ON EXCEPTIONAL SETS FOR SUPERHARMONIC FUNCTIONS IN A HALFSPACE: AN INVERSE PROBLEM

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1. Introduction.

The problem of the present paper can be stated in the following way: assume that the set where a potential is large is known to be “small”. Can we say anything about the Riesz mass?

This is of course far too general. Let us consider $D = \{x \in \mathbb{R}^p : x_1 > 0\}$ where $x = (x_1, \ldots, x_p)$ and $p \geq 2$. We shall discuss superharmonic functions of the form $u = P\nu + G\mu$, where $P\nu$ is the Poisson integral of a measure $\nu$ on the (euclidean) boundary $\partial D$ of $D$ and $G\mu$ is the Green potential of a measure $\mu$ on $D$: we assume that these integrals are convergent. It is known that

$$(P\nu(x) + G\mu(x))/x_1 \to 0, \ x \to \infty \ \text{in} \ D, \ x \in D \setminus E,$$

$$(P\nu(x) + G\mu(x))/|x| \to 0, \ x \to \infty \ \text{in} \ D, \ x \in D \setminus F,$$

where the exceptional set $E$ is minimally thin at infinity in $D$ (cf. [13]) and the exceptional set $F$ is rarefied at infinity in $D$ (cf. [7]).

If $\beta \in [0, 1]$ is given, we define $S_\beta$ to be the class of positive superharmonic functions $u = P\nu + G\mu$ in $D$ which are such that

$$\int_{\partial D} (1 + |y|)^{1-p-\beta} \ d\nu(y) + \int_D (1 + |y|)^{1-p-\beta} y_1 \ d\mu(y) < \infty,$$

(cf. [7, Definition 4.1]).

It follows from the convergence of the integrals in the representation $u = P\nu + G\mu$ of $u$ that $u \in S_1$. We quote the following result from [7, Theorem 4.7]:

**Theorem A.** Let $\beta \in [0, 1]$ be given and assume that $D \subset \mathbb{R}^p$. If $u \in S_\beta$, there exists a set $E \subset D$ which is rarefied or minimally thin at infinity in $D$ such that

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\[ \lim u(x)/|x|^\beta = 0, \ x \to \infty, \ x \in D \setminus E, \]
\[ \lim u(x)/(x_1|x|^\beta-1) = 0, \ x \to \infty, \ x \in D \setminus E, \]
respectively.

Conversely, this result is best possible in the sense that if \( E \) is rarefied or minimally thin at infinity in \( D \), then there exists \( u \in S_\rho \) such that \( E \) is contained in the exceptional set for \( u \).

We wish to discuss the following problem: let \( u \) be given and assume that the set \( \{ x \in D : u(x) > |x|^\beta \} \) or the set \( \{ x \in D : u(x) > x_1 |x|^\beta-1 \} \) is rarefied or minimally thin at infinity in \( D \). Does it follow that \( u \in S_\rho \)?

The answer is known to be affirmative for rarefied sets in one special case: if \( p = 2, \beta = 0 \) and \( u = G\mu \) where \( \mu \) is a sum of point masses (cf. J. S. Hwang [11, Lemma 1]).

In his proof, J. S. Hwang uses the following fact:

There exists a sequence \( \{R_n\} \) such that \( \lim_{n \to \infty} R_n/2^n = 1 \) and such that the rarefied set does not intersect the circle \( \{|x| = R_n\} \) for any \( n \).

**Remark 1.1.** This statement is not true for minimally thin sets, neither for rarefied sets in the case \( p \geq 3 \). However, we shall prove a weaker statement (cf. Lemmas 4.2 and 4.3) which we can use in the proof in the general case.

2. The main results.

**Theorem 2.1.** Let \( \beta \in [0, 1) \) be given and let \( u \in S_1 \) be of the form \( u = Pv + G\mu \). We assume that there exists a set \( E_0 \subset D \) which is minimally thin at infinity in \( D \) and which is such that \( \mu(D \setminus E_0) = 0 \). Then the set \( \{ x \in D : u(x) > x_1 |x|^\beta-1 \} \) is minimally thin at infinity in \( D \) if and only if \( u \in S_\rho \).

The analogous result in the rarefied case is different. To explain the difference, we mention that if \( \mathcal{O} \subset \mathbb{R}^p \) is an open set such that \( \partial D \subset \mathcal{O} \), then \( \mathcal{O} \cap D \) is not rarefied at infinity. At the same time, the set \( \{ x \in D : 0 < x_1 < 1 \} \) is minimally thin at infinity in \( D \).

**Theorem 2.2.** Let \( \beta \in [0, 1) \) and let \( u = Pv + G\mu \in S_1 \). Suppose that there exist sets \( E_0 \subset D \) and \( F_0 \subset \partial D \) which are such that \( \mu(D \setminus E_0) = 0 \), \( \nu(\partial D \setminus F_0) = 0 \), \( E_0 \) is rarefied at infinity in \( D \), and furthermore that

\[ \int_{F_0} (1 + |y|)^{-p} \, dy < \infty. \]

Then the set \( \{ x \in D : u(x) > |x|^\beta \} \) is rarefied at infinity in \( D \) if and only if \( u \in S_\rho \).

For examples showing that our conditions on the sets of concentration of the
measures are best possible, we refer to Section 7. In the case of general distributions of mass, the conclusions of Theorems 2.1 and 2.2 need no longer be true. As an example, let us take the superharmonic functions \( h(x) = x_1 |x|^\beta - 1 \) and \( u = \frac{1}{2}(h + v) \) where \( v \in S_\beta \). The set \( \{ u > h \} = \{ v > h \} \) is minimally thin at infinity in \( D \) (cf. Theorem A), but the set \( \{ u > \frac{1}{2}h \} = D \) is not minimally thin at infinity in \( D \) and \( u \notin S_\beta \).

For a discussion of the general case where there are no restrictions on the supports of the measures, we refer to Section 8.

In the proofs of Theorems 2.1 and 2.2, we consider first two special cases which we state as

**Theorem 2.3.** Let \( u \) and \( \beta \) be as in Theorems 2.1 or 2.2 and assume furthermore that the measures \( \nu \) and \( \mu \) are sums of point masses. We can then conclude that

i) the set \( \{ x \in D : u(x) > x_1 |x|^\beta - 1 \} \) is minimally thin at infinity if and only if \( u \in S_\beta \);

ii) the set \( \{ x \in D : u(x) > |x|^\beta \} \) is rarefied at infinity if and only if \( u \in S_\beta \).

