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EMBEDDINGS OF VALUED GROUPS

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1. Introduction.

In this paper we investigate the problem of embedding a valued group into an adequate Hahn group. The study of groups with general valuation – an ultrametric function into an ordered set with a least element – was initiated in [7].

If G is a group with a normal valuation, then one can construct a Hahn product of the factor groups formed from the closed 0-centered balls over the respective open balls. Since the Hahn group consists of those elements of the general product group whose supports are dually well-ordered, its elements may be regarded as a kind of generalized sequences and thus their operational behaviour is reasonably regular. Furthermore, it is worth noting that, in most cases, the Hahn products are continuously closed and Cauchy complete; see [8].

The group G is embeddable into the Hahn group mentioned above only if the valuation of G is conjugation invariant. Especially, G must be a valued group in the sense of [7]. The embedding is furthermore expected to be immediate in the sense of [8]; this means that the elements of the Hahn group are pseudo-limits from the elements of G. Pseudo-convergence and immediate extensions as well as relationships between them are studied in [8].

As a first we give a necessary and sufficient condition for the existence of the desired embedding. Utilizing this we formulate a sufficient condition lattice-theoretically: the group G is immediately embeddable into the Hahn group if in the lattice of normal subgroups of G there exists a Banaschewski function, that is, an antitonic complement function, defined on the set of all convex subgroups. This formulation gives further the possibility to use the existence results for Banaschewski functions presented in [6] to find new solutions to the embedding problem. All together, our results form a generalization to the embedding theory for ordered groups, well-known both in the commutative and non-commutative case; see [1]–[5]. As an application we get also the classical embedding theorem of Hahn.

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2. Preliminaries.

On valuation theoretical concepts we refer to the articles [7] and [8]. We recall here only the most relevant definitions.

Let G denote a not necessarily commutative group, which anyway will be written additively, and let Γ denote an ordered set, which has a least element 0. Denote the set $\Gamma \setminus \{0\}$ by Γ^*.

A mapping $|\cdot|: G \to \Gamma$ is a *valuation*, if

(1) $|g| = 0$ if and only if $g = 0$;

(2) $|g - h| \leqq \max\{|g|, |h|\}$ for all $g, h \in G$.

If $|\cdot|$ is a valuation, then the triple $(G, \Gamma, |\cdot|)$, or simply G, is called a *group with valuation*. A valuation $|\cdot|: G \to \Gamma$ is

(1) *conjugation invariant*, if $|g + h - g| = |h|$ for all $g, h \in G$;

(2) *normal*, if $|g| < |h|$ implies $|h + g - h| < |h|$.

A group H with valuation is an *immediate extension* of its subgroup G if for every $h \in H$ with $|h| > 0$ there exists $g \in G$ such that $|h - g| < |h|$ and $|-g + h| < |h|$.

The groups G and G' with valuations $|\cdot|$ and $|\cdot|'$, respectively, are *valuation isomorphic* if there is a group isomorphism $\varphi: G \to G'$ and an order isomorphism $\eta: |G| \to |G'|'$ such that $|\varphi(g)|' = \eta(|g|)$ for all $g \in G$.

Let A be an ordered set and for each $\alpha \in A$ let $(G_\alpha, \Gamma_\alpha, |\cdot|_\alpha)$ be a group with valuation. For $g \in \prod_A G_\alpha$ set $\operatorname{supp}(g) = \{\alpha \in A : |g(\alpha)|_\alpha \neq 0\}$. Define further:

$$G = \left\{ g \in \prod_A G_\alpha : \operatorname{supp}(g) \text{ is dually well-ordered in } A \right\},$$

$$\Gamma = \{0\} \oplus \sum_A \Gamma_\alpha^* \ \text{(ordered sum)},$$

$$|g| = \begin{cases} 0, & \text{if } \operatorname{supp}(g) = \emptyset, \\ |g(\alpha)|_\alpha, & \text{if } \operatorname{supp}(g) \neq \emptyset \text{ and } \max \operatorname{supp}(g) = \alpha. \end{cases}$$

Then the set $\mathbf{H}(A; G_\alpha) = \mathbf{H}(A; G_\alpha, \Gamma_\alpha, |\cdot|_\alpha) = (G, \Gamma, |\cdot|)$ is a group with valuation; it is called a *Hahn group (with valuation)*.

