A NOTE ON INNER COALGEBRA MEASURING AND DERIVATIONS

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Introduction and results.

Recently, A. Masuoka [2] proved the following:

THEOREM. Let $k$ be a field, $C$ a $k$-coalgebra, $A$ a $k$-algebra, and $B \subseteq A$ a $k$-subalgebra. For any measuring $\mu: C \otimes B \to A$ there is a unique maximal subcoalgebra $C''$ of $C$ for which the restricted measuring $C'' \otimes B \to A$ is inner.

Another proof was found by M. Takeuchi and the author (see [2]). In this note the theorem is proved a third time. The new proof is rather constructive and gives some insight to the structure of $C''$. Also, it utilizes a connection with derivations $B \to A$, which enables us to deduce the following as a corollary to the proof:

COROLLARY. Let $A$, $B$, and $C$ be as in the theorem and assume that all $k$-derivations $B \to A$ are inner. Then a measuring $\mu: C \otimes B \to A$ is inner if and only if its restriction $C_0 \otimes B \to A$ is inner, where $C_0$ is the coradical of $C$.

Another case where the question of $\mu$ being inner can be reduced to the coradical, was given in [1, 3.2].

In [3, 4.2] A. Nowicki showed that if $A$ is an algebra over a commutative ring $k$ and if all $k$-derivations $A \to A$ are inner, then so are all higher $k$-derivations. (Actually, Nowicki worked with rings, but the same proof applies.) Since a higher derivation can be regarded as a measuring by a certain coalgebra $C$, where $C_0$ acts trivially [4, p. 140], and since an inner higher derivation corresponds to an inner measuring (as follows easily from [3, 3.2]), the corollary generalizes Nowicki's theorem when $k$ is a field.

Preliminaries.

For the preliminaries on coalgebras we refer to [4]; here we mention only some basic facts and notations. Let $A$, $B$, and $C$ be as in the theorem. The coalgebra
structure maps of $C$ are denoted by $A$ and $\varepsilon$, and as in [4], we write $A(c) = \sum c_{(1)} \otimes c_{(2)}$. We denote $W^+ = W \cap \text{Ker} \varepsilon$ for any subspace $W$ of $C$. A $k$-linear map $\mu: C \otimes B \to A$ is a measuring if $\mu(c \otimes 1) = \varepsilon(c) 1$ and $\mu(c \otimes bb') = \sum \mu(c_{(1)} \otimes b) \mu(c_{(2)} \otimes b')$ for $c \in C$, $b, b' \in B$. The space $\text{Hom}(C, A)$ of $k$-linear maps $C \to A$ is a $k$-algebra with respect to the convolution product defined by $(\sigma * \tau)(c) = \sum \sigma(c_{(1)}) \tau(c_{(2)})$ for $\sigma, \tau \in \text{Hom}(C, A)$, $c \in C$; the identity element is the map sending $c$ to $\varepsilon(c) 1$. In this paper $\sigma^{-1}$ always means the inverse of $\sigma$ under the convolution product. Any convolution invertible $\sigma \in \text{Hom}(C, A)$ implements a measuring $\iota_\sigma: C \otimes B \to A$ by the rule $\iota_\sigma(c \otimes b) = \sum \sigma(c_{(1)}) b^\sigma(c_{(2)})$; these are called inner measurements.

Given a measuring $\mu: C \otimes B \to A$ and a subcoalgebra $D \subseteq C$, we say briefly that $\mu$ is inner on $D$ when the restricted measuring $D \otimes B \to A$ is inner.

A $k$-derivation $\delta: B \to A$ is a $k$-linear map that satisfies $\delta(bb') = b\delta(b') + \delta(b)b'$ for $b, b' \in B$, and inner derivations $B \to A$ are maps that send $b \in B$ to $xb - bx$ for some fixed $x \in A$.

Construction of $C^\mu$.

Let $A$, $B$, $C$, and $\mu$ be as in the theorem. We are going to find inductively certain subcoalgebras $C_0^\mu \subseteq C_1^\mu \subseteq \ldots \subseteq C_n^\mu \subseteq \ldots$ of $C$ and certain convolution invertible maps $\sigma_n \in \text{Hom}(C, A)$. On each $C_n^\mu$ the measuring $\mu$ will be inner, implemented by the restriction of $\sigma_n$ to $C_n^\mu$. The maps $\sigma_n$ will be compatible in the sense that $\sigma_n$ and $\sigma_m$ coincide on $C_n^\mu$ if $n \leq m$.

When $C_n^\mu$ and $\sigma_n$ are all found, the subcoalgebra $C^\mu$ is defined as $C^\mu = \cup_n C_n^\mu$. The maps $\sigma_n$ determine then a map $\sigma: C^\mu \to A$ that makes $\mu$ inner on $C^\mu$.