**Corollary 2.1.** Let \( u \) and \( \beta \) be as in Theorem 2.3. Then

i) \( u \in S_\beta \) if and only if the set \( \{ x \in D : u(x) > cx_1 |x|^\beta - 1 \} \) is minimally thin at infinity for some \( c > 0 \).

ii) \( u \in S_\beta \) if and only if the set \( \{ x \in D : u(x) > c |x|^\beta \} \) is rarefied at infinity in \( D \) for some \( c > 0 \).

**Remark 2.1.** In the corollary, we can replace “some” by “every”. It follows that if the set \( \{ x \in D : u(x) > c |x|^\beta \} \) is rarefied at infinity for some \( c > 0 \), then such sets will be rarefied at infinity for all \( c > 0 \).

There is a similar remark for minimally thin sets.

**Remark 2.2.** It is possible to generalize our results to functions superharmonic in cones in \( \mathbb{R}^p \), \( p \geq 2 \). The machinery which we need here can be found in Ronkin [14]. The class analogous to our class \( S_1 \) is defined by formulas (49) and (50) in [14, p. 68].

The crucial property of potentials of point masses is that for all \( c > 0 \), there is no Riesz mass in the complements of the sets \( \{ x \in D : u(x) > cx_1 |x|^\beta - 1 \} \) and \( \{ x \in D : u(x) > c |x|^\beta \} \) with respect to \( \hat{D} \).

As corollaries, we obtain certain results on the logarithms of moduli of \( H^\infty \)-functions in the unit disc \( U \). When \( p = 2 \), we map \( D \) conformally onto \( U \). As a normalization we assume that the point at infinity in \( D \) is mapped onto the point \( z = 1 \). This means that we work with the mapping \( z = (w - 1)/(w + 1) \), where \( w \) denotes the variable in the \( D \)-plane.

Since minimal thinness is defined in terms of minimal harmonic functions (cf. [3, p. 397] or [6, Section 0]), this concept is conformally invariant. There is no
such analogue for rarefiedness. Therefore, we shall define a set \( S \subset U \) to be \textit{rarefiedly thin} (or \textit{rarefied}) at \( 1 \in \partial U \) if the image \( w(S) \) is rarefied at infinity in \( D \). We say that a function \( f \) has the rarefied fine limit \( a \) at \( 1 \) if there is a set \( S \subset U \) which is rarefied at \( 1 \) such that \( f(z) \to a \) whenever \( z \to 1 \) and \( z \in U \setminus S \).

It is well known that if \( f \in H^\omega(U) \) with \( \| f \|_\infty \leq 1 \), we have the representation formula

\[
\log |f(z)| = -(G_0 \mu_0(z) + P_0 v_0(z)),
\]

where \( G_0 \mu_0 \) is a Green potential of a measure \( \mu_0 \) on \( U \) and \( P_0 v_0 \) is the Poisson integral of a measure \( v_0 \) on \( \partial U \). Furthermore, \( \mu_0 \) is the sum of unit masses at points \( \{ z_n \} \) in \( D \) which are such that \( \sum (1 - |z_n|) < \infty \) (cf. [5, Ch. 2]).

Mapping \( U \) onto the right half-plane with \( z = 1 \) going to \( w = \infty \), we see that

\[
-\log |f((w - 1)/(w + 1))| = P_\alpha(w) + G_\mu(w) + \alpha w_1,
\]

where \( dv_\theta = |1 - e^{i\theta}|^2 dv \cot(\theta/2), \theta \neq 0 \) (the boundary correspondence is given by \( v = \cot(\theta/2) \) and we have \( d\mu_0(z) = d\mu(w) \)). Thus the superharmonic function defined by (2.2) will be in \( S_\beta, 0 \leq \beta \leq 1 \), if and only if \( \alpha = 0 \) and

\[
\int_{-\pi}^{\pi} (\sin(\theta/2))^{\beta-1} dv_\theta(e^{i\theta}) + \sum_n (1 - |z_n|)|1 - z_n|^{\beta-1} < \infty.
\]

We obtain the following two corollaries.

**Corollary 2.2.** Let \( \beta \in [0, 1) \) and let \( f \in H^\omega(U) \) satisfy (2.1). We assume that there exists a set \( F_1 \subset \partial U \) such that \( v_\theta(\partial U \setminus F_1) = 0 \) and that \( \int_{F_1} d\theta/|\theta| < \infty \). Then the function \( |1 - z|^{\beta} \log |f(z)| \) has the rarefied fine limit 0 at 1 if and only if (2.3) holds.

**Remark 2.3.** The case \( \beta = 0 \) is due to J. S. Hwang (cf. [11, Theorem 2]) for the case when \( f \) is a Blaschke product. In this situation, condition (2.3) is reduced to

\[
\sum_n (1 - |z_n|)|1 - z_n|^{-1} < \infty,
\]

which is a classical condition considered by Frostman (cf. [4, Theorem 2.13]; cf. also [6, Section 6]).

We say that a function \( g \) has the minimal fine limit \( a \) at 1 if there exists a minimally thin set \( S \subset U \) such that \( g(z) \to a \) whenever \( z \to 1 \) in \( U \) and \( z \in U \setminus S \).

**Corollary 2.3.** Let \( \beta \in [0, 1) \) and let \( f \in H^\omega(U) \) satisfy (2.1). Then the function \( (1 - |z|)^{-1}|1 - z|^{\beta+1} \log |f(z)| \) has the minimal fine limit 0 at 1 if and only if (2.3) holds.

These corollaries are immediate consequences of Theorems 2.2 and 2.1.
3. Basic properties of rarefied and minimally thin sets.

Given \( E \subset D \), suppose there exists a measure \( \lambda = \lambda_E \) whose Green potential is \( G^\lambda_E = \hat{R}^E_{\lambda} \), where \( \hat{R}^E_{\lambda} \) is the regularized reduced function of \( x_1 \) on \( E \) with respect to the cone of positive superharmonic functions on \( D \). We call \( \lambda_E \) the fundamental distribution on \( E \). We define \( \gamma(E) = \int G^\lambda_E(x) \, d\lambda_E(x) = \int x_1 \, d\lambda_E(x) \) and call \( \gamma(E) \) the Green energy of \( E \) (cf. [13, p. 129], [7, p. 237]).

