1^2 \leqq e_1^1$, $0 \neq e_2^2 \leqq e_2^1$ and $\|Q_{e_2^2}B_2Q_{e_1^2}\|_{L(M)} < \varepsilon/2$, and so on. The projections $p_1 = e_1^m$ and $p_2 = e_2^m$ obtained in such a way on the mth step satisfy the conditions of the Lemma.

LEMMA 2.5. *If R is the hyperfinite factor of type* II_1, *then $R \in$* (SOB).

PROOF. It follows from Lemma 2.4 that R satisfies the conditions (a) and (b) from the definition of (SOB). Since $Q_{p_i}R = p_iRp_i$, $i = 1, 2$, are again injective factors of type II_1 and all such factors are mutually $*$-isomorphic, we have $Q_{p_i}R \simeq R$. Thus, the condition (c) from the definition of (SOB) also is valid.

Lemma 2.2, 2.5 and Theorem 1.4 imply the following

THEOREM 2.6. *The general linear group of the hyperfinite factor of type* II_1 *is contractible to a point.*

The following theorem shows that the injectivity assumption for factors of type III is redundant.

THEOREM 2.7. *The linear group of a σ-finite factor M of type* III *is contractible to a point.*

The assertion of Theorem 2.7 follows immediately from the two lemmas below and Theorem 1.4

LEMMA 2.8. $M \in$(SOB).

PROOF. The result of Lemma 2.4 is valid for factors of type III, i.e. the conditions (a) and (b) from the definition of (SOB) hold. Since M is a σ-finite factor of type III, we have that p is equivalent to 1_M for every projection $p \in M$ ([10]). This means that $pMp \simeq M$.

LEMMA 2.9. $M \in$(ID).

PROOF. Let $E = \left\{x \in M: x = (\mathrm{so})\text{-}\sum_{n=1}^{\infty} p_n x p_n\right\}$, where $(p_n)_{n=1}^{\infty}$ is a family of mutually orthogonal projections from $\mathsf{P}_M \setminus \{0\}$. As in the proof of Lemma 2.2 we can assert that E is a complemented subspace of M. For every p_n there exists a partial isometry $w_n \in M$ such that $p_n = w_n^* w_n$, $1_M = w_n w_n^*$. It is clear that the mapping $T: l_\infty(M) \to E$ defined by $T((x_n)_{n=1}^{\infty}) = (\mathrm{so})\text{-}\sum_{n=1}^{\infty} w_n^* x w_n$, is an isometry of $l_\infty(M)$ onto E. Therefore, $M \in$(ID).

3. Contractibility to a point of the linear group of the predual spaces.

For any W^*-algebra M, are projection $p \subset \mathsf{P}_M$ and for any functional $f \in M_*$ we define the functional pfp by the equality $pfp(x) = f(pxp)$ and put $pM_*p = \{pfp, f \in M_*\}$. Since $(pM_*p)^* = pMp$ (see [1]) we have the following

LEMMA 3.1. *Let $p \in \mathsf{P}_M \setminus \{0\}$ be such that pMp is $*$-isomorphic to M. Then $pM_*p \simeq M_*$.*

It is proved in [7], [8] that the preduals of the von Neumann algebras $L_\infty(0, 1)$

and $B(H)$ have linear groups which are contractible to a point. The following theorem extends this assertion to some algebras of typ e II and III.

THEOREM 3.2. *If M is a hyperfinite* II_1*-factor or a σ-finite factor of type* III, *then* $\mathrm{GLG}(M_*)$ *is contractible to a point.*

PROOF. It is sufficient to prove that $M_* \in (\mathrm{SOB})$ and that $M_* \in (\mathrm{ID})$. For $i = 1,2$ and for $p_i \in \mathsf{P}_M$ we define the projection S_p: $M_* \to M_*$ by the equality $S_{p_i} f = p_i f p_i$, where $f \in M_*$, and we define $S'_{p_i}: M \to M$ via the duality $\langle S_{p_i} f, x\rangle = \langle f, S'_{p_i} x\rangle$, where $x \in M$. Then $S'_{p_i} x = p_i x p_i$, i.e. $S'_p = Q_p$ for each $p \in \mathsf{P}_M$.

Let $\mathscr{B} \subset L(M_*)$ be a compactum and let B': $M \to M$ be the conjugate operator for B: $M_* \to M_*$. Then $\mathscr{B}' = \{B' \colon B \in \mathscr{B}\}$ is a compactum in $L(M)$. By Lemmas 2.5 and 2.8, for any $\varepsilon > 0$ there exist non-zero nutually orthogonal projections $p_i \in \mathsf{P}_M$, $i = 1,2$, such that Q_{p_i} satisfies the conditions (a), (b) and (c) in the definition of (SOB). It is easy to see (using Lemma 3.1) that $S_{p_i}, i = 1,2$ also satisfy the analogous conditions. Thus, $M_* \in (\mathrm{SOB})$.

