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A NOTE ON INNER COALGEBRA MEASURING AND DERIVATIONS

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Introduction and results.

Recently, A. Masuoka [2] proved the following:

THEOREM. *Let k be a field, C a k-coalgebra, A a k-algebra, and $B \subseteq A$ a k-subalgebra. For any measuring $\mu: C \otimes B \to A$ there is a unique maximal subcoalgebra C^{μ} of C for which the restricted measuring $C^{\mu} \otimes B \to A$ is inner.*

Another proof was found by M. Takeuchi and the author (see [2]). In this note the theorem is proved a third time. The new proof is rather constructive and gives some insight to the structure of C^{μ}. Also, it utilizes a connection with derivations $B \to A$, which enables us to deduce the following as a corollary to the proof:

COROLLARY. *Let A, B, and C be as in the theorem and assume that all k-derivations $B \to A$ are inner. Then a measuring $\mu: C \otimes B \to A$ is inner if and only if its restriction $C_0 \otimes B \to A$ is inner, where C_0 is the coradical of C.*

Another case where the question of μ being inner can be reduced to the coradical, was given in [1, 3.2].

In [3, 4.2] A. Nowicki showed that if A is an algebra over a commutative ring k and if all k-derivations $A \to A$ are inner, then so are all higher k-derivations. (Actually, Nowicki worked with rings, but the same proof applies.) Since a higher derivation can be regarded as a measuring by a certain coalgebra C, where C_0 acts trivially [4, p. 140], and since an inner higher derivation corresponds to an inner measuring (as follows easily from [3, 3.2]), the corollary generalizes Nowicki's theorem when k is a field.

Preliminaries.

For the preliminaries on coalgebras we refer to [4]; here we mention only some basic facts and notations. Let A, B, and C be as in the theorem. The coalgebra

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structure maps of C are denoted by Δ and ε, and as in [4], we write $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$. We denote $W^+ = W \cap \operatorname{Ker} \varepsilon$ for any subspace W of C. A k-linear map $\mu: C \otimes B \to A$ is a *measuring* if $\mu(c \otimes 1) = \varepsilon(c)1$ and $\mu(c \otimes bb') = \sum \mu(c_{(1)} \otimes b)\mu(c_{(2)} \otimes b')$ for $c \in C, b, b' \in B$. The space $\operatorname{Hom}(C, A)$ of k-linear maps $C \to A$ is a k-algebra with respect to the *convolution product* defined by $(\sigma * \tau)(c) = \sum \sigma(c_{(1)})\tau(c_{(2)})$ for $\sigma, \tau \in \operatorname{Hom}(C, A)$, $c \in C$; the identity element is the map sending c to $\varepsilon(c)1$. In this paper σ^{-1} always means the inverse of σ under the convolution product. Any convolution invertible $\sigma \in \operatorname{Hom}(C, A)$ *implements* a measuring $\iota_\sigma: C \otimes B \to A$ by the rule $\iota_\sigma(c \otimes b) = \sum \sigma(c_{(1)})b\sigma^{-1}(c_{(2)})$; these are called *inner measurings*.

Given a measuring $\mu: C \otimes B \to A$ and a subcoalgebra $D \subseteq C$, we say briefly that μ is *inner on* D when the restricted measuring $D \otimes B \to A$ is inner.

A *k-derivation* $\delta: B \to A$ is a k-linear map that satisfies $\delta(bb') = b\delta(b') + \delta(b)b'$ for $b, b' \in B$, and *inner derivations* $B \to A$ are maps that send $b \in B$ to $xb - bx$ for some fixed $x \in A$.

Construction of C^μ.

Let A, B, C, and μ be as in the theorem. We are going to find inductively certain subcoalgebras $C_0^\mu \subseteq C_1^\mu \subseteq \ldots \subseteq C_n^\mu \subseteq \ldots$ of C and certain convolution invertible maps $\sigma_n \in \operatorname{Hom}(C, A)$. On each C_n^μ the measuring μ will be inner, implemented by the restriction of σ_n to C_n^μ. The maps σ_n will be compatible in the sense that σ_n and σ_m coincide on C_n^μ if $n \leqq m$.

When C_n^μ and σ_n are all found, the subcoalgebra C^μ is defined as $C^\mu = \cup_n C_n^\mu$. The maps σ_n determine then a map $\sigma: C^\mu \to A$ that makes μ inner on C^μ.