Suppose that there exist measures \( \nu_1 \) and \( \mu_1 \) on \( \partial D \) and \( D \), respectively, which are such that

\[
\hat{R}^E_{\nu_1}(x) = \int_{\partial D} P(x, y) \, d\nu_1(y) + \int_D G(x, y) \, d\mu_1(y).
\]

The Green mass of \( E \) is defined as \( \lambda'(E) = \nu_1(\partial D) + \int_D y_1 \, d\mu_1(y) \) (cf. [7, p. 239]).

These two set functions are first defined for compact sets and then extended to capacities defined for general sets in a standard way (cf. [7, p. 243]). They are both monotone and countably subadditive. Furthermore, we have \( \lambda(E) = \lambda'(E) \) for any set \( E \subset D \) (cf. [7, Lemma 2.5]).

Let \( E^{(n)} \) be the intersection of \( E \) and the half-annulus \( \{ x \in D \cup \partial D : 4^n \leq |x| \leq 4^{n+1} \} \). Then the set \( E \) is minimally thin at infinity in \( D \) if and only if (cf. [13])

\[
\sum_{i=1}^{\infty} \gamma(E^{(n)}) 4^{-np} < \infty.
\]

(3.1)

A set \( E \) is rarefied at infinity if and only if (cf. [7])

\[
\sum_{i=1}^{\infty} \lambda'(E^{(n)}) 4^{n(1-p)} = \sum_{i=1}^{\infty} \lambda(E^{(n)}) 4^{n(1-p)} < \infty.
\]

(3.2)

**Remark 3.1.** i) In (3.1) and (3.2) we work with powers of 4 to define our annuli. We could just as well considered annuli \( \{ a^n \leq |x| \leq a^{n+1} \} \) for any \( a > 1 \): the definitions are independent of what \( a \) we would like to use.

ii) The basic definitions and results on rarefied and minimally thin sets do not depend on the general assumption in [7] that the dimension \( p \) is at least 3: everything is correct also in the case \( p = 2 \).

4. Three lemmas.

**Lemma 4.1.** Let \( u \) be harmonic and positive in a half-ball \( \{ x \in \mathbb{R}^p : |x| < 4 \} \cap D \) and vanishing on \( \partial D \). If there exists \( y \in D \) such that \( |y| \leq 3 \) and \( u(y) \leq y_1 \), then there exists a constant \( C'_p \) such that

\[
u(x) \leq C'_p x_1, \quad x \in D, \quad |x| \leq 2.
\]

(4.1)
The proof is an elementary consequence of Poisson’s formula for a half-ball and is omitted.

After having decomposed \( u = P v + G \mu \) into two superharmonic functions, we shall use the following two lemmas to construct spheres with large radii which do not intersect the exceptional sets for these two functions.

In the sequel we denote by \( c_p \) constants, which depend only on the dimension \( p \) and which may be different from line to line.

Let \( X_n = \{ x \in \mathbb{R}^p : 4^n \leq |x| < 4^{n+1} \} \), \( n = 1, 2, \ldots \). If \( X = \bigcup_1^{\infty} X_{2n} \), we consider \( v_1 = v|_X, \mu_1 = \mu|_X \) and \( u_1 = P v_1 + G \mu_1 \).

**Lemma 4.2.** Assume that \( \beta \in [0, 1) \) and that the set \( \{ x \in D : u(x) > x_1 |x|^{\beta-1} \} \) is minimally thin at infinity. Then there exists a constant \( c_p > 1 \) such that

\[
\{ x \in D : u_1(x) > c_p x_1 |x|^{\beta-1} \} \cap \{ |x| = 2 \cdot 4^{2n+1} \} = \emptyset
\]

for all sufficiently large values of \( n \).

**Proof.** The set \( E = \{ x \in D : u_1(x) > x_1 |x|^{\beta-1} \} \) is minimally thin at infinity in \( D \). Thus we can cover \( E \) by balls \( \{ B_k \} \) such that \( \sum (r_k/R_k)^p < \infty \), where \( r_k \) is the radius of \( B_k \) and \( R_k \) is the distance from the centre of \( B_k \) to the origin (cf. [8, Corollary 3, p. 397]).

If \( \varepsilon > 0 \) is given, there exists \( N \) such that we have

\[
(4.2) \quad \sum (r_k/R_k)^p < \varepsilon,
\]

where we sum over all indices \( k \) such that \( B_k \cap \left( \bigcup_{n \geq N} X_n \right) \neq \emptyset \).

We wish to cover the half-sphere \( H_n = D \cap \{ |x| = 2 \cdot 4^{2n+1} \} \) by a finite number of balls of radius at least \( 4^{2n-2} \). The balls must be chosen in such a way that \( u_1 \) is harmonic in the intersection of the doubled balls and \( D \).

Let us first cover that part of \( H_n \) which is near \( \partial D \) by finitely many balls \( \{ B'_k \} \) of radius \( 4^{2n} \) and with centres on \( \partial D \cap \bar{H}_n \). Let \( B \in \{ B'_k \} \), let \( x_B \in \partial D \) be the centre of \( B \). Furthermore, if \( \varepsilon > 0 \) and \( n > N \), it follows from (4.2) that \( B \) is not covered by \( \bigcup B_k \). Thus, there exists \( y \in B \) such that \( u_1(y) \leq y_1 |y|^{\beta-1} \leq y_1 4^{(2n+1)(\beta-1)} \). According to Lemma 4.1, we have

\[
u_1(x) \leq C_p x_1 4^{(2n+1)(\beta-1)} \leq c_p x_1 |x|^{\beta-1}, \quad x \in B. \]

It follows that

\[
u_1(x) < c_p x_1 |x|^{\beta-1}, \quad x \in \cup B'_k. \]

The next step is to cover the rest of \( H_n \) by finitely many balls of radius \( 4^{2n-1} \) with centres on \( H_n \). The same argument as above shows that for all large \( n \), there exists in each one of these balls a point \( y \) with \( u_1(y) \leq y_1 |y|^{\beta-1} \). Since the doubles of all these balls are contained in \( X_{2n+1} \), where \( u_1 \) is harmonic, it follows from
Harnack’s inequality that we have $u_1(x) \leq c_p x_1 |x|^{\beta - 1}$ in the union of all these balls and thus on $H_n$. This concludes the proof of Lemma 4.2.

