ON KLEIN SURFACES AND DIHEDRAL GROUPS

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1. Introduction.

In this paper we study the following problem. Given an NEC group \( \Gamma \) and the dihedral group \( D_p \), with \( p \) a prime, how many non conjugate normal subgroups of \( \Gamma \) has \( D_p \) as quotient group? This is equivalent to asking how many non-biconformally equivalent Klein surfaces that are coverings of the orbifold whose fundamental group is \( \Gamma \) admit \( D_p \) as a group of automorphisms. A related question is the classification of actions of \( D_p \) on a Riemann surface. Natanzon [9] gives a classification for \( D_2 \)-actions on Riemann surfaces.

This paper is a generalization to NEC groups of the paper of Lloyd [6].

2. Klein surfaces and their groups.

Let \( H \) be the hyperbolic plane. A non-Euclidean crystallographic NEC group is a discrete subgroup \( \Gamma \) of automorphisms of \( H \) (Iso \( H \)) with compact quotient space (2-orbifold). Equivalently, \( \Gamma \) is a group that acts properly discontinuously on \( H \). If \( \Gamma \) is an NEC group containing orientation-reversing elements, then \( \Gamma \) is called a proper NEC group; otherwise \( \Gamma \) is called a Fuchsian group and is a subgroup of \( \text{Iso}^+ H \), where \( \text{Iso}^+ H \) denotes the subgroup of \( \text{Iso} H \) formed by the orientation-preserving automorphisms of \( H \). If \( \Gamma \) is a proper NEC group then \( \Gamma \cap \text{Iso}^+ H = \Gamma^+ \) is a Fuchsian group called the canonical Fuchsian group of \( \Gamma \).

DEFINITION ([8]). A 2-orbifold \( M \) is a connected Hausdorff space which admits a folding atlas \( \mathcal{A} \) formed by folding charts \( (U_i, \phi_i, G_i, A_i) \) where \( U_i \) is an open subset of \( \mathbb{C} \), \( G_i \) is a finite group, and the mapping \( \phi_i: A_i \to U_i \) is such that \( A_i/G_i \approx U_i \), with the following compatibility condition that for all \( x \in A_i \) and \( y \in A_j \) such that \( \phi_i(x) = \phi_j(y) \), there exist open subsets \( V_i \) of \( A_i \) and \( V_j \) of \( A_j \), and a diffeomorphism \( \phi: V_i \to V_j \) such that \( \phi_j = \phi_j \phi \).

Moreover, if \( p \in M \), \( p = \phi_i(x), x \in A_i \), then the group \( \text{Stb}(p) = \{ g \in G_i : xg = x \} \)
depends only on \( p \) and is independent of the choice of \( x \) or \( U_i \), so we can distinguish the following points on the 2-orbifold \( M \).

- \( p \) is a **regular point** if \( \text{Stb}(p) = I_d \).
- \( p \) is a **cone point** if \( \text{Stb}(p) \) is a cyclic rotation group of order \( n \).
- \( p \) belongs to a **mirror line** if \( \text{Stb}(p) \) is a cyclic rotation group of order 2 generated by one reflection,
- \( p \) is a **corner point** if \( \text{Stb}(p) \) is a dihedral group of order \( 2n \) generated by 2 reflections.

The above number \( n \) is called the **order of \( p \)**.

**Definition ([8])**. Let \( M, N \) be 2-orbifolds and let \( h : N \to M \) be a continuous mapping. \( h \) is called an **(orbifold-)covering** if there exists a folding atlas \( \mathcal{A} = \{ (U_i, \phi_i, G_i, A_i) \} \) for \( M \) such that for every connected component \( V \) of \( h^{-1} U_i \) there exists a folding chart \( f_j : A_i \to V \) in the maximal atlas of \( N \) such that \( hf_i = \phi_i \).

**Example.** Let \( S \) be a Riemann surface, \( S \) has a 2-orbifold structure whose points are all regular. If now \( G \) is a group acting properly discontinuously on \( S \), then the quotient space \( S/G \) has orbifold structure and the projection map \( \pi : S \to S/G \) is an (orbifold-)covering.

A surface \( S \) with boundary is a 2-orbifold without cone or corner points. The connected components of the boundary correspond to the mirror lines of the 2-orbifold.

A 2-orbifold \( M \) is called a **good 2-orbifold** if \( M \) has a covering which is a surface. We denote by \( \mathcal{U} \) the universal covering of the orbifold \( M \) (\( \mathcal{U} \) is either the 2-sphere \( S^2 \), the Euclidean plane \( E^2 \) or the hyperbolic plane \( H \)). A good 2-orbifold \( M \) is the quotient of \( \mathcal{U} \) by a group \( \Gamma \) acting properly discontinuously on \( \mathcal{U} \). The group \( \Gamma \) is called the **fundamental group of \( M \)** and we write \( \Gamma = \pi(M) \) since, if \( M \) is a closed surface, then \( \Gamma \) is the fundamental group of \( M \). Notice that the fundamental group, as a 2-orbifold, of a surface \( M \) with boundary is not the fundamental group of \( M \) as a 2-manifold.