3. Immediate embeddings.

Let $G = (G, \Gamma, |\cdot|)$ be a group with valuation. If the valuation $|\cdot|$ is normal then the open balls $B(\gamma) = \{g \in G : |g| < \gamma\}$ are normal subgroups of the corresponding closed balls $B'(\gamma) = \{g \in G : |g| \leqq \gamma\}$. Hence we can form the quotient groups $G(\gamma) = B'(\gamma)/B'(\gamma)$ for all $\gamma \in \Gamma^*$. We define a simple valuation on the group $G(\gamma)$ by the rule

$$|g + B(\gamma)|_\gamma = \begin{cases} 0, & \text{if } g \in B(\gamma), \\ \gamma, & \text{if } g \notin B(\gamma); \end{cases}$$

thus the value set may be chosen to $\Gamma_\gamma = \{0, \gamma\}$ for $\gamma \in |G| \setminus \{0\}$, and $\Gamma_\gamma = \{0\}$ otherwise. We now constitute the Hahn group $\mathbf{H}(\Gamma^*; G(\gamma)) = \mathbf{H}(\Gamma^*; G(\gamma), \Gamma_\gamma, |\cdot|_\gamma)$ and ask when the original group G may be considered as a subgroup of this Hahn group in the following sense.

3.1. DEFINITION. Let G and H be groups with valuations. The group G is *immediately embeddable* into the group H if there is a subgroup G' of H such that G is value isomorphic to G' and H is an immediate extension of G'.

Since the valuations $|\cdot|_\gamma$ considered above are conjugation invariant, also the valuation of the Hahn group $\mathbf{H}(\Gamma^*; G(\gamma))$ has the same property; see [7], Proposition 2.7. Hence the group G is embeddable into this Hahn group only if the valuation of G is conjugation invariant. Expecially, G must be a valued group; see [7], Theorem 2.8. A sufficient and necessary condition for the existence of an immediate embedding is expressed in the following result, which is a generalization of [1], Corollary to Lemma 2.5.

3.2. THEOREM. *Let G be a group with a conjugation invariant valuation. Then G is immediately embeddable into the Hahn group $\mathbf{H}(\Gamma^*; G(\gamma))$ if and only if there is a collection $\{T_\gamma\}_{\gamma \in \Gamma^*}$ of normal subgroups of G such that*

(1) *$B(\gamma) = B'(\gamma) \cap T_\gamma$ for all $\gamma \in \Gamma^*$;*

(2) *$G = B'(\gamma) + T_\gamma$ for all $\gamma \in \Gamma^*$;*

(3) *the set $\Gamma_g = \{\gamma \in \Gamma^*: g \notin T_\gamma\}$ is dually well-ordered for all non-zero $g \in G$.*

PROOF. Suppose first that G is immediately embeddable into the Hahn group $H = \mathbf{H}(\Gamma^*; G(\gamma))$. Then there is a value preserving monomorphism $\varphi: G \to H$ such that H is an immediate extension of $\varphi(G)$. For all $\gamma \in \Gamma^*$ choose $N_\gamma = \{h \in H : h(\gamma) = 0\}$ and observe that the normal subgroups N_γ fulfil the conditions (1)–(3) in H. As in [1], Lemma 2.5, one can prove that the collection $\{N_\gamma \cap \varphi(G)\}_{\gamma \in \Gamma^*}$ fulfils these conditions in $\varphi(G)$. Finally, choose $T_\gamma = \varphi^{-1}(N_\gamma)$ for all $\gamma \in \Gamma^*$.

Conversely, suppose that there is a collection $\{T_\gamma\}_{\gamma \in \Gamma^*}$ of normal subgroups of G with the properties (1)–(3). On each group G/T_γ we define a valuation by the rule

$$|g + T_\gamma|_\gamma = \begin{cases} 0, & \text{if } g \in T_\gamma, \\ \gamma, & \text{if } g \notin T_\gamma. \end{cases}$$

Form now the Hahn group $H = \mathbf{H}(\Gamma^*; G/T_\gamma)$. Along with the valuations $|\cdot|_\gamma$ the valuation of H is also conjugation invariant.