**Lemma 4.3.** Assume that $\beta \in [0, 1)$ and that the set $\{x \in D: u(x) > |x|^{\beta}\}$ is rarefied at infinity in $D$. Then there exists a constant $c_p$ such that we have

$$\{x \in D: u_1(x) \geq c_p |x|^{\beta}\} \cap \{|x| = 2 \cdot 4^{2n+1}\} = \emptyset$$

for all sufficiently large values of $n$.

**Proof.** The set $E = \{x \in D: u_1(x) \geq |x|^{\beta}\}$ is rarefied at infinity in $D$ (cf. [7, Theorem 4.6]). Thus there exists a covering of $E$ by balls $\{B_k\}$ such that $\sum (r_k/R_k)^{\beta - 1} < \infty$ (cf. [1] and [8, p. 397]; we use the notations as in the proof of Lemma 4.2.). Consequently, we have also $\sum (r_k/R_k)^{\beta} < \infty$, and we can use the same method as in the proof of Lemma 4.2.

Let $B \in \{B_k\}$, and consider the ball $B^* = \{|x - (x_B + 4^{2n-1}(2, 0, \ldots, 0)| < 4^{2n-1}\} \subset B$, where $x_B$ is the centre of $B$. It is clear that for sufficiently large $n$ the ball $B^*$ is not covered by $\bigcup B_k$. Hence, there exists $y \in B^*$ such that $u_1(y) \leq |y|^{\beta} \leq c_p y_1 |y|^{\beta - 1}$. The rest of the proof is the same as in the proof of Lemma 4.2.

**Remark 4.1.** When $p = 2$, the argument can be simplified. In this case a set $E$ rarefied at infinity in $D$ can be covered by discs $\{B_k\}$ such that $\sum r_k/R_k < \infty$. It is now easy to see that there exists sequence $\{R_n\}$ increasing to infinity such that $\lim R_n/4^n = 1$ and such that $E$ does not intersect the circle $\{|x| = R_n\}$ for any $n$.

5. **Proof of Theorem 2.3.**

With $\{X_n\}$ and $X$ as in Section 4, we also define $X' = \bigcup_0^\infty X_{2n+1}$. We write $u = u_1 + u_2$, where $u_1$ is the function associated with the measures $\nu_1 = v|_X$ and $\mu_1 = \mu|_X$ and $u_2 = u - u_1$: here the corresponding measures are $\nu_2 = v|_{X'}$ and $\mu_2 = \mu|_{X'}$.

Let us first prove Theorem 2.3 ii). The two sets $\{u_1(x) > |x|^{\beta}\}$ and $\{u_2(x) > |x|^{\beta}\}$ are rarefied at infinity in $D$. From now on, we discuss $u_1 = P\nu_1 + G\mu_1$: the details in the proof are similar in the discussion of $u_2$.

Let $c_p > 1$ be the constant in Lemma 4.3. We consider the two rarefied and relatively open sets $E = \{x \in D: P\nu_1(x) > c_p |x|^{\beta}\}$ and $F = \{x \in D: G\mu_1(x) > c_p |x|^{\beta}\}$. According to Lemma 4.3, they can for large $n$ be divided into disjoint subsets

$$E_n = E \cap \{2 \cdot 4^{2n-1} \leq |x| \leq 2 \cdot 4^{2n+1}\},$$

$$F_n = F \cap \{2 \cdot 4^{2n-1} \leq |x| \leq 2 \cdot 4^{2n+1}\},$$

which are such that $P\nu_1(x) = c_p |x|^{\beta}$ on $\partial E_n \cap D$ and $G\mu_1(x) = c_p |x|^{\beta}$ on $\partial F_n \cap D$. 
Let us first discuss \( \{F_n\} \). Since \( \mu_1 \) is discrete, \( G\mu_1 \) is infinite on the set of concentration of \( \mu_1 \) and thus \( \mu_1 \) is concentrated on the open set \( F \). From Lemma 4.3, we see that \( F_n \) is also open. If \( \lambda_n \) is the fundamental distribution on \( F_n \), we have \( G\lambda_n(x) = x_1 \) on \( F_n \) and \( \lambda_n \) is concentrated on \( \partial F_n \cap D \). It follows that

\[
\int_{F_n} x_1 \, d\mu_1(x) = \int_{D} G\lambda_n(x) \, d\mu_1(x) = \int_{\partial F_n \cap D} G\mu_1(x) \, d\lambda_n(x) \\
= \int_{\partial F_n \cap D} c_p \, |x|^\beta \, d\lambda_n(x) \leq c4^{2n\beta} \lambda_n(\partial F_n \cap D) = c4^{2n\beta} \lambda_n'(D).
\]

Let \( F_0 = \bigcup_{n=N}^{\infty} \bar{F}_n \), where \( N \) is so large that Lemma 4.3 holds for \( n \geq N \). Our estimate above implies that

\[
(5.1) \quad \int_{F_0} x_1 (1 + |x|)^{1-p-\beta} \, d\mu_1(x) \leq c \sum_{n=N}^{\infty} \lambda_n'(D)4^{2n(1-p)} < \infty.
\]

The last sum is finite since \( F_0 \) is rarefied at infinity in \( D \) (cf. (3.2)). Since \( G\mu_1 \) is harmonic outside \( F \), we have proved that \( G\mu_1 \in S_\beta \).

It remains to study \( \{E_n\} \). Let \( \lambda_n^0 \) be the fundamental distribution on \( E_n \). Since \( E_n \) is relatively open in \( D \), we know that \( (\text{supp} \, \lambda_n^0) \cap D = \partial E_n \cap D \) and that \( \lambda_n^0 \) has no mass on \( \partial D \).

Let \( f = P\nu_1 \). We claim that \( \tilde{R}_f^E = f \). To see this, let \( v \) be a nonnegative superharmonic function in \( D \) which dominates \( f \) on \( E \). Our claim will follow if we can prove that \( v \) dominates \( f \) on \( D \). Using that \( \nu_1 \) is a sum of point masses, we define \( f_j \) to be the Poisson integral of the first \( j \) masses. Since \( v \) dominates \( f \) on \( E \), \( v \) dominates the smaller function \( f_j \) on the smaller set \( A_j = \{ x \in D : f_j(x) \geq c_p |x|^\beta \} \). Furthermore, \( f_j \) vanishes on \( \partial D \setminus \partial A_j \). Hence \( v - f_j \) is a nonnegative superharmonic function on \( D \) (cf. [10, p. 232]). Letting \( j \to \infty \), we conclude that \( v - f \) is also nonnegative on \( D \) which finishes this part of the proof.