**Example.** Let \( M = H/\Gamma \) be a good hyperbolic 2-orbifold. The 2-sheeted covering \( M^+ \) of \( M \) given by \( H/\Gamma^+ \), where \( \Gamma^+ \) is the canonical Fuchsian subgroup of \( \Gamma \), is called the **complex double** of \( M \).

The algebraic structure of \( \Gamma \) or, equivalently, the geometrical structure of the quotient 2-orbifold \( M = H/\Gamma \) is determined by the signature:

\[
(2.1) \quad s(M) = s(\Gamma) = (g; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{p1}, \ldots, n_{ps_p})\})
\]

where \( H/\Gamma \) is a 2-orbifold lying on a compact surface of genus \( g \) and having \( p \) mirror lines. If the orbifold is orientable we use the sign \( + \), and the sign \( - \) if it is non-orientable. The integers \( m_1, \ldots, m_r \) are called the **periods** of \( s(\Gamma) \); these are the orders the cone points of \( H/\Gamma \). The bracket \( (n_{k1}, \ldots, n_{k_s}) \) is called the \( k \)th **period**
cyke, it is associated to the kth mirror line of \( H/\Gamma \). The integers \( n_{k1}, \ldots, n_{kn} \), called link periods, are the orders of the corner points on the kth mirror line.

Associated with the signature of a 2-orbifold \( M = H/\Gamma \) is a presentation for the NEC group \( \Gamma \) with:

**GENERATORS:** \( x_1, \ldots, x_r, c_{k,0}, \ldots, c_{k,n_k}, 1 \leq k \leq p, \)

(If \( H/\Gamma \) is non-orientable) \( (1) \ a_h, 1 \leq h \leq g, \)

(If \( H/\Gamma \) is orientable) \( (2) \ a_h, b_h, 1 \leq h \leq g. \)

**RELATORS:** \( x_i^{m_i}, 1 \leq i \leq r, c_{k,j}^2, 1 \leq k \leq p, 0 \leq j \leq n_k, \)

\( (c_{k,j-1}c_{k,j})^{s_{kj}}, 1 \leq k \leq p, 1 \leq j \leq s_k, c_{k,0} = e_k^{-1}c_{k,n_k}e_k, 1 \leq k \leq p, \)

(2.2) \( (1) \ \prod_{i=1}^{r} x_i \prod_{k=1}^{p} e_k \prod_{h=1}^{g} a_h^2 \) or \( (2) \ \prod_{i=1}^{r} x_i \prod_{k=1}^{p} e_k \prod_{h=1}^{g} [a_h, b_h]. \)

(Where \([a_h, b_h]\) is the commutator of \( a_h \) and \( b_h \))

Let \( \Gamma \) be an NEC group. \( \Gamma \) is called a surface group if \( s(\Gamma) = (g, \pm, \{ - \}, \{ - \}) \), and called a bordered surface group if \( s(\Gamma) = (g, \pm, \{ - \}, \{ - \}, \ldots, \{ - \}) \). Shortening we say that \( \Gamma \) is a surface group in both cases.

Every good 2-orbifold \( M = \mathcal{U}/\Gamma \) admits a finite-sheeted (orbifold-)covering \( S \) that is a surface. This allows us to generalize the Euler characteristic to a 2-orbifold \( M \):

\[
\chi(M) = 2 - \omega g - p - \sum_i \left( 1 - \frac{1}{m_i} \right) - \frac{1}{k} \sum_{j} \left( 1 - \frac{1}{n_{kj}} \right),
\]

where \( \omega = 1 \) if \( \mathcal{U}/\Gamma \) is non-orientable and \( \omega = 2 \) if \( \mathcal{U}/\Gamma \) is orientable.

Let \( \mathcal{U} \) be the universal covering of a 2-orbifold \( M \). Then \( \mathcal{U} = S^2 \) if \( \chi(M) < 0 \), \( \mathcal{U} = E^2 \) if \( \chi(M) = 0 \) and \( \mathcal{U} = H \) if \( \chi(M) > 0 \).

The hyperbolic area \( \mu(M) \) of a 2-orbifold \( M = H/\Gamma \) depends only on \( s(\Gamma) \). It is calculated by the Gauss-Bonnet formula:

(2.3) \( \mu(\Gamma) = \mu(M) = -2\pi\chi(M), \)

Let \( \Gamma \) be an NEC group with quotient orbifold \( M \). If \( \Gamma' \) is a subgroup of \( \Gamma \) of finite index \( [\Gamma : \Gamma'] = n \) in \( \Gamma \), then \( \Gamma' \) is an NEC group whose quotient orbifold \( M' \) is an \( n \)-sheeted covering of \( M \), whose monodromy map is the representation of the action of \( \Gamma \) on the \( \Gamma \)-cosets. We have the Riemann-Hurwitz formula:

(2.4) \( \mu(\Gamma') = [\Gamma : \Gamma']\mu(\Gamma). \)

Let \( \Gamma \) be an NEC group with quotient 2-orbifold \( M = H/\Gamma \). Then \( M^+ = H/\Gamma^+ \) is a 2-orbifold without mirror lines called the complex double of \( M \). The algebraic genus \( g_a \) of \( M \) is defined to be the topological genus of its complex double, i.e.
(2.5) \[ g_a = \alpha g + k - 1, \]

where \( \alpha = 1 \) if \( M \) is non-orientable and \( \alpha = 2 \) if \( M \) is orientable, \( g \) is the genus of \( M \) and \( k \) is the number of mirror lines in \( M \).