To start the construction, let $C_0 = \oplus_\alpha D_\alpha$ be the coradical as the sum of simple subcoalgebras. We define C_0^μ to be the sum of those D_α's on which μ is inner. Then μ is inner on C_0^μ, implemented by some $\sigma_0 \in \operatorname{Hom}(C_0^\mu, A)$. We extend σ_0 to the whole of C: first we let it be $c \mapsto \varepsilon(c)1$ on the remaining D_α's, and then we extend it to a linear map $C \to A$ in an arbitrary way. By [5, Lemma 14] σ_0 is then convolution invertible in $\operatorname{Hom}(C, A)$.

If $C_0^\mu = 0$ then we set $C_n^\mu = 0$ for all n, i.e., $C^\mu = 0$. Assume now that $C_0^\mu \neq 0$ and fix $c_0 \in C_0^\mu$ with $\varepsilon(c_0) = 1$.

Let $n \geqq 0$ be fixed and assume that we have found C_n^μ and σ_n. Denote $W = C_n^\mu \wedge C_0^\mu$. Then $C_n^\mu \subseteq W \subseteq C$ and W is a subcoalgebra [4, 9.0.0 (i)]. Since $(C_n^\mu)^+$ is a coideal of C_n^μ, and hence of W, the space $\bar{W} = W/(C_n^\mu)^+$ is a coalgebra and the natural map $W \to \bar{W}$ is a coalgebra map; we denote the map by $c \mapsto \bar{c}$. If $c \in C_n^\mu$ then $\bar{c} = \varepsilon(c)\bar{c}_0$. Easily follows $\Delta(\bar{c}_0) = \bar{c}_0 \otimes \bar{c}_0$, hence

$$(*) \qquad \Delta(c_0) \equiv c_0 \otimes c_0 \pmod{(C_n^\mu)^+ \otimes W + W \otimes (C_n^\mu)^+}.$$

More generally, if $c \in W$ then $\Delta(c) \in C_n^\mu \otimes W + W \otimes C_0^\mu$, hence $\Delta(\bar{c}) \in \bar{c}_0 \otimes \bar{W} +$

$\bar{W} \otimes \bar{c}_0$, and one obtains easily that $\Delta(\bar{c}) = -\varepsilon(c)\bar{c}_0 \otimes \bar{c}_0 + \bar{c}_0 \otimes \bar{c} + \bar{c} \otimes \bar{c}_0$. For $c \in W^+$ this gives

$$(**) \qquad \Delta(c) \equiv c_0 \otimes c + c \otimes c_0 \pmod{(C_n^\mu)^+ \otimes W + W \otimes (C_n^\mu)^+}.$$

Define a measuring $v: C \otimes B \to A$ by

$$(***) \qquad v(c \otimes b) = \sum \sigma_n^{-1}(c_{(1)})\mu(c_{(2)} \otimes b)\sigma_n(c_{(3)}).$$

Since σ_n implements μ on C_n^μ, we have $v(c \otimes b) = \varepsilon(c)b$ for $c \in C_n^\mu$, $b \in B$. In particular, $v(c_0 \otimes b) = b$ and $v((C_n^\mu)^+ \otimes B) = 0$. Then from $(**)$ follows for any $c \in W^+$ that

$$v(c \otimes bb') = \sum v(c_{(1)} \otimes b)v(c_{(2)} \otimes b') = bv(c \otimes b') + v(c \otimes b)b' \quad \text{for} \quad b, b' \in B,$$

i.e., $v(c \otimes -)$ is a derivation $B \to A$. Let $V = \{c \in W^+ \mid v(c \otimes -) \text{ is an inner derivation}\}$. Notice that V is a subspace and $(C_n^\mu)^+ \subseteq V$.

We define C_{n+1}^μ to be the unique maximal subcoalgebra of W contained in $C_n^\mu + V$. Then $C_n^\mu \subseteq C_{n+1}^\mu$.

Let $C_{n+1}^\mu = C_n^\mu \oplus V'$ where $V' \subseteq V$. Fix a basis $\{v_1, v_2, \ldots\}$ of V', and for each v_i fix $x_i \in A$ such that $v(v_i \otimes b) = x_i b - bx_i$ for $b \in B$. Let $\tau: C_{n+1}^\mu \to A$ be the linear map with $\tau(c) = \varepsilon(c)1$ for $c \in C_n^\mu$ and $\tau(v_i) = x_i$ for each i. Using $(*)$ and $(**)$ and the facts that $C_{n+1}^\mu = kc_0 \oplus (C_n^\mu)^+ \oplus V'$ and $V' \subseteq W^+$, one sees easily that τ has a convolution inverse given by $\tau^{-1}(c) = \varepsilon(c)1$ for $c \in C_n^\mu$ and $\tau^{-1}(v_i) = -x_i$ for each i. Similarly one sees that v is inner on C_{n+1}^μ, implemented by τ. As σ_0 above, τ extends to a convolution invertible map $C \to A$. When we set $\sigma_{n+1} = \sigma_n * \tau$, then for any $c \in C_{n+1}^\mu$, $b \in B$, we have