Using the condition (3) we can define a mapping $\varphi: G \to H$ by the rule $\varphi(g)(\gamma) = g + T_\gamma$ for all $g \in G$ and $\gamma \in \Gamma^*$. Clearly φ is a group homomorphism and $|\varphi(g)| = \max \Gamma_g$ for all $g \in G \backslash \{0\}$. The condition (1) guarantees that φ is value preserving: If $g \in G \backslash \{0\}$, then $g \notin T_{|g|}$ and hence $|g| \in \Gamma_g$. On the other hand, if $\gamma \in \Gamma_g$ then $g \notin T_\gamma$ and hence $|g| \geqq \gamma$. Therefore $|g| = \max \Gamma_g = |\varphi(g)|$.

To show that H is an immediate extension of $\varphi(G)$, let $h \in H$ with $\gamma = |h| \neq 0$ be arbitrary. Then $h(\gamma) = g + T_\gamma$ for some $g \in G$. Let g be decomposed $g = b' + t$ according to (2). Then $(h - \varphi(b'))(\gamma) = (g + T_\gamma) - (b' + T_\gamma) = (g - b') + T_\gamma = T_\gamma$. Furthermore, if $\delta > \gamma$ then $b' \in T_\delta$ and thus $(h - \varphi(b'))(\delta) = T_\delta - (b' + T_\delta) = T_\delta$. Therefore $|h - \varphi(b')| < \gamma = |h|$. Analogously one can prove that $|-\varphi(b') + h| < |h|$. Hence H is an immediate extension of $\varphi(G)$, and the group G is immediately embeddable into the Hahn group $H = \mathbf{H}(\Gamma^*; G/T_\gamma)$.

To complete the proof it is sufficient to note that, by (1), (2) and a fundamental theorem of group homomorphisms, the groups G/T_γ and $G(\gamma)$ are isomorphic for each $\gamma \in \Gamma^*$. Clearly, they are isomorphic also as groups with valuations. Hence also their Hahn products are valuation isomorphic.

REMARK. The first two properties in Theorem 3.2 guarantee that the subgroups T_γ are strictly closed, that is, they are τ_r-closed and the infima $|g; T_\gamma| = \inf\{|g - t| : t \in T_\gamma\}$ are always attained. Thus one can form the quotient valuations $|\cdot; T_\gamma|$ on the groups G/T_γ in the sense of [7], Section 3. Interestingly enough, these valuations coincide with the above defined valuations $|\cdot|_\gamma$.

In the following we give some sufficient conditions for the group G to be immediately embeddable into the Hahn group. For all $\Delta \subset \Gamma^*$ we denote

$$B(\Delta) = \bigcap_{\delta \in \Delta} B(\delta).$$

Let the set of all subsets of Γ^* be preordered by defining $\Delta \leqq \Delta'$ if $B(\Delta) \subset B(\Delta')$. We write $G = H \oplus K$ for normal subgroups H and K if $G = H + K$ and $H \cap K = \{0\}$.

3.3. THEOREM. *Let G be a group with a conjugation invariant valuation. Suppose that there is a collection $\{G_\Delta\}_{\Delta \subset \Gamma^*}$ of normal subgroups of G such that*

(1) *$G = B(\Delta) \oplus G_\Delta$ for all $\Delta \subset \Gamma^*$;*

(2) *if $\Delta, \Delta' \subset \Gamma^*$ with $\Delta \leqq \Delta'$ then $G_\Delta \supset G_{\Delta'}$.*

Then G is immediately embeddable into the Hahn group $\mathbf{H}(\Gamma^*; G(\gamma))$.

PROOF. For all $\gamma \in \Gamma^*$ let $G_{\gamma+} = G_{\Delta_\gamma}$, where $\Delta_\gamma = \{\delta \in \Gamma^* : \delta > \gamma\}$; hence $G = B'(\gamma) \oplus G_{\gamma+}$. Choose $T_\gamma = B(\gamma) \oplus G_{\gamma+}$ for all $\gamma \in \Gamma^*$. It is straightforward to show that these normal subgroups fulfil the first two conditions listed in Theorem 3.2.