Let \( G \) be a finite group. If \( G \) acts faithfully on a Klein surface \( S \) as a group of automorphisms, where \( S \cong H/A \), then \( G \cong \Gamma/A \), where \( A \) is a normal subgroup of an NEC group \( \Gamma \). Now, the following are equivalent:

a) The number \( \phi(\Gamma, G) \) of normal surface subgroups \( A \) of an NEC group \( \Gamma \) such that \( \Gamma/A \cong G \).

We say that an epimorphism \( \phi: \Gamma \to G \) is a surface kernel epimorphism if \( \ker \phi \) is a group without other elliptic elements than reflections, i.e. \( H/\ker \phi \) is a surface. Then the number \( \phi(\Gamma, G) \) given in a) yields us

b) the number of non-equivalent surface kernel epimorphisms from \( \Gamma \) onto \( G \).

If we consider that two orbifold-coverings of the orbifold \( M = H/\Gamma \) are isomorphic if and only if their fundamental groups are conjugate in \( \Gamma \) (the fundamental group of \( M \)) [8], then we can express a) or b) with geometrical words as follows:

c) the number \( \phi(\Gamma, G) \) of non-equivalent surfaces that are regular (orbifold-)coverings of \( H/\Gamma \), where \( G \) is the group of transformations of the covering.

After the following lemma, our aim is to calculating the above number \( \phi(\Gamma, G) \) when \( G \) is a dihedral group \( D_p \), with \( p \) a prime number. In this case all link-periods in \( \Gamma \) must be equal to the prime \( p \). This is the natural generalization to NEC groups of [6].

**Lemma 2.1.** If \( \Gamma \) is a proper NEC group with some link-period of odd order \( p \), then \( G \) does not admit a cyclic group \( C_p \) as a quotient group by a surface group.

**Proof.** Let \( c_{k,j-1} \) and \( c_{k,j} \) be the reflections associated to the odd link-period \( n_{kj} \). Let \( \phi: \Gamma \to C_p \) be an epimorphism from \( \Gamma \) to \( C_p \), where \( C_p = \langle u/u^p = 1 \rangle \). As \( c_{k,j-1} \) and \( c_{k,j} \) are elements of order 2 in \( \Gamma \), both \( c_{k,j-1} \) and \( c_{k,j} \) must be in \( \ker \phi \). Then \( c_{k,j-1}c_{k,j} \) is an elliptic element in \( \ker \phi \) and \( \ker \phi \) is not a surface group.

Lemma 2.1 does not apply for \( p = 2 \).

**Lemma 2.2.** Let \( \Gamma \) be an NEC group with \( s(\Gamma) = (0, +, [-], \{(2, \ldots, 2), \ldots, (2, \ldots, 2)\}) \), where the \( r \) period cycles are of even length. Then there are \( 2^{2r-1} \) non-equivalent surface coverings of \( \mathcal{U}/\Gamma \) which admit \( C_2 \) as a group of automorphisms.

**Note.** \( \mathcal{U} = S^2 \) if \( s(\Gamma) = (0, +, [-], \{(2, 2)\}) \) and \( \mathcal{U} = E^2 \) if \( s(\Gamma) = (0, +, [-], \{(2, 2, 2)\}) \).
Proof. Let us write $C_2 = \{1, u\}$. As Aut $C_2$ is trivial, to prove the lemma we must count $\phi(\Gamma, C_2)$, i.e. in how many ways can we define $\phi: \Gamma \to C_2$ on the generators $c_{k,j}$ and $e_k$ of $\Gamma$ such that $\phi$ is a surface kernel epimorphism?

Now, we can define $\phi$ for the generators of each period cycle in two ways:

i) $\phi(c_{k,2j}) = 1 = \phi(c_{k,s_k})$, $\phi(c_{k,2j+1}) = u$, $\phi(e_k) = 1 = 1 \leq j \leq (s_k - 2)/2$, or

ii) $\phi(c_{k,2j}) = u = \phi(c_{k,s_k})$, $\phi(c_{k,2j+1}) = 1$, $\phi(e_k) = 1 = 1 \leq j \leq (s_k - 2)/2$, but $\phi(e_k)$ is given by the condition $\prod \phi(e_k) = 1$. So $\phi(\Gamma, C_2) = 2, 4^{r-1} = 2^{2r-1}$.