To start the construction, let $C_0 = \bigoplus D_a$ be the coradical as the sum of simple subcoalgebras. We define $C_0^\mu$ to be the sum of those $D_a$'s on which $\mu$ is inner. Then $\mu$ is inner on $C_0^\mu$, implemented by some $\sigma_0 \in \text{Hom}(C_0^\mu, A)$. We extend $\sigma_0$ to the whole of $C$: first we let it be $c \mapsto \varepsilon(c) 1$ on the remaining $D_a$'s, and then we extend it to a linear map $C \to A$ in an arbitrary way. By [5, Lemma 14] $\sigma_0$ is then convolution invertible in $\text{Hom}(C, A)$.

If $C_0^\mu = 0$ then we set $C_n^\mu = 0$ for all $n$, i.e., $C^\mu = 0$. Assume now that $C_0^\mu \neq 0$ and fix $c_0 \in C_0^\mu$ with $\varepsilon(c_0) = 1$.

Let $n \geq 0$ be fixed and assume that we have found $C_n^\mu$ and $\sigma_n$. Denote $W = C_n^\mu \wedge C_0^\mu$. Then $C_n^\mu \subseteq W \subseteq C$ and $W$ is a subcoalgebra [4, 9.0.0(i)]. Since $(C_n^\mu)^+$ is a coideal of $C_n^\mu$, and hence of $W$, the space $\tilde{W} = W/(C_n^\mu)^+$ is a coalgebra and the natural map $W \to \tilde{W}$ is a coalgebra map; we denote the map by $c \mapsto \tilde{c}$. If $c \in C_n^\mu$ then $\tilde{c} = \varepsilon(c) c_0$. Easily follows $A(\tilde{c}_0) = \tilde{c}_0 \otimes \tilde{c}_0$, hence

$$(*) \quad A(c_0) \equiv c_0 \otimes c_0 \pmod{(C_n^\mu)^+ \otimes W + W \otimes (C_n^\mu)^+}.$$  

More generally, if $c \in W$ then $A(c) \in C_n^\mu \otimes W + W \otimes C_0^\mu$, hence $A(\tilde{c}) \in \tilde{c}_0 \otimes \tilde{W} +$
\[ \bar{W} \otimes \bar{c}_0, \] and one obtains easily that \( \Delta(\bar{c}) = -\varepsilon(c)\bar{c}_0 \otimes \bar{c}_0 + \bar{c}_0 \otimes \bar{c} + \bar{c} \otimes \bar{c}_0. \) For \( c \in W^+ \) this gives

\((**)\quad \Delta(c) \equiv c_0 \otimes c + c \otimes c_0 \mod (C_n^n)^+ \otimes W + W \otimes (C_n^n)^+).\)

Define a measuring \( \nu: C \otimes B \rightarrow A \) by

\[(***)\quad \nu(c \otimes b) = \sum \sigma_n^{-1}(c_{(1)}) \mu(c_{(2)} \otimes b) \sigma_n(c_{(3)}),\]

since \( \sigma_n \) implements \( \mu \) on \( C_n^n \), we have \( \nu(c \otimes b) = \varepsilon(c)b \) for \( c \in C_n^n, b \in B \). In particular, \( \nu(c_0 \otimes b) = b \) and \( \nu((C_n^n)^+ \otimes B) = 0 \). Then from \((***)\) follows for any \( c \in W^+ \) that

\[\nu(c \otimes bb') = \sum \nu(c_{(1)} \otimes b) \nu(c_{(2)} \otimes b') = b\nu(c \otimes b') + \nu(c \otimes b)b' \quad \text{for} \quad b, b' \in B,\]
i.e., \( \nu(c \otimes -) \) is a derivation \( B \rightarrow A \). Let \( V = \{ c \in W^+ | \nu(c \otimes -) \text{ is an inner derivation} \} \). Notice that \( V \) is a subspace and \((C_n^n)^+ \subseteq V\).

We define \( C_{n+1}^+ \) to be the unique maximal subcoalgebra of \( W \) contained in \( C_n^n + V \). Then \( C_n^n \subseteq C_{n+1}^+ \).

Let \( C_{n+1}^+ = C_n^n \oplus V' \) where \( V' \subseteq V \). Fix a basis \( \{v_1, v_2, \ldots\} \) of \( V' \), and for each \( v_i \) fix \( x_i \in A \) such that \( \nu(v_i \otimes b) = x_i b - bx_i \) for \( b \in B \). Let \( \tau: C_{n+1}^+ \rightarrow A \) be the linear map with \( \tau(c) = \varepsilon(c)1 \) for \( c \in C_n^n \) and \( \tau(v_i) = x_i \) for each \( i \). Using \((*)\) and \((***)\) and the facts that \( C_{n+1}^+ = k C_0 \oplus (C_n^n)^+ \oplus V' \) and \( V' \subseteq W^+ \), one sees easily that \( \tau \) has a convolution inverse given by \( \tau^{-1}(c) = \varepsilon(c)1 \) for \( c \in C_n^n \) and \( \tau^{-1}(v_i) = -x_i \) for each \( i \). Similarly one sees that \( \nu \) is inner on \( C_{n+1}^+ \), implemented by \( \tau \). As \( \sigma_0 \) above, \( \tau \) extends to a convolution invertible map \( C \rightarrow A \). When we set \( \sigma_{n+1} = \sigma_n \ast \tau \), then for any \( c \in C_{n+1}^+ \), \( b \in B \), we have