To show that also the third condition of Theorem 3.2 is satisfied one may argue as follows. For a given $g \in G \setminus \{0\}$ let A be a non-empty subset of $\Gamma_g = \{\gamma \in \Gamma^* : g \notin T_\gamma\}$. Choose $\Delta = \{\gamma \in \Gamma^* : \gamma > \alpha \text{ for all } \alpha \in A\}$; then $B(\Delta) = \cup_{\alpha \in A} B'(\alpha)$. According to the assumption there are decompositions

$$G = B(\Delta) \oplus G_\Delta = B'(\alpha) \oplus G_{\alpha+}$$

for all $\alpha \in A$. Let $g = x + y$, where $x \in B(\Delta)$ and $y \in G_\Delta$. Let further these elements be decomposed

$$g = g_\alpha + g_\alpha^+, \quad x = x_\alpha + x_\alpha^+, \quad y = y_\alpha + y_\alpha^+,$$

where $g_\alpha, x_\alpha, y_\alpha \in B'(\alpha)$ and $g_\alpha^+, x_\alpha^+, y_\alpha^+ \in G_{\alpha+}$. As $G_\Delta \subset G_{\alpha+}$, we get $y_\alpha = 0$, and thus $g_\alpha = x_\alpha$. On the other hand, since $x \in B(\Delta)$, there exists $\alpha_0 \in A$ such that $x \in B'(\alpha_0)$. This element α_0 is in fact the greatest element of A: if there where an element $\alpha \in A$ such that $\alpha > \alpha_0$, then $x \in B(\alpha)$ and consequently $g = x + y_\alpha^+ \in B(\alpha) + G_{\alpha+} = T_\alpha$, a contradiction. Thus the set Γ_g is dually well-ordered.

An application of Theorem 3.2 completes the proof.

We can give a lattice-theoretical formulation for Theorem 3.3. Recall first from [6] that for a bounded lattice $L = (L, \wedge, \vee)$ and for a non-empty subset X of L a function $\tau: X \to L$ is a *Banaschewski function* on X if the statements (1) $x \wedge \tau(x) = 0$, (2) $x \vee \tau(x) = 1$, and (3) $x \leqq y$ implies $\tau(x) \geqq \tau(y)$, hold for all x, $y \in X$. Let $N(G)$ denote the lattice of all normal subgroups of G and let $C(G)$ denote the set of all convex subgroups of G; a subset C of G is *convex in* G if $c \in C$, $g \in G$ and $|g| \leqq |c|$ imply $g \in C$. Apparently $C(G) = \{B(\Delta)\}_{\Delta \subset \Gamma^*}$ and if the valuation of G is conjugation invariant then (and only then) $C(G) \subset N(G)$. Hence Theorem 3.3 may be reformulated in the following way.

3.4. THEOREM. *Let G be a group with a conjugation invariant valuation. If in the lattice $N(G)$ there exists a Banaschewski function on $C(G)$, then G is immediately embeddable into the Hahn group* $\mathbf{H}(\Gamma^*; G(\gamma))$.

The following theorems give sufficient conditions for the existence of appropriate Banaschewski functions. The first one is an application of a general result on lattices [6], Theorem 5.4. Recall that the lattice $N(G)$ is modular, which means that $A \cap (B + C) = A \cap B + C$ for all $A, B, C \in N(G)$ such that $C \subset A$. Note also that $C(G)$ is always a chain, that is, a totally ordered set.

3.5. THEOREM. *Let G be a group with a conjugation invariant valuation. Suppose that the lattice $N(G)$ satisfies the following condition: if $A \in N(G)$ and $A \neq G$ then there is $P \in N(G)$, $P \neq \{0\}$, such that $A \cap P = \{0\}$ and for each $B \in N(G)$ either the subgroups B and P are distinct or one of them contains the other. Then in $N(G)$ there is a Banaschewski function on $C(G)$, and hence G is immediately embeddable into the Hahn group* $\mathbf{H}(\Gamma^*; G(\gamma))$.