Let \( B_E \) be the set of points on \( \partial D \) where \( E \) is not minimally thin. Since \( \tilde{R}_f^E = f \), it follows that \( \nu_1(\partial D \setminus B_E) = 0 \) (cf. [2, p. 129]). Furthermore, it follows from [7, Lemma 2.3] that

\[
P^*\lambda_n^0(x) := c_p \int_{B_E} \frac{y_1}{|x-y|^{1-p}} \, d\lambda_n^0(y) = 1, \quad x \in B_E \cap \bar{E}_n.
\]

Let \( \nu_n \) be the restriction of \( \nu_1 \) to \( \bar{E}_n \). Using the two facts above and that (supp \( \nu_1 \)) \( \subset \partial D \cap \bar{E}_n \), we deduce that \( \nu_n(\partial D) = \nu_n(B_E \cap \bar{E}_n) \) and

\[
\nu_n(\partial D) \leq \int_{\partial D} P^*\lambda_n^0(y) \, d\nu_1(y) = \int_{D} P\nu_1(x) \, d\lambda_n^0(x) \leq c_p4^{2n\beta}\lambda_n^0(D).
\]
Let $E_0 = \bigcup_{n=-N}^{\infty} E_n \cap B_E$, where $N$ is so large that Lemma 4.3 holds for $n \geq N$. Using the estimate of $v_n(\partial D)$ and our assumption that $E$ is rarefied at infinity in $D$, we deduce

$$\int_{E_0} (1 + |y|)^{1-p-\beta} \, dv_1(y) \leq c_p \sum_{n=N}^{\infty} \lambda_n^{0}(D) 4^{2n(1-p)} < \infty.$$ 

Hence $Pv_1 \in S_p$ and we have proved Theorem 2.3 ii).

The proof of Theorem 2.3 i) is similar. After having separated the Riesz masses in the even and the odd annuli, we define the sets $E = \{ x \in D : Pv_1(x) > c_p x_1 |x|^\beta x_1 \} \cap B_E$ and $F = \{ x \in D : G\mu_1(x) > c_p x_1 |x|^\beta x_1 \} \cap B_E$. We divide the sets into disjoint subsets $\{ E_n \}$ and $\{ F_n \}$, each one of these associated with an annulus containing $|x| = 4^2n$. Then the sets $\{ F_n \}$ can be handled in exactly the same way as before.

Let $\lambda_n$ be the fundamental distribution on $E_n$. Arguing in the same way as in the proof of Theorem 2.3 ii), we deduce that

$$v_n(\partial D) \leq \int_{\partial D} P^*\lambda_n(y) \, dv_1(y) = \int_D Pv_1(x) \, d\lambda_n(x) \leq c_p 4^{2n(\beta-1)} \gamma(E_n).$$ 

Since $E$ is minimally thin at infinity in $D$, we know that (3.1) holds. It is now easy to prove that $v_1 \in S_\beta$.

6. Proof of Theorems 2.1 and 2.2.

In the proof of Theorems 2.1 and 2.2, it is convenient to introduce the kernel

$$K(y, x) = \begin{cases} G(y, x)/y_1, & \text{on } D \times D \\ c_p x_1 |x - y|^{-p}, & \text{on } \partial D \times D, \end{cases}$$

where $c_p = 2(p - 2)$, $p > 2$; $c_2 = 2$. We note that if $x \in D$, then $K(\cdot, x)$ is continuous $\bar{D} \setminus \{x\}$. We write our superharmonic function in the form

$$u(x) = Pv(x) + G\mu(x) = \int_{\bar{D}} K(y, x) \, d\eta(y) = K\eta(x),$$

where

$$d\eta(y) = y_1 \, d\mu(y), \quad y \in D,$$

$$d\eta(y) = dv(y), \quad y \in \partial D$$

(cf. [7, Section 2]).

We shall say that two positive real valued functions $f$ and $g$ are comparable and write $f \approx g$ if there exist positive constants $A \leq B$ such that $Ag \leq f \leq Bg$. 
Let us first prove Theorem 2.1. We assume that the sets \( E = \{ x \in D : u(x) > |x|^\beta - 1 \} \) and \( E_0 \) are minimally thin at infinity in \( D \). From the subadditivity of the Green energy [7, Lemma 2.1] and the definition of minimal thinness [7, Definition 3.1], it follows that the set \( \mathcal{E} := E \cup E_0 \cup \{ x \in D : 0 < x_1 < 1 \} \) is also minimally thin at infinity in \( D \). Then there exists an open set \( \mathcal{O} \subset D \) such that \( \mathcal{O} \supset \mathcal{E} \) and \( \mathcal{O} \) is minimally thin at infinity in \( D \) [7, Lemma 2.1]. For every \( y \in \mathcal{O} \cup \partial D \), we select a ball \( B_y \) with centre \( y \) and radius \( r(y) < 1 \) such that \( B_y \cap D \subset \mathcal{O} \). It is clear that \( \mathcal{O} \cup \partial D \subset \cup_i B_{y_i} \). According to Besicovitch-Ahlfors-Landkof's lemma (see [9, p. 2–6; 12, p. 197]), there exists a countable subfamily of balls \( \{ B_{y_i} \} \) such that \( \mathcal{O} \cup \partial D \subset \cup_i B_{y_i} \) and every \( y \in \mathcal{O} \cup \partial D \) belongs to at most \( N(p) \) balls \( B_{y_i} \), where \( N(p) \) depends only on the dimension \( p \). We need the inequality

\[
(6.1) \quad K(y, x) \geq c_1 K(y_i, x), \quad x \in D \setminus B_{y_i}, \quad y \in \overline{B}_{y_i} \cap \partial D, \quad c_1 = c_1(p).
\]

The proof will be given at the end of this section.