3. Generating epimorphisms $\phi: \Gamma \to D_p$, with $p$ an odd prime.

We are only interested in those NEC groups $\Gamma$ whose quotient orbifold lies on the sphere and without conic points. So $s(\Gamma) = (0, +, [-], \{C_k\}_{k=1,r})$, where each period cycle $C_k$ have $s_k$ link-periods equal to $p$ and a presentation for $\Gamma$ is the following:

$$\Gamma = \langle e_k, c_{k,j}, k = 1, r, r, j = 0, s_k/c_{k,j}, (c_{k,j-1}c_{k,j})^p, c_{k,0}e_{k,-1}c_{k,x}e_k, \prod e_k \rangle.$$

$D_p = \langle c, u/c^2, u^p, cuc \rangle$. If $p$ is a prime distinct from 2, there is one conjugacy class of elements of order 2 and one conjugacy class of elements of order $p$. If $p = 2$, there are 3 conjugacy classes of elements of order 2, namely $\{c\}, \{u\}$ and $\{cu\}$.

We consider in this paragraph the cases when $p$ is an odd prime.

If $\phi: \Gamma \to D_p$ is a surface kernel epimorphism, then $o(\phi(c_{k,j}))$ divides 2 ($o(z)$ denotes the order of an element $z$ in $D_p$), $\phi(c_{k,j-1}c_{k,j}) = u^y$, for some $y \in \{1, \ldots, p - 1\}$ and $\prod \phi(e_k) = 1$. So the conditions to be satisfied by the images of the generators of $G$ are:

\begin{enumerate}
  \item[(3.1)] i) $g_{kj} = \phi(c_{k,j-1}) = cu^y$, $y \in \{0, \ldots, p - 1\}$, $\phi(c_{k,j}) = cu^x$, where $x \neq y$,
  \item[(3.2)] ii) $g_k = \phi(e_k)$ is such that $\phi(c_{k,0})(\phi(e_k))^{-1} \phi(c_{k,x})\phi(e_k) = 1$ and $\prod \phi(e_k) = 1$.
\end{enumerate}

Since a set $G_\phi = \{g_{kj}, g_k/satisfying (3.1) and (3.2)\}$ is a generating set for $D_p$, the number $\phi(\Gamma, D_p)$ is the number of orbits of the sets $G_\phi$ under the action of Aut $D_p$.

But Aut $D_p = C_p \bowtie C_{p-1}$, so $|\text{Aut } D_p| = p(p - 1)$.

Theorem 3.1. Let $\Gamma$ be an NEC group with $s(\Gamma) = (0, +, [-], \{(p, \ldots, p)\})$, where there are $s \geq 3$ link-periods. Then there are $\phi(\Gamma, D_p) = (p - 1)^r - \sum_{j=0}^{s-3} \binom{s-2}{j} p^{s-3-j}(-1)^j$ non equivalent surface coverings of $\mathcal{U}/\Gamma$ and the surfaces admit $D_p$ as a group of automorphisms.

Again, $\mathcal{U} = E^2$ if $s(\Gamma) = (0, +, [-], \{(3, 3, 3)\})$, otherwise $\mathcal{U} = H$. 


Proof. We begin by calculating the number $Q_s$ of different sets $G_{\phi} = \{\phi(c_i), 0 \leq i \leq s - 1\}$, where $c_i, 0 \leq i \leq s$ are the generating reflections of $\Gamma$. Notice that $c_0 = c_s$.

For $s = 3$, we have that $\phi(c_0) = cu^x, \phi(c_1) = cu^y$, with $y \neq x$ and $\phi(c_2) = cu^z$, with $z \neq x$ and $z \neq y$. So there are $Q_3 = p(p - 1)(p - 2)$ different sets.

For $s = 4$, we have $\phi(c_0), \phi(c_1)$ and $\phi(c_2) = cu^z$ as for $s = 3$. For $\phi(c_3)$, we have, respectively, $p - 1$ choices if $z = x$ or $p - 2$ if $z \neq x$. So $Q_4 = p(p - 1)[(p - 1) + (p - 2)^2]$.

For $s \geq 5$, we have $p - 2$ choices for $\phi(c_{s - 1})$ if $\phi(c_{s - 2})$ is distinct from $\phi(c_0)$ and $p - 1$ choices if $\phi(c_{s - 2})$ is equal to $\phi(c_0)$. But $\phi(c_{s - 2})$ is equal to $\phi(c_0)$ if $\phi(c_{s - 3})$ is distinct from $\phi(c_0)$. So $Q_s$ satisfies the equation:

$$Q_s = (p - 2)Q_{s - 1} + (p - 1)Q_{s - 2}.$$  

Equation (3.3) is a homogeneous difference equation with characteristic polynomial

$$r^2 - (p - 2)r - (p - 1).$$

The zeros of (3.4) are $p - 1$ and $-1$. So, the general solution is $Q_s = (p - 1)^s A + (-1)^s B$. Using $Q_3$ and $Q_4$ to determine the constants $A$ and $B$, we get $A = 1, B = p - 1$. So