PROOF. We shall apply [6], Theorem 5.4, to the lattice $N(G)$ and the set $C(G)$ by using freely the terminology of [6]. By assumption the existence of basic supplementary elements is fulfilled in $N(G)$. Using this together with [6], Proposition 5.5, and the fact that the set $C(G)$ forms a chain one can show the existence of non-trivial regular quasi-complement functions functions on $C(G)$. Since the lattice $N(G)$ is join-continuous and modular, the set $C(G)$ is join-continuous at zero and satisfies the general disjointness property, see [6], Section 3 and Proposition 4.2. Hence there is a Banaschewski function on $C(G)$.

Besides some isolated examples it would be interesting to find some larger class of groups which satisfy the above condition on the lattice of normal subgroups.

In the following theorem it is supposed that the group G has a concrete structure as a direct sum of normal subgroups, in which each summand has a sufficiently complemented subgroup lattice or, on the other hand, for any intersection of open balls each summand either is contained in or is disjoint from the sum of the previous summands and this intersection.

3.6. THEOREM. *Let G be a group with a conjugation invariant valuation and suppose that the group G has a direct sum representation $G = \Sigma_{\lambda \in \Lambda} G_\lambda$, where Λ is a well-ordered set and G_λ are normal subgroups of G. For all $\lambda \in \Lambda$ denote $K_\lambda = \Sigma_{\mu < \lambda} G_\mu$ and $H_\lambda = \Sigma_{\mu \leqq \lambda} G_\mu$. Assume further that for every $\lambda \in \Lambda$ there is in $N(G_\lambda)$ a Banaschewski function τ_λ on $G_\lambda \cap (K_\lambda + C(H_\lambda))$. Then in $N(G)$ there is a Banaschewski function on $C(G)$, and hence G is immediately embeddable into the Hahn group $\mathbf{H}(\Gamma^*; G(\gamma))$.*

Before proving the theorem we shall first prove an auxiliary result:

3.7. LEMMA. *Let the assumptions of Theorem 3.6 be fulfilled. Then for every $\lambda \in \Lambda$ there is in $N(H_\lambda)$ a Banachewski function $\sigma_\lambda \colon C(H_\lambda) \to N(H_\lambda)$ such that $\mu \leqq \lambda$ implies $\sigma_\mu \leqq \sigma_\lambda$, that is, $\sigma_\mu(H_\mu \cap C) \subset \sigma_\lambda(C)$ for all $C \in C(H_\lambda)$.*

PROOF. In the course of the proof we use the following easily provable facts:

(1) $C(K) = K \cap C(H)$ for any subgroups H and K such that $K \subset H$.

(2) If (H_ι) is an ascending chain of subgroups and each H_ι has a normal subgroup N_ι such that (N_ι) forms also an ascending chain, then $\cup N_\iota$ is a normal subgroup of $\cup H_\iota$.

(3) $H_\lambda = K_\lambda \oplus G_\lambda$ for all $\lambda \in \Lambda$.

(4) $K_\lambda = \bigcup_{\mu < \lambda} H_\mu$ for all $\lambda \in \Lambda$.

We use a transfinite induction with respect to λ. One can clearly presuppose that $G_0 = \{0\}$, and hence in the case $\lambda = 0$ we may choose $\sigma_\lambda = 0$. Let then $\lambda \in \Lambda$ be fixed and assume that the assertion holds for all $\mu < \lambda$.

Define $\phi_\lambda \colon C(K_\lambda) \to N(K_\lambda)$ by

$$\phi_\lambda(C) = \bigcup_{\mu<\lambda} \sigma_\mu(H_\mu \cap C) \text{ for all } C \in C(K_\lambda).$$

By (2) the values of this function are indeed normal subgroups of K_λ, and one can show that this function is a Banachewski function: the requirements $C \cap \phi_\lambda(C) = \{0\}$ and $C + \phi_\lambda(C) = K_\lambda$ are consequences of (4) and the fact that the functions σ_μ are Banachewski functions, and in view of (1) the antitony of ϕ_λ is inherited from the functions σ_μ. Furthermore, for all $\mu < \lambda$ clearly $\sigma_\mu \leqq \phi_\lambda$, that is, $\sigma_\mu(H_\mu \cap C) \subset \phi_\lambda(C)$ for all $C \in C(K_\lambda)$.