\[\mu(c \otimes b) = \sum \sigma_n(c_{(1)}) \nu(c_{(2)} \otimes b) \sigma_n^{-1}(c_{(3)}) = \sum \sigma_n(c_{(1)}) \tau(c_{(2)}) b \tau^{-1}(c_{(3)}) \sigma_n^{-1}(c_{(4)}) = \sum \sigma_{n+1}(c_{(1)}) b \sigma_{n+1}^{-1}(c_{(2)}).\]

Finally, on \( C_n^n \) the maps \( \sigma_n \) and \( \sigma_{n+1} \) coincide. Thus, \( C_{n+1}^+ \) and \( \sigma_{n+1} \) are as required. This completes the construction of \( C_n^n \).

**Maximality of \( C_n^n \).**

Let now \( D \) be any subcoalgebra of \( C \) such that \( \mu \) is inner on \( D \). Let \( D_0 \subseteq D_1 \subseteq \ldots \) be the coradical filtration of \( D \). We are going to show that \( D_n \subseteq C_n^n \) for all \( n \), where \( C_n^n \) is as constructed above; this will then imply \( D \subseteq C_n^n \), thus completing the proof of the theorem.

Trivially \( D_0 \subseteq C_0^n \). Fix \( n \geq 0 \) and assume that \( D_n \subseteq C_n^n \). Now, \( \mu \) is implemented on \( D \) by some \( \rho \in \text{Hom}(D, A) \) and on \( C_n^n \) by the restriction of \( \sigma_n \), hence for \( d \in D_n \subseteq C_n^n \), \( b \in B \),
\[ \mu(d \otimes b) = \sum \rho(d^{(1)})b \rho^{-1}(d^{(2)}) = \sum \sigma(d^{(1)})b \sigma^{-1}(d^{(2)}). \]

So, if we denote \( \beta = \sigma^{-1}_n \cdot D \ast \rho \in \text{Hom}(D, A) \), then \( \beta(D_n) \) and \( \beta^{-1}(D_n) \) centralize \( B \).

By (***), for any \( d \in D \), \( b \in B \),

\[ \nu(d \otimes b) = \sum \sigma_n^{-1}(d^{(1)})\rho(d^{(2)})b \rho^{-1}(d^{(3)})\sigma_n(d^{(4)}) = \sum \beta(d^{(1)})b \beta^{-1}(d^{(2)}). \]

Let \( d \in D_{n+1} = D_n \wedge D_0 \) (wedge in \( D \)). Then \( \Delta(d) = \sum d_i \otimes d'_i + \sum d''_i \otimes d'''_i \)

where \( d_i \in D_n, d''_i \in D_0 \), and \( d'_i, d'''_i \in D \). We have

\[ \nu(d \otimes b) = \sum \beta(d_i)b \beta^{-1}(d'_i) + \sum \beta(d''_i)b \beta^{-1}(d'''_i) \]

since \( \beta(d_i) \) and \( \beta^{-1}(d'''_i) \) centralize \( B \). Hence,

\[ \nu(d \otimes b) = bx + yb \quad \text{for} \quad b \in B, \]

where \( x, y \in A \) depend on \( d \) but not on \( b \). If \( \varepsilon(d) = 0 \), then \( \nu(d \otimes 1) = \varepsilon(d)1 = 0 \), hence \( y = -x \), which gives \( \nu(d \otimes b) = bx - xb \). In other words, for any \( d \in D_{n+1} \), \( \nu(d \otimes -) \) is an inner derivation. Since also \( D_{n+1} \subseteq C_n^\mu \wedge C_0^\mu = W \), we conclude that \( D_{n+1} \subseteq C_{n+1}^\mu \).

**Proof of the corollary.**

Assume that all derivations \( B \to A \) are inner. Then in the construction of \( C_n^\mu \) we have \( V = W^+ \), hence \( C_n^\mu + V = W \), and \( C_{n+1}^\mu = W = C_n^\mu \wedge C_0^\mu \). It follows that \( C_n^\mu = \bigwedge^{n+1} C_0^\mu \), so,

\[ C_n^\mu = \bigcup_n C_n^\mu = \bigcup_n \bigwedge^n C_0^\mu \]

is the unique maximal subcoalgebra with coradical \( C_0^\mu \). This implies the corollary.

**REFERENCES**