At every point \( y_i \), we place the point mass \( \eta(B_{y_i}) \) which is nonnegative (we might have \( \eta(B_{y_i}) = 0 \)). Let \( \eta_1 \) be the sum of all these point masses and let \( u_1(x) = K \eta_1(x) \). If \( x \in D \setminus \mathcal{O} \), then

\[
\begin{align*}
u(x) &= \int_{D} K(y, x) \, d\eta(y) 
\geq N(p)^{-1} \sum_i \int_{B_{y_i}} K(y, x) \, d\eta(y) 
\geq N(p)^{-1} c_1 \sum_{i} K(y_i, x) \eta(B_{y_i}) 
= c_2 u_1(x),
\end{align*}
\]

where \( c_2 = c_2(p) = c_1/N(p) \). It follows that

\[
u_1(x) \leq \nu(x)/c_2 \leq c_2^{-1} x_1 \, |x|^{\beta - 1}, \quad x \in D \setminus \mathcal{O}.
\]

Hence

\[
\{ x \in D : \nu_1(x) > c_2^{-1} x_1 \, |x|^{\beta - 1} \} \subset \mathcal{O}.
\]

Since \( \mathcal{O} \) is minimally thin at infinity in \( D \), we can apply Corollary 2.1 to deduce that \( \nu_1 \in S_\beta \) and thus that \( \nu \in S_\beta \). Here we use that by assumption, \( \mu(D \setminus \mathcal{O}) = 0 \). Therefore, no mass is lost when we go from \( \eta \) to \( \eta_1 \).

In the opposite direction, Theorem 2.1 follows from Theorem A.

To prove Theorem 2.2, we assume that the sets \( E = \{ x \in D : u(x) > |x|^\beta \} \) and \( E_0 \) are rarefied at infinity in \( D \). Since the set \( \mathcal{E}_1 = E \cup E_0 \) is rarefied at infinity in \( D \), there exists an open set \( \mathcal{O}_1 \subset D \) such that \( \mathcal{E}_1 \subset \mathcal{O}_1 \) and \( \mathcal{O}_1 \) is also rarefied at infinity in \( D \) [7, Lemma 2.4]. Let \( \mathcal{O}_2 \) be a relatively open subset of \( \partial D \) such that \( F \subset \mathcal{O}_2 \) and
\[
\int_{\mathcal{C}_2} (1 + |y|)^{1-p} \, dy < \infty.
\]

For every \( y \in \mathcal{C}_1 \cup \mathcal{C}_2 \), we select a ball \( B_y \) with centre \( y \) and radius \( r(y) < 1 \) such that \( B_y \subset \mathcal{C}_1 \), \( y \in \mathcal{C}_1 \), and \( B_y \cap \partial D = \mathcal{C}_2 \), \( y \in \mathcal{C}_2 \). Again, we apply the lemma of Besicovitch, Ahlfors and Landkof and obtain a family of balls \( \{B_{y_i}\} \) such that \( \mathcal{C}_1 \cup \mathcal{C}_2 \subset \bigcup_i B_{y_i} \) and every \( y \in \mathcal{C}_1 \cup \mathcal{C}_2 \) belongs to at most \( N(p) \) balls \( B_{y_i} \). We see that

\[
\sum_{y_i \in \mathcal{C}_2} \frac{r(y_i)}{|y_i|^{p-1}} \approx \int_{\mathcal{C}_2} (1 + |y|)^{1-p} \, dy < \infty,
\]

and that the set \( \mathcal{C}_3 = \{(\bigcup B_{y_i}) \cap D : y_i \in \mathcal{C}_2 \} \) is rarefied at infinity in \( D \) (cf. [7, Theorem 1.1]). Let \( \mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_3 \). Then \( \mathcal{C} \) is rarefied at infinity in \( D \). By assumption, \( \mu(D \setminus \mathcal{C}) = \nu(\partial D \setminus \mathcal{C}) = 0 \). The rest of the argument is the same as in the proof of Theorem 2.1.

It remains to prove (6.1). Assuming that \( x \in D \setminus B_{y_i} \) and \( y \in \partial B_{y_i} \cap \mathcal{D} \), we see that \( |y - x| \leq |y - y_i| + |y_i - x| \leq 2|y_i - x| \). Let \( x' \) be the reflection of \( x \) in the plane \( \{x_1 = 0\} \). We note that \( |x' - y| \leq 2|x' - y_i| \). If \( p > 2 \) and \( x = (x_1, x_2, \ldots, x_p) \), then

\[
K(y, x) \approx x_1 |x' - y|^{-2} |x - y|^{2-p} \geq c_3(p)x_1 |x' - y_i|^{-2} |x - y_i|^{2-p} \approx K(y_i, x).
\]

If \( p = 2 \) and \( y_1 \) is the first coordinate of \( y \), then

\[
K(y, x) = \begin{cases} 
\frac{1}{2y_1} \log \left(1 + \frac{4x_1y_1}{|x - y|^2}\right) \geq \frac{1}{2y_1} \log \left(1 + \frac{x_1y_1}{|x - y_1|^2}\right), & y_1 > 0, \\
2x_1 |x - y|^{-2}, & y_1 = 0.
\end{cases}
\]

Let \( y_i = (y_{1i}, \ldots, y_{pi}) \). If \( 0 < y_1 < 2r(y_1) = 2r_1 \), then

\[
\frac{x_1y_1}{|y_i - x|^2} \leq \frac{2(|x_1 - y_1| + y_1)r}{|y_i - x| r} \leq 2 \frac{|x - y_1| + (y_1 + r)}{|y_i - x|} \leq 2 \left(1 + \frac{3r}{r}\right) = 8.
\]

Hence, for \( y_{1i} > 0 \),

\[
K(y, x) \geq c_4 x_1 |y_i - x|^{-2} > c_5K(y_i, x),
\]

where \( c_5 \) is an absolute constant.

If \( y_1 > 2r \), then \( y_1 \approx y_{1i} \) and

\[
K(y, x) \geq \frac{1}{2y_1} \log \left(1 + \frac{x_1y_1}{|x - y_1|^2}\right) \approx K(y_i, x).
\]
Since $K(\cdot, x)$ is continuous on $\bar{D}$, (6.1) holds also in the remaining cases when $y_1 = 0$ or $y_{11} = 0$.

7. Examples.

To show that our results are best possible, we construct examples using the theory in [7, Section 4]. If $s > 1$ is fixed, we define $I_n = \{ x \in \bar{D} : s^n \leq |x| < s^{n+1} \}$.