Define now $\sigma_\lambda: C(H_\lambda) \to N(H_\lambda)$ by

$$\sigma_\lambda(C) = \phi_\lambda(K_\lambda \cap C) + \tau_\lambda(G_\lambda \cap (K_\lambda + C)) \text{ for all } C \in C(H_\lambda).$$

By (3) the sum $A + B$ belongs to $N(H_\lambda)$ for all $A \in N(K_\lambda)$ and $B \in N(G_\lambda)$; hence the range of σ_λ is indeed contained in $N(H_\lambda)$. To prove that σ_λ is a Banachewski function we first show that $C \cap \sigma_\lambda(C) = \{0\}$. For any $C \in C(H_\lambda)$ one has

$$C \cap \sigma_\lambda(C) = C \cap (K_\lambda + C) \cap [\phi_\lambda(K_\lambda \cap C) + \tau_\lambda(G_\lambda \cap (K_\lambda + C))].$$

Since the lattice $N(H_\lambda)$ is modular and $\phi_\lambda(K_\lambda \cap C) \subset K_\lambda + C$, we get

$$C \cap \sigma_\lambda(C) = C \cap [\phi_\lambda(K_\lambda \cap C) + (K_\lambda + C) \cap \tau_\lambda(G_\lambda \cap (K_\lambda + C))].$$

Now the result is achieved by using twice the complementary properties of Banachewski functions, firstly for τ_λ and then for ϕ_λ.

Analogously, by using the complementary properties of the functions ϕ_λ and τ_λ, one can prove that $C + \sigma_\lambda(C) = H_\lambda$ for all $C \in C(H_\lambda)$.

The antitony of σ_λ and the fact that $\sigma_\mu \leqq \sigma_\lambda$ for all $\mu \leqq \lambda$ follow from (1) and the antitony of the functions ϕ_λ and τ_λ.

Proof of Theorem 3.6. We define the function σ: $C(G) \to N(G)$ by the rule

$$\sigma(C) = \bigcup_{\lambda \in \Lambda} \sigma_\lambda(H_\lambda \cap C) \text{ for all } C \in C(G),$$

where the functions σ_λ are as in Lemma 3.7. Using the same methods as in the proof of Lemma 3.7 and the fact that $G = \bigcup_{\lambda \in \Lambda} H_\lambda$ one can show that σ is a well-defined Banaschewski function on $C(G)$.

As an application of Theorem 3.6 we have the following result. First, recall the notation $B(\Delta) = \bigcap_{\delta \in \Delta} B(\delta)$ for $\Delta \subset \Gamma^*$.

3.8. Theorem. *Let G be a group with a conjugation invariant valuation and suppose that the group G has a direct sum representation $G = \sum_{\lambda \in \Lambda} G_\lambda$, where Λ is a well-ordered set and G_λ are non-trivial normal subgroups of G. For all $\lambda \in \Lambda$ denote $K_\lambda = \sum_{\mu<\lambda} G_\mu$. Assume further that for any $\lambda \in \Lambda$ and $\Delta \subset \Gamma^*$ either*

$G_\lambda \subset K_\lambda + B(\Delta)$ or $G_\lambda \cap (K_\lambda + B(\Delta)) = \{0\}$. Then G is immediately embeddable into the Hahn group $\mathbf{H}(\Gamma^*; G(\gamma))$.