$$Q_s = (p - 1)^s + (-1)^s(p - 1)$$

i.e. $Q_s = [(p - 1)^{s - 1} + (-1)^s](p - 1)$, where $(p - 1)^{s - 1} + (-1)^s = p \sum_{k=0}^{s-2} \binom{s - 1}{k} p^{s - 2 - k}(-1)^k$, so

$$Q_s = (p - 1)p \sum_{k=0}^{s-2} \binom{s - 1}{k} p^{s - 2 - k}(-1)^k =$$

$$= \left[(p - 1)^{s - 2} - \sum_{j=0}^{s-3} \binom{s - 2}{j} p^{s - 3 - j}(-1)^j\right](p - 1)p.$$

$$\phi(\Gamma, D_p) = \frac{Q_s}{p(p - 1)} = (p - 1)^{s - 2} - \sum_{j=0}^{s-3} \binom{s - 2}{j} p^{s - 3 - j}(-1)^j.$$

Note 1. By the Riemann-Hurwitz formula and using the representation of $D_p$ as permutation group, there are different biconformal structures on a non-orientable surface or orientable surface of genus $(s - 2)(p - 1)$ or $(s - 2)(p - 1)/2$ respectively without boundary components. The subgroups $A$ of $\Gamma$ associated to them have signature $s(A) = ((s - 2)(p - 1), -[, -], \{ - \})$ or $s(A) = ((s - 2)(p - 1)/2, +, [ - ], \{ - \})$.  


The minimal genus surfaces with $D_p$ as a group of automorphism occur when $\Gamma$ is an NEC group with signature $s(\Gamma) = (0, +, [\, - \], \{(p, p, p)\})$. The number of such non-equivalent surface coverings is $p - 2$ according to theorem 3.1. In particular, there is a unique biconformal structure for a torus admitting $D_3$ as a group of automorphisms. A fundamental region for this torus is shown in figure 1. Its fundamental group $\Lambda$ is a normal subgroup of an NEC group with signature $s(\Gamma) = (0, +, [\, - \], \{(3, 3, 3)\})$ and with the following permutation representation: $\phi: \Gamma \to \Sigma_6$ defined by $f(c_0) = (1, 2)(3, 4)(5, 6)$, $f(c_1) = (1, 3)(2, 5)(4, 6)$, $f(c_2) = (1, 6)(3, 5)(2, 4)$. Notice that $(1, 2)(3, 4)(5, 6)$ and $(1, 5, 4)(2, 3, 6) = f(c_0)f(c_1)$ generate $D_3$. The map $\phi$ is also the monodromy map of the (orbifold-)covering $F: E^2/\Lambda \to E^2/\Gamma$.

Note 2. We can extrapolate to $Q_2 = p(p - 1)$, $Q_1 = 0$, $Q_0 = p$. They have geometrical interpretation. For instance, the signature $s(\Gamma) = (0, +, [\, - \], \{(p)\})$ is not admissible since the orbifold lying on a disc with one corner point is not a good orbifold.

With the same calculations as in theorem 3.1:

**Corollary 3.1.** Let $\Gamma$ be an NEC group with $s(\Gamma) = (0, +, [\, - \], \{(p, \ldots, p), \ldots, (p, \ldots, p)\})$, where at least one period cycle has 2 or more link-periods and all cycles are non-empty. The number $\phi(\Gamma, D_p)$ of non-equivalent epimorphisms from $\Gamma$ onto $D_p$ is $p^{r-2}(p - 1)^{s-1}$, where $s$ is the number of link-periods and $r$ is the number of period cycles in $s(\Gamma)$.

**Proof.** We assume that the last period cycle has more than 1 link-period. Let $s_i$ be the number of link-periods in the $i$-th cycle, with $\sum s_i = s$. Now $\phi(c_{i,s_i})$ is
conjugate, but not necessarily equal to \( \phi(c_{i,o}) \). So for all cycles except the last one the number of choices of \( \{ \phi(c_{i,j}), 0 \leq j \leq s, 1 \leq i \leq r - 1 \} \) are \( Q'_r = p(p - 1)^s \).

To calculate the number of choices for the last period cycle we must distinguish two cases:

a) If \( \phi(e_r) = 1 \) from the relator \( \prod \phi(e_i) = 1 \), then \( \phi(c_{i,s_i}) = \phi(c_{i,s_r}) \) and \( Q''_r = Q_s \), where \( Q_s = (p - 1) p \sum_{k=0}^{s-2} \binom{s - 1}{k} p^{s - 2 - k}(-1)^k \) is given in theorem 3.1.

b) If \( \phi(e_r) \neq 1 \), then \( \phi(e_r) = u^z \) with \( z \neq 0 \). If \( \phi(c_{r,0}) = cu^x \), then \( \phi(c_{r,s_r}) = cu^x \), where \( x \) satisfies the equation \( x - y = -2z \mod(p) \).

If \( s_r = 2 \), then \( Q''_r = p(p - 2)(\phi(c_{r,0}) = cu^x, \phi(c_{r,2}) = cu^x, \phi(c_{r,1}) = cu^x \), with \( x' \) distinct from \( x \) and \( y \).