$$\begin{aligned}
\mu(c \otimes b) &= \sum \sigma_n(c_{(1)})v(c_{(2)} \otimes b)\sigma_n^{-1}(c_{(3)}) \\
&= \sum \sigma_n(c_{(1)})\tau(c_{(2)})b\tau^{-1}(c_{(3)})\sigma_n^{-1}(c_{(4)}) \\
&= \sum \sigma_{n+1}(c_{(1)})b\sigma_{n+1}^{-1}(c_{(2)}).
\end{aligned}$$

Finally, on C_n^μ the maps σ_n and σ_{n+1} coincide. Thus, C_{n+1}^μ and σ_{n+1} are as required. This completes the construction of C^μ.

Maximality of C^μ.

Let now D be any subcoalgebra of C such that μ is inner on D. Let $D_0 \subseteq D_1 \subseteq \ldots$ be the coradical filtration of D. We are going to show that $D_n \subseteq C_n^\mu$ for all n, where C_n^μ is as constructed above; this will then imply $D \subseteq C^\mu$, thus completing the proof of the theorem.

Trivially $D_0 \subseteq C_0^\mu$. Fix $n \geq 0$ and assume that $D_n \subseteq C_n^\mu$. Now, μ is implemented on D by some $\rho \in \mathrm{Hom}(D, A)$ and on C_n^μ by the restriction of σ_n, hence for $d \in D_n \subseteq C_n^\mu$, $b \in B$,

$$\mu(d \otimes b) = \sum \rho(d_{(1)}) b \rho^{-1}(d_{(2)}) = \sum \sigma_n(d_{(1)}) b \sigma_n^{-1}(d_{(2)}).$$

So, if we denote $\beta = \sigma_n^{-1}|_D * \rho \in \mathrm{Hom}(D, A)$, then $\beta(D_n)$ and $\beta^{-1}(D_n)$ centralize B. By (***), for any $d \in D$, $b \in B$,

$$\nu(d \otimes b) = \sum \sigma_n^{-1}(d_{(1)}) \rho(d_{(2)}) b \rho^{-1}(d_{(3)}) \sigma_n(d_{(4)}) = \sum \beta(d_{(1)}) b \beta^{-1}(d_{(2)}).$$

Let $d \in D_{n+1} = D_n \wedge D_0$ (wedge in D). Then $\Delta(d) = \sum_i d_i \otimes d_i' + \sum_i d_i'' \otimes d_i'''$ where $d_i \in D_n$, $d_i''' \in D_0$, and $d_i', d_i'' \in D$. We have

$$\nu(d \otimes b) = \sum_i b \beta(d_i) \beta^{-1}(d_i') + \sum_i \beta(d_i'') \beta^{-1}(d_i''') b$$

since $\beta(d_i)$ and $\beta^{-1}(d_i''')$ centralize B. Hence,

$$\nu(d \otimes b) = bx + yb \quad \text{for} \quad b \in B,$$

where $x, y \in A$ depend on d but not on b. If $\varepsilon(d) = 0$, then $\nu(d \otimes 1) = \varepsilon(d)1 = 0$, hence $y = -x$, which gives $\nu(d \otimes b) = bx - xb$. In other words, for any $d \in D_{n+1}^+$, $\nu(d \otimes -)$ is an inner derivation. Since also $D_{n+1} \subseteq C_n^\mu \wedge C_0^\mu = W$, we conclude that $D_{n+1} \subseteq C_{n+1}^\mu$.

Proof of the corollary.

Assume that all derivations $B \to A$ are inner. Then in the construction of C^μ we have $V = W^+$, hence $C_n^\mu + V = W$, and $C_{n+1}^\mu = W = C_n^\mu \wedge C_0^\mu$. It follows that $C_n^\mu = \bigwedge^{n+1} C_0^\mu$, so,

$$C^\mu = \bigcup_n C_n^\mu = \bigcup_n \bigwedge^n C_0^\mu$$

is the unique maximal subcoalgebra with coradical C_0^μ. This implies the corollary.

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