PROOF. Let $\lambda \in \Lambda$ be arbitrary. Since $C(G) = \{B(\Delta)\}_{\Delta \subset \Gamma^*}$ and $C(H_\lambda) = H_\lambda \cap C(G)$, then every $C \in C(H_\lambda)$ can be written in the form $H_\lambda \cap B(\Delta)$ with a suitable $\Delta \subset \Gamma^*$. Using the modularity of the lattice $N(H_\lambda)$ it is seen that $G_\lambda \cap (K_\lambda + C) = G_\lambda \cap (K_\lambda + B(\Delta))$. Thus we can simply define

$$\tau_\lambda(G_\lambda \cap (K_\lambda + C)) = \begin{cases} G_\lambda \ , \text{ if } G_\lambda \cap (K_\lambda + C) = \{0\} \\ \{0\} \ , \text{ if } G_\lambda \subset K_\lambda + C \end{cases}$$

for all $C \in C(H_\lambda)$. Evidently τ_λ is a well-defined Banaschewski function on $G_\lambda \cap (K_\lambda + C(H_\lambda))$ with values in $N(G_\lambda)$. An application of Theorem 3.6 gives now the desired result.

Restricted for Abelian groups the content of Theorem 3.6 is – as far as we can see – relevantly the same as the content of [3]; see especially the formulation on p. 287.

To demonstrate further the usefulness of lattice-theoretical methods developed in [6] we apply them together with Theorem 3.3 to divisible Abelian groups.

3.9. DEFINITION. A valuation $|\cdot|$ on a group G is $\mathbf{Z}$-*invariant* if $|ng| = |g|$ for all non-zero $n \in \mathbf{Z}$ and $g \in G$.

Obviously a normal valuation is $\mathbf{Z}$-invariant if and only if the quotient groups $G(\gamma) = B'(\gamma)/B(\gamma)$ are torsion-free for all $\gamma \in \Gamma^*$.

3.10. THEOREM. *Let G be a divisible Abelian group with a $\mathbf{Z}$-invariant valuation. Then G is immediately embeddable into the Hahn group $\mathbf{H}(\Gamma^*; G(\gamma))$. Furthermore, G has an ordering which is compatible with the valuation in the sense that if $0 < g < g'$ in G then $|g| \leqq |g'|$.*

PROOF. For a subgroup A of G denote by A^d the *division closure* of A, that is, $A^d = \{g \in G\colon ng \in A$ for some positive integer $n\}$. If $A = A^d$ then A is called *division closed*. As the $\mathbf{Z}$-invariance of the valuation guarantees that the group G is torsion-free and thus orderable, we can apply [6], Theorem 6.2, by which the lattice L_d consisting of the division closed subgroups of G has a Banaschewski function $\sigma\colon L_d \to L_d$. Especially $G = A \vee \sigma(A) = (A + \sigma(A))^d$ for all $A \in L_d$. The $\mathbf{Z}$-invariance implies further that the open balls $B(\gamma)$ are elements of L_d. By selecting $G_\Delta = \sigma(B(\Delta))$ for all $\Delta \subset \Gamma^*$ we obtain a collection which satisfies the assumptions of Theorem 3.3. Indeed, the only critical point is to show that $B(\Delta) + G_\Delta = (B(\Delta) + G_\Delta)^d$; but this follows from the divisibility of G. Therefore G is immediately embeddable into the Hahn group $H = \mathbf{H}(\Gamma^*; G(\gamma))$.

The groups $G(\gamma)$ are torsion-free Abelian groups, and are thus orderable. Choose on H a lexicographic product of these orderings, see for instance [5], p. 4, and observe that this induces on G an ordering which is compatible with the original valuation.

3.11. COROLLARY (Hahn's embedding theorem). *Let G be an ordered Abelian group. Then G is embeddable into the Hahn group* $\mathbf{H}(\Gamma^*; \mathsf{R})$, *where Γ is the set of Archimedean classes of G and* R *is the additive group of real numbers.*

PROOF. We can suppose that G is division closed; otherwise we can embed it to its division closure. The natural valuation of G which maps each element of G to its Archimedean class is a Z-invariant valuation. Hence G can be embedded into the Hahn group $\mathbf{H}(\Gamma^*; G(\gamma))$. The quotient order in $G(\gamma) = B'(\gamma)/B(\gamma)$ induced by the order of G is Archimedean and thus $G(\gamma)$ can be embedded into the additive group R. Putting these facts together we get an embedding of G into the Hahn group $\mathbf{H}(\Gamma^*; \mathsf{R})$. A short calculation shows that this embedding is order preserving.

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