**Proposition 7.1.** Let $\beta \in [0, 1)$ be given and assume that the set $E$ is not rarefied (or minimally thin) at infinity in $D$. Then there exists a measure $\eta$ in $D$ such that $\sup \eta \subset E$, $u = K\eta \in S_1$ and

$$|x|^{-\beta} u(x) \to 0 \quad \text{(or } x_1^{-1} |x|^{-\beta} u(x) \to 0 \text{)} \quad \text{as } x \to \infty \text{ in } D,$$

but $u \notin S_\beta$.

**Proof.** Let $h_\beta$ be the Poisson integral in $D$ with boundary values $|x|^\beta$ and let $r = |x|$. It is easy to see that $h_\beta(x) \approx |x|^\beta$ in $D$. Thus there exists a positive constant such that $\hat{R}_{r\beta}^E \leq \text{Const. } h_\beta$ for any set $E \subset \bar{D}$. In particular we must have $\hat{R}_{r\beta}^E \in S_1$ (cf. [7, Remark 4.2]). Let the measure $\mu_0$ be defined by

$$\hat{R}_{r\beta}^E = \int_D K(y, \cdot) d\mu_0(y).$$

Since $E$ is not rarefied at infinity in $D$, we have

$$\sum_n \mu_0(\bar{I}_n)s^{-n(p+\beta-1)} = \infty$$

(cf. [7, Theorem 4.4]). We find a sequence of positive numbers $\{\epsilon_n\}$ in $(0, 1)$ tending to zero such that

$$\sum_n \epsilon_n \mu_0(\bar{I}_n)s^{-n(p+\beta-1)} = \infty.$$

If $\mu_n$ is the restriction of $\mu_0$ to $I_n$, we define $\eta = \sum_{n=1}^\infty \epsilon_n \mu_n$ and $u(x) = K\eta(x)$. Since we know that $\hat{R}_{r\beta}^E \in S_1$, it is clear that we have also $u \in S_1$.

Furthermore, we have

$$u(x) \leq \sum_1^N \epsilon_n \int_D K(y, x) d\mu_n(y) + \epsilon \text{ Const. } h_\beta(x)$$

assuming that $\epsilon_n \leq \epsilon$ for $n > N$. Consequently,

$$\lim \sup |x|^{-\beta} u(x) \leq \text{Const. } \epsilon,$$

and we conclude that $|x|^{-\beta} u(x) \to 0$ as $x \to \infty$. It is also clear that $u \notin S_\beta$. 
In the minimally thin case, the same argument works, if we replace [7, Theorem 4.4] by [7, Theorem 4.5] and \( h_\beta \) by \( x_1 |x|^{\beta - 1} \) which is a positive superharmonic function in \( D \).

**Proposition 7.2.** Let \( F_0 \subset \partial D \) be a closed set such that \( \int_{F_0} (1 + |y|)^{1-p} dy = \infty \). Then there exists a measure \( \nu \) on \( \partial D \) with \( \text{supp} \nu \subset F_0 \) such that \( u = P\nu \in S_1 \), \( |x|^{-\beta} u(x) \to 0 \) as \( |x| \to \infty \) but \( u \notin S_\beta \).

**Proof.** Let \( T_n = \{ x \in \partial D : 2^n - 1 \leq |x| < 2^n \} \) and let \( M_n \) be the \((p - 1)\)-dimensional measure of \( T_n \cap F_0 \). Our condition on \( F_0 \) can equivalently be written \( \sum_n M_n 2^{n(1 - p)} = \infty \). Again, we find a sequence \( \{ \varepsilon_n \} \) in \((0, 1)\) tending to zero such that

\[
\sum_n \varepsilon_n M_n 2^{n(1 - p)} = \infty.
\]

(7.1)

We define a measure \( \nu \) on \( \partial D \) as \( dv = f d\sigma \), where \( \sigma \) is \((p - 1)\)-dimensional measure on \( \partial D \) and

\[
f = \sum_n \varepsilon_n 2^{\beta n} \chi_{F_0 \cap T_n} \]

(\( \chi_{F_0 \cap T_n} \) is the characteristic function of the set \( F_0 \cap T_n \)). It is clear that \( \text{supp} \nu \subset F_0 \).

If \( u = P\nu \), we note that \( u \leq \text{Const.} \, h_\beta \). Hence we have \( u \in S_1 \) (cf. [7, Remark 4.2]). Arguing as in the proof of Proposition 7.1, we see that if \( \varepsilon_n < \varepsilon \) for \( n > N \), then

\[
u(x) \leq \sum_{i=1}^{N} \varepsilon_n 2^{\beta n} \chi_{F_0 \cap T_n}(x) + \text{Const.} \, h_\beta(x).
\]

It follows that \( |x|^{-\beta} u(x) \to 0 \) as \( x \to \infty \). It is clear from (7.1) that \( u \notin S_\beta \).

From Propositions 7.1 and 7.2, we conclude that the conditions on the measures \( \mu \) and \( \nu \) in Theorems 2.1 and 2.2 are best possible.

8. **Measures with arbitrary sets of concentration.**

Let \( \beta \in [0, 1) \) be given and let \( u \in S_1 \) be of the form \( u = P\nu + G\mu = K\eta \). We consider sets of the form \( E_\varepsilon = \{ x \in D : u(x) > c x_1 |x|^{\beta - 1} \} \) (\( E_\varepsilon \) is a set of the same type as the set \( E \) used in Section 5; we forget here the notation \( E_n \) used in Section 5).

**Proposition 8.1.** Assume that the set \( E_1 \) is minimally thin at infinity in \( D \). Then \( u \in S_{\beta + \varepsilon} \) for each \( \varepsilon > 0 \).

**Proof.** We consider the superharmonic function \( w(x) = \min \{ u(x), x_1 |x|^{\beta - 1} \} \). We have \( w = u \) in \( D \setminus E_1 \). It is easy to see that \( w \leq h_\beta \) in \( D \). Since \( h_\beta \in S_{\beta + \varepsilon} \) for each
$\varepsilon > 0$, we have also $w \in S_{\beta + \varepsilon}$ for each $\varepsilon > 0$ (cf. [7, pp. 250–251]). This means that we can control the size of the mass $\eta$ in $D \setminus E_1$.

It remains to estimate the size of the remaining part of $\eta$. Let $\mu_0$ be the restriction of $\mu$ to $E_1$. If $u_0 = P\nu + G\mu_0 = K\eta_0$, it is clear that

$$E' := \{ x \in D : u_0(x) > x_1 |x|^{\beta - 1} \} \subset E_1.$$  

Since $E_1$ is minimally thin at infinity in $D$, $\mu_0$ satisfies the assumptions of Theorem 2.1: we know that $\mu_0(D \setminus E_1) = 0$. Applying Theorem 2.1, we see that $u_0 \in S_{\beta}$. Hence, we can control the size of the mass $\eta$ also in $\partial D \cup E_1$, and Proposition 8.1 is proved.