Let $\{p_i\}_{i=1}^{\infty}$ be a countable family of mutually orthogonal projections from $\mathsf{P}_M \backslash \{0\}$. For every positive functional $f \in M_*$ we have

$$\left\| \sum_{n=k}^{m} p_n f p_n \right\|_{M_*} = \left(\sum_{n=k}^{m} p_n f p_n \right)(1) = \sum_{n=k}^{m} f(p_n) = f\left(\sum_{n=k}^{m} p_n \right) \to 0$$

as $k, m \to \infty$. Hence the series $\sum_{n=k}^{\infty} p_n f p_n$ converges in M_*. Since every functional from M_* is a linear combination of positive functionals from M_*, the series $\sum_{n=1}^{\infty} p_n f p_n$ converges in M_* for any $f \in M_*$. Furthermore,

$$\left\| \sum_{n=1}^{\infty} p_n f p_n \right\|_{M_*} = \sup_{\|x\|_M \leqq 1} \left| f\left(\sum_{n=1}^{\infty} p_n x p_n \right) \right| \leqq \|f\|_{M_*}$$

Thus $\Phi \colon f \to \sum_{n=1}^{\infty} p_n f p_n$ is a continuous projection in M_*, so $E = \left\{ f = \sum_{n=1}^{\infty} p_n f p_n,\ f \in M_* \right\}$ is a complemented subspace in M_*.

We claim now that $E \simeq l_1(M_*)$. Indeed, by Lemma 3.1, there exist isometries $\varphi_n \colon p_n M_* p_n \xrightarrow{\sim} M_*$. Therefore the series $\sum_{n=1}^{\infty} \varphi_n^{-1}(f_n)$ converges for $\{f_n\}_{n=1}^{\infty} \in l_1(M_*)$. Define the map $T \colon l_1(M_*) \to M_*$ by $T(\{f_n\}_{n=1}^{\infty}) = \sum_{n=1}^{\infty} \varphi_n^{-1}(f_n)$. It is clear that T is a linear isometry of $l_1(M_*)$ onto E. Therefore, $M_* \in (\mathrm{ID})$.

4. Contractibility to a point of the linear groups of a matroid C^*-algebra.

Let us denote by M_n the C^*-algebras of all complex matrices with the usual operator norm. A C^*-algebra A is called a matroid if there exists a sequence $(A_n)_{n=1}^{\infty}$ of C^*-subalgebras of A (possibly, with different units) such that

(a) $A_k \subset A_{k+1}$ (strictly), $k = 1, 2, 3, \ldots$;

(b) A_k is $*$-isomorphic to $M_{n(k)}$ for some $n(k) \in \mathbf{Z}_+$, $k = 1, 2, 3, \ldots$;

(c) $\bigcup\limits_{k=1}^{\infty} A_k$ is dense in A in the norm topology.

J. Arazy [2] proved that every matroid is linearly isomorphic as a Banach space either to $C(l_2)$ (the C^*-algebra of compact operators on l_2) or to the Fermion algebra $F = \bigotimes\limits_{n=1}^{\infty} M_2$.

THEOREM 4.1. *The linear group* GLG(A) *of any matroid A is contractible to a point.*

PROOF. The contractibility to a point of the general linear group of the space $C(l_2)$ is proved in [8]. Therefore it is sufficient to prove only that GLG(F) is contractible to a point. Since $A \simeq c_0(A)$ for every matroid A (see [2]) we have $F \in$ (ID). Therefore we must prove only that $F \in$ (SOB). The proof of the last inclusion is similar to proof of the $R \in$ (SOB) (see Lemmas 2.3, 2.4), but for the sake of completeness and clarity we give more self-contained proof. We preserve the notation Q_p for operator from $L(F)$ defined by $Q_p(x) = pxp$, $x \in F$. Denote $A_\infty = \bigcup\limits_{k=1}^{\infty} A_k$, where $A_k = \underbrace{M_2 \otimes M_2 \otimes \ldots \otimes M_2}_{k\text{-times}} \otimes 1 \otimes 1 \ldots.$ Notice that $pFp \simeq F$ for any non-zero projection $p \in A_\infty$. We now claim that for every pair of non-zero projections $e_1, e_2 \in A_\infty$, every $\varepsilon > 0$ and every $B \in L(F)$ there exists a pair non-zero projections $p_1, p_2 \in A_\infty$, $p_i \leqq e_i$, such that $\|Q_{p_2} B Q_{p_1}\|_{L(F)} < \varepsilon$. Proving this, we complete the proof the Theorem 4.1 by using the same arguments as in Lemma 2.4.