If \( s_r = 3 \), then \( Q''_r = p(p - 1) + p(p - 2)^2 \).

For \( s_r \geq 4 \), we have \( p - 2 \) choices for \( \phi(c_{r,s_r - 2}) \) is distinct from \( \phi(c_{r,s_r}) \) and \( p - 1 \) choices if \( \phi(c_{r,s_r - 2}) \) is equal to \( \phi(c_{r,s_r}) \). But \( \phi(c_{r,s_r - 2}) \) is equal to \( \phi(c_{r,s_r}) \) if \( \phi(c_{r,s_r - 3}) \) is equal to \( \phi(c_{r,s_r}) \). So \( Q''_r \) satisfies the equation (3.3) with characteristic polynomial (3.4). The general solution is \( Q''_r = (p - 1)^x A' + (-1)^y B' \). Using \( Q''_r \) for \( s_r = 2 \), \( s_r = 3 \) to determine the constants \( A' \) and \( B' \), we get \( A' = 1, B' = -1 \). So

\[
Q''_r = (p - 1)^{s_r} + (-1)^{s_r + 1} = p \sum_{j=0}^{s_r - 1} \binom{s_r}{j} p^{s_r - 1 - j}(-1)^j
\]

So the number of choices for the last cycle is:

\[
Q' = \frac{Q_s}{p} + \frac{(p - 1)Q''_r}{p} = (p - 1) \left[ \sum_{j=0}^{s_r - 2} \binom{s_r - 1}{j} p^{s_r - 2 - j}(-1)^j \right] + \sum_{j=0}^{s_r - 1} \binom{s_r}{j} p^{s_r - 1 - j}(-1)^j \right] = (p - 1)(p - 1)^{s_r - 1} = (p - 1)^{s_r}.
\]

and \( \phi(\Gamma, D_p) = \frac{\Pi Q'_i}{p(p - 1)} = \frac{p^{r-1}(p - 1)^s}{p(p - 1)} = p^{r-2}(p - 1)^{s-1} \).

**Corollary 3.2.** Let \( \Gamma \) be an NEC group with \( s(\Gamma) = (0, +, [-], \{p, \ldots, p\}, \ldots, (p, \ldots, p)) \), with \( s \) link-periods in \( r \) period cycles, where at least one period cycle is empty. Then there are \( \phi(\Gamma, D_p) = (p - 1)^{s-1} p^{r-2} \) non equivalent surface coverings of \( \mathcal{U}/\Gamma \) which admit \( D_p \) as a group of automorphisms.

**Proof.** We can assume that the empty period cycles are the \( r - r' \) last ones and each of the \( r' \) first period cycles has \( s_i \) link-periods, with \( \sum s_i = s \). So for all non-empty period cycles the number of choices of \( \{ \phi c_{i,j}, 0 \leq j \leq s, 1 \leq i \leq r' \} \) are
$Q_i = p(p - 1)^{r_i}$. $\phi(e_r)$ is given as the commutator of $\phi(c_{i,0})$ and $\phi(c_{i,s})$. The number of choices for all empty period cycles except the last one is $p$.

Finally, $\phi(c_r)$ and $\phi(e_r)$ are fixed by the relators $\prod \phi(e_i) = 1_d$, $\phi(c_r)(\phi(e_r))^{-1}$ $\phi(c_i)\phi(e_i) = 1_d$. Therefore the number of sets $G_\phi$ is $p^{r-1}(p - 1)^s$, and $\phi(\Gamma, D_p) = p^{r-2}(p - 1)^{s-1}$.

**Remark.** To calculate the number $\phi(\Gamma, D_p)$ of non equivalent surface coverings of $H/\Gamma$ with $D_p$ as the group of covering-transformations is slightly different from theorem 3.1 for the groups $\Gamma$ when all period cycles have exactly one link-period. If $\Gamma$ is such a group, then, as in note 2, $s(\Gamma) = (0, +, [-], \{(p), \ldots, (p)\})$, with at least 2 period cycles. The generators of $\Gamma$ are $c_{i,0}, c_{i,1}, e_i$, $1 \leq i \leq r$, where $r$ is the number of period cycles. To calculate the different sets $G_\phi = \{g_{k,j, g_k/satisfying (3.1) and (3.2)}\}$, we must consider that $\phi(c_{i,0}) \neq \phi(c_{i,1})$, $1 \leq i \leq r$, so $\phi(e_i) \neq 1_d$.

The case when $r = 2$, $\phi(c_{1,0}) = cu^x$, $\phi(c_{1,1}) = cu^y$, with $y \neq x$, $\phi(e_1) = u^z$ is the commutator of $cu^x$ and $cu^y$, so $z \neq 0$. We have choices for $\phi(c_{2,0})$, but $\phi(c_{2,1})$ and $\phi(e_2)$ are fixed. The number of sets $G_\phi$ is $I_2 = p^2(p - 1)$, and $\phi(\Gamma, D_p) = p$.