In our next result, we consider sets of the type $F_\varepsilon = \{ x \in D : u(x) > c |x|^\beta \}$.

**Proposition 8.2.** Assume that the set $F_\varepsilon$ is rarefied at infinity in $D$. Then $u \in S_{\beta + \varepsilon}$ for each $\varepsilon > 0$.

**Proof.** We consider the superharmonic functions $v = \bar{R}_{\varepsilon}^E$ and $w = \min(u, v + h_\beta)$. In $D \setminus F_1$ we have $u \leq |x|^{\beta} \leq v + h_\beta$. It follows that $w = u$ in $D \setminus F_1$. Again, $w \leq \mathrm{Const.} \ h_\beta$ in $D$ with $h_\beta \in S_{\beta + \varepsilon}$ for each $\varepsilon > 0$ and we must have $w \in S_{\beta + \varepsilon}$ for each $\varepsilon > 0$.

Since $F_1$ is rarefied at infinity in $D$, we know according to a result of V. Azarin (cf. [8, p. 397]) that $F_1$ can be covered by a union of balls $\{B_i = B(x^{(i)}, r_i)\}$ such that

$$\sum_i (r_i/R_i)^{p-1} < \infty,$$

where $R_i = |x^{(i)}|$.

Let $\mathcal{O}$ be the orthogonal projection of $\bigcup B_i$ onto $\partial D$. Let $\mu_0$ be the restriction of $\mu$ to $F_1$ and let $\nu_0$ be the restriction of $\nu$ to $\mathcal{O}$. If $\mu_1 = \mu - \mu_0$ and $\nu_1 = \nu - \nu_0$, we define $u_0 = P\nu_0 + G\mu_0$ and $u_1 = P\nu_1 + G\mu_1$. Since $u = u_0 + u_1 \leq u_0 + w$, it remains to study $u_0$. It is clear that

$$\{ x \in D : u_0(x) > |x|^\beta \} \subset F_1,$$

where $F_1$ is rarefied at infinity in $D$. Since $\mu_0$ is concentrated on $F_1$ and $\nu_1$ is concentrated on $\mathcal{O}$ which according to (8.1) satisfies the condition

$$\int_{\mathcal{O}} (1 + |x|)^{1-p} \, dx < \infty,$$

we can apply Theorem 2.2 and conclude that $u_0 \in S_{\beta}$. We have proved Proposition 8.2.

What can we say about the mass $\eta$ if we assume that the set
$E_c = \{ x \in D : u(x) > cx_1 |x|^\beta - 1 \}$ is minimally thin at infinity in $D$ for all $c > 0$. Let $m(\cdot)$ be Lebesgue measure in $\mathbb{R}^p$ and let $n(r) = \eta(B(r))$, where $B(r) = \{ x \in \mathbb{R}^p : |x| < r \}$.

**Proposition 8.3.** Let $u \in S_1$ and assume that $E_c$ is minimally thin at infinity in $D$ for all $c > 0$. Then

$$n(r)(1 + r)^{1 - p - \beta} \to 0, \quad r \to \infty. \quad (8.2)$$

We note that if $E_c$ is minimally thin at infinity in $D$, then

$$\lim_{r \to \infty} m(E \cap B(r)) r^{-p} = 0$$

(cf. [8, Corollary 3]). Thus Proposition 8.3 is a direct consequence of

**Lemma 8.1.** Let $u \in S_1$ and assume that

$$\limsup_{r \to \infty} n(r)(1 + r)^{1 - p - \beta} > 0. \quad (8.3)$$

Then there exists a constant $c > 0$ and a set $E \subset D$ such that

i) $u(x) > cx_1 |x|^\beta - 1, \ x \in E,$

ii) $\limsup_{r \to \infty} m(E \cap B(r)) r^{-p} > 0.$

**Proof.** Let $\eta_i = n(2^{i+1}) - n(2^i)$. It is easy to see that (8.3) holds if and only if

$$\limsup_{i \to \infty} \eta_i (1 + 2^i)^{1 - p - \beta} > 0.$$

Hence there exists a subsequence $\{i_j\}$ and a positive number $d$ such that

$$\eta_{i_j}(1 + 2^{i_j})^{1 - p - \beta} > d > 0, \quad j = 1, 2, \ldots.$$

If $x \in X_{i_j} = \{ x \in D : 2^{i_j} \leq |x| < 2^{i_j + 3} \}$, then

$$u(x) = K \eta(x) \approx \int_D |x' - y|^{-2} |x - y|^{2 - p} x_1 d\eta(y) > c' x_1 2^{-p} \eta_i \geq cx_1 |x|^\beta - 1,$$

where the positive constants $c'$ and $c$ depend only on $p$ and $d$. Now define $E$ by $E = \cup_j X_{i_j}$. We have proved Lemma 8.1 and also Proposition 8.3.

Our next example shows that even if the set $\{ x \in D : u(x) > cx_1 |x|^\beta - 1 \}$ is bounded for every $c > 0$, we can not prove anything stronger then (8.2).

**Proposition 8.4.** Let $\beta \in [0, 1)$ be given and let $n_0(r)$ be a nondecreasing function such that $n_0(r)(1 + r)^{1 - p - \beta} \to 0$ as $r \to \infty$. Then there exists a measure $\mu$ in $D$ such that $G\mu \in S_1$ and
\[
\int_{B(x')} y_1 d\mu(y) \approx n_0(r), \\
\text{ii) } x_1^{r-1} |x|^{-\beta} G_\mu(x) \to 0 \text{ as } |x| \to \infty.
\]

**Proof.** Let \( V_i = \{ x \in D : 2x_1 > |x|, \ 2^{i-1} \leq |x| < 2^i \} \). Let \( \mu \) be the measure supported by \( \{ x \in D : 2x_1 \geq |x| \} \) which has constant density \( d_i = (n_0(2^i) - n_0(2^{i-1})) 2^{-i(p+1)} \) on \( V_i \). It is not difficult to verify that all conditions of Proposition 8.4 are satisfied.

There are similar results for rarefied sets. We omit the details.

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