Suppose that there exists an $\varepsilon_0 > 0$, $B \in L(F)$ and projections $e_1, e_2 \in A_\infty$ such that $\|Q_{p_2} B Q_{p_1}\|_{L(F)} > \varepsilon_0$ for every pair of non-zero projections $p_i \in A_\infty$, $i = 1, 2$, $p_1 \leqq e_1$, $p_2 \leqq e_2$. Fix an integer $N > 17\,\|B\|_{L(F)}/\varepsilon_0$. Choose non-zero projections $e_1^1, e_1^2, \ldots, e_1^N$ so that $e_1^i \in A_\infty$, $i = \overline{1, N}$ and $e_1 = e_1^1 + e_1^2 + \ldots + e_1^N$, $e_1^i e_1^j = 0$, $i \neq j$. Using the same arguments as in Lemma 2.3 we can assert that there exists $x_1 \in F$, $\|x_1\|_F \leqq 1$ such that $\|Q_{e_2} \tilde{x}_1\|_F \geqq \varepsilon_0/4$, where $\tilde{x}_1 = \mathrm{Re}(BQ_{e_1^1} x_1)$.

Select $y_1 = y_1^* \in A_\infty$ such that

$$\|Q_{e_2}(y_1 - \tilde{x}_1)\|_F < \|B\|_{L(F)}/(16N).$$

Then we have $\|Q_{e_2} y_1\|_F > \varepsilon_0/8 - \varepsilon_0/16 = \varepsilon_0/16$. Therefore there exists a projection $e_2^1 \in F$, $e_2^1 \leqq e_2$ such that $\delta_1 Q_{e_2^1} y_1 \geqq \varepsilon_0 e_2^1/16$ where $\delta_1 = \pm 1$. Moreover, since $y_1 \in A_\infty$ and $e_2 \in A_\infty$ we can choose e_2^1 so that e_2^1 also belongs to A_∞. Replacing e_1^1 by e_1^2 and repeating the above arguments we find $x_2 \in F$, $\|x_2\|_F \leqq 1$, $\tilde{x}_2 = \operatorname{Re} BQ_{e_2^1}(x_2)$, $y_2 \in A_\infty$, $e_2^2 \in \mathrm{P}_{A_\infty}$, $e_2^2 \leqq e_2^1$ such that $\|Q_{e_2^1} \tilde{x}_2\|_F \geqq \varepsilon_0/4$,

$$\|Q_{e_2^1}(y_2 - \tilde{x}_2)\|_F < \|B\|_{L(F)}/(16N), \quad \text{and} \quad \delta_2 Q_{e_2^2} y_2 \geqq \varepsilon_0 e_2^2/16,$$

where $\delta_2 = \pm 1$. After N steps we obtain operators $y_i \in A_\infty$, $x_i \in F$, $\|x_i\|_F \leqq 1$, $\tilde{x}_i = \operatorname{Re} BQ_{e_i^1}(x_i)$, $i = 1, \ldots, N$ and a projection $e_2^N \in A_\infty$ such that, $\|Q_{e_2^N}(y_i - \tilde{x}_i)\|_F < \|B\|_{L(F)}/(16N)$, $i = 1, \ldots, N$, and

$$\sum_{n=1}^{N} \delta_i Q_{e_2^N} y_i \geqq N\varepsilon_0 e_2^N/16 > (17\,\|B\|_{L(F)}/16) e_2^N,$$

whence $\left\| \sum_{i=1}^{N} \delta_i \mathrm{Q}_{e_2^N} \mathrm{y}_i \right\|_F > 17\,\|\mathrm{B}\|_{L(F)}/16$. On the other hand,

$$\left\| \sum_{i=1}^{N} \delta_i Q_{e_2^N} y_i \right\|_F \leqq \left\| \sum_{i=1}^{N} \delta_i Q_{e_2^N} \tilde{x}_i \right\|_F + \left\| \sum_{i=1}^{N} \delta_i Q_{e_2^N}(y_i - \tilde{x}_i) \right\|_F \leqq$$

$$\leqq \left\| \sum_{i=1}^{N} \delta_i \tilde{x}_i \right\|_F + \|B\|_{L(F)}/16 =$$

$$= \left\| \sum_{i=1}^{N} \operatorname{Re}(\delta_i B Q_{e_1^i} x_i) \right\|_F + \|B\|_{L(F)}/16 \leqq 17\,\|B\|_{L(F)}/16.$$

This contradiction completes the proof of Theorem 4.1.

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