The case when $r = 3$. $\phi(c_{1,0}) = cu^x$, $\phi(c_{1,1}) = cu^y$, with $y \neq x$, $\phi(e_1) = u^z$, where $u^z$ is the commutator of $cu^x$ and $cu^y$, so $z \neq 0$. To choose $\phi(e_2)$ and $\phi(e_3)$, we must satisfy the condition

(3.7) $\phi(e_2)\phi(e_3) = u^{-z}$, with $\phi(e_2) \neq 1_d$ and $\phi(e_3) \neq 1_d$.

$\phi(c_{2,1})$ and $\phi(c_{3,1})$ are given by the condition $c_{i,0} = e_i^{-1}c_{i,1}e_i$.

Condition (3.7) is equivalent to the following: counting ordered pairs $(z', z'')$ of numbers between 1 and $p - 1$ such that $z' + z'' = -z \mod(p)$.

There are $(p - 2)$ such pairs, hence $I_3 = p^3(p - 1)(p - 2)$, and $\phi(\Gamma, D_p) = p^2(p - 2)$.

If $r \geq 4$, then $\phi(c_{i,0}) = cu^x$, $\phi(c_{i,1}) = cu^y$, with $y \neq x$, $\phi(e_i) = u^z$, $1 \leq i \leq r - 2$, where $u^z$ is the commutator of $cu^x$ and $cu^y$, so $z \neq 0$. If $\prod_{i=1}^{r-2} e_i = 1_d$, then we do as in the case $r = 2$ for the two last cycles. If $\prod_{i=1}^{r-2} e_i \neq 1_d$, then we do as in the case $r = 3$ for the last cycles.

Hence $I_r = p^{r-2}(p - 1)^{r-2}p^2\left[\frac{(p - 1)(p - 2)}{p} + \frac{(p - 1)}{p}\right] = p^{r-1}(p - 1)^{r-1}$ $(p - 2 + 1) = p^{r-2}(p - 1)^{r-1}$. Therefore $\phi(\Gamma, D_p) = p^{r-2}(p - 1)^{r-1}$.

**Note 3.** For the NEC groups $\Gamma$ in theorem 3.1 and corollaries 3.1, 3.2 and the previous remark, $\text{Ker } \phi$ is a normal surface subgroup of index $2p$ in $\Gamma$. This is just twice the minimal index for surfaces subgroups calculated in [3].
4. Surfaces with $D_2$ as a group of automorphisms.

If $p = 2$, there are 3 conjugacy classes of elements of order 2, namely $\{c\}$, $\{u\}$ and $\{cu\}$. If $\phi: \Gamma \to D_2$ is a surface kernel epimorphism, then $o(\phi(c_{k,j}))$ divides 2 ($o(z)$ denotes the order of an element $z$ in $D_2$), $\phi(c_{k,j-1}c_{k,j})$ must be $c$, $u$ or $cu$, and $\prod \phi(e_k) = 1_d$. So the conditions to be satisfied by the images of the generators of $G$ are:

i) $g_{kj} = \phi(c_{k,j-1}) = z$, $z \in \{c, u, cu, 1_d\}$, $\phi(c_{k,j}) = z'$, $z' \in \{c, u, cu, 1_d\}$, where $z \neq z'$,

ii) $g_k = \phi(e_k)$ is such that $\phi(c_{k,0})(\phi(e_k))^{-1}\phi(c_{k,s})\phi(e_k) = 1_d$ and $\prod \phi(e_k) = 1_d$.

Since a set $G_\phi = \{g_{kj}, g_k/satisfying i) and ii)\}$ may generate $D_2$ or any of the 3 cyclic subgroups of order 2, we consider the sets $G'_\phi = \{g_{kj}, g_k/satisfying i) and ii)\}$, generating $D_2$. The number $\phi(\Gamma, D_2)$ is the number of orbits of the sets $G'_\phi$ under the action of Aut $D_2$. But Aut $D_2 = S_3$, so $|Aut D_2| = 6$.

**Lemma 4.1.** Let $\Gamma$ be an NEC group with all link-periods equal to 2. If $\Gamma$ has some period cycle with exactly one link-period, then $D_2$ is not a quotient group of $\Gamma$ by a surface group.

**Proof.** Let $\phi: \Gamma \to D$ an epimorphism from $\Gamma$ to $D_2$. Let $C_k$ be the period cycle with exactly one link-period. $\Gamma$ has, among others, the generators $c_{k,0}$, $c_{k,1}$ and $e_k$ with the relation $c_{k,0} = e_k^{-1}c_{k,1}e_k$. But the elements of order 2, $\phi(c_{k,0})$ and $\phi(c_{k,1})$, are conjugate in $D_2$ if and only if $\phi(c_{k,0}) = \phi(c_{k,1})$. So $\phi(c_{k,0}c_{k,1}) = 1$, and Ker $\phi$ is not a surface group.

In the following, we consider signatures of NEC groups where there are at least two link-periods in each cycle.

**Theorem 4.1.** Let $\Gamma$ be an NEC group with $s(\Gamma) = (0, +, [-], \{(2, \ldots, 2)\})$, where there are $s$ link-periods all equal to 2. The number of non equivalent surface coverings of $\mathcal{U}/\Gamma$ which admit $D_2$ as a group of automorphisms is:

\[\phi(\Gamma, D_2) = 2 \sum_{j=0}^{s-2} \binom{s-1}{j} 4^{s-2-j}(-1)^j \text{ if } s \text{ is odd, or}\]

\[\phi(\Gamma, D_2) = 2 \sum_{j=0}^{s-2} \binom{s-1}{j} 4^{s-2-j}(-1)^j - 1 \text{ if } s \text{ is even.}\]

**Note.** $\mathcal{U} = S^2$ if $s(\Gamma) = (0, +, [-], \{(2, 2)\})$ or $s(\Gamma) = (0, +, [-], \{(2, 2, 2)\})$, $\mathcal{U} = E^2$ if $s(\Gamma) = (0, +, [-], \{(2, 2, 2, 2)\})$.

**Proof.** We must distinguish the cases when the number of link-periods $s$ is odd or even.

a) $s$ is odd. Then $G_\phi = G'_\phi$. The number $Q_s$ of different sets $G_\phi = \{\phi(c_i),$
where $c_i$, $0 \leq i \leq s$ ($c_0 = c_s$), are the generating reflections of $\Gamma$, is calculated as in theorem 3.1, but now $p = 4$. So

$$Q_s = (4 - 1)^s + (-1)^s(4 - 1) = 12 \sum_{j=1}^{s-2} \binom{s - 1}{j} p^{s - 2 - j}(-1)^j,$$

and

$$\phi(\Gamma, D_2) = \frac{Q_s}{6} = 2 \sum_{j=1}^{s-2} \left( \binom{s - 1}{j} p^{s - 2 - j}(-1)^j \right).$$

b) $s$ is even. Then the number of sets $G'_\phi$ is the number of sets $G_\phi$ minus 3 times the number $C_s$ of different sets that generate any of the cyclic subgroups of $D_2$. We have calculated this number $C_s$ in lemma 2.2. We have $C_s = 2$.

So

$$Q_s = (4 - 1)^s + (-1)^s(4 - 1) - 6 = 6 \left[ 2 \sum_{j=1}^{s-2} \left( \binom{s - 1}{j} p^{s - 2 - j}(-1)^j - 1 \right) \right].$$

Therefore,

$$\phi(\Gamma, D_2) = \frac{Q_s}{6} = 2 \sum_{j=1}^{s-2} \left( \binom{s - 1}{j} p^{s - 2 - j}(-1)^j - 1 \right).$$

Corollary 4.1. Let $\Gamma$ be an NEC group with $s(\Gamma) = (0, +, [-], \{2, \ldots, 2\}), \ldots, (2, \ldots, 2)$), where there $r$ period cycles, each of them with $s_i$ link-periods equal to 2. The number of non equivalent surface coverings of $\mathcal{U}/\Gamma$ that admit $D_2$ as a group of automorphisms is:

a) $\phi(\Gamma, D_2) = \left( 3^{r-1} \prod_{i=1}^{r} [3^{s_i - 1} + (-1)^{s_i}] \right)/2$ if $s_i$ is odd for some $i$, or

b) $\phi(\Gamma, D_2) = \left( 3^{r-1} \prod_{i=1}^{r} [3^{s_i - 1} + (-1)^{s_i}] \right)/2 - 4^{r-1}$ if all $s_i$ are even.

Proof. First of all, as two elements $\phi(c_{i,0})$ and $\phi(c_{i,s_i})$ of order 2 in $D_2$ are conjugate if and only if $\phi(c_{i,0}) = \phi(c_{i,s_i})$, the number of sets $G_\phi$ for the $i$th cycle is $Q_{s_i}$, where $Q_{s_i}$ is given in theorem 4.1.

a) Some $s_i$ is odd. Then $G_\phi = G'_\phi$, $Q_{s_i} = 3[3^{s_i - 1} + (-1)^{s_i}]$, we notice that 4, and hence 2, divides $3^{s_i - 1} + (-1)^{s_i}$.

So

$$\phi(\Gamma, D_2) = \frac{\prod Q_{s_i}}{6} = \left( 3^{r-1} \prod_{i=1}^{r} [3^{s_i - 1} + (-1)^{s_i}] \right)/2.$$

b) All $s_i$ are even. Then the number of sets $G'_\phi$ is the number of sets $G_\phi$ minus 3 times the number $C_r$ of different sets that generate any of the cyclic subgroups of $D_2$. We have calculated this number $C_r$ in lemma 2.2. We have $C_r = 2^{2r-1}$. 
\[
\phi(\Gamma, D_2) = \frac{3^r \prod_{i=1}^{r} [3^{s_i} - 1 + (-1)^{s_i}] - 3(2^{2r} - 1)}{6} = \\
= \left(3^{r-1} \prod_{i=1}^{r} [3^{s_i} - 1 + (-1)^{s_i}] \right) / 2 - 4^{r-1}.
\]

REFERENCES