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EXTENSIONS OF FATOU THEOREMS IN PRODUCTS OF UPPER HALF-SPACES

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Abstract.

We consider a multi-parameter maximal function and give a necessary and sufficient condition for its boundedness on L^p, $p > 1$. We also prove that the maximal function with suitable restrictions on the parameters is of weak type $(1, 1)$.

0. Introduction.

The purpose of this paper is to generalize Nagel and Stein's extension of Fatou theorems to the multi-parameter case. We start with a brief description of their one-parameter results. Let $Pf(x,t)$ be the Poisson integral of a function f in $L^p(\mathsf{R}^n)$, and let Ω be a region in R^{n+1}_+ with $0 \in \bar{\Omega}$, and put $\Omega^x = x + \Omega$.

The problem is to characterize those regions Ω for which $Pf(x,t)$ have a limit a.e. as (x,t) in Ω approaches the boundary $\mathsf{R}^n \times \{0\}$, i.e. when is it true that

$$(1) \qquad \lim_{\substack{(x,t)\in\Omega^{x_0} \\ (x,t)\to(x_0,0)}} Pf(x,t) = f(x_0), \quad \text{a.e.}$$

A classical theorem of Fatou asserts that (1) is true when Ω is the cone $C_\alpha = \{(x,t) \in \mathsf{R}^{n+1}_+ : |x| \leqq \alpha t\}$. On the other hand, Littlewood showed that (1) will not be true when Ω contains a curve which approaches the boundary tangentially.

But there are many regions Ω not contained in any cone C_α for which the boundary limits exist a.e., as Nagel and Stein showed in [NS]. They gave a characterization of the regions Ω for which the associated maximal function $M_\Omega Pf(x) = \sup_{(y,t)\in\Omega^x} |Pf(y,t)|$ is suitably bounded. The sufficient condition on Ω to guarantee that $f \to M_\Omega Pf$ is of weak type $(1,1)$, and bounded on L^p, $p > 1$, is that

$$(2) \qquad |\Omega(t)| = |\{x \in \mathsf{R}^n ; (x,t) \in \Omega\}| \leqq Ct^n,$$

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and

$$\Omega + C_\alpha \subset \Omega. \tag{3}$$

Conversely, if $f \to M_\Omega Pf$ is of weak type (p, p), for some $p \geqq 1$, then $\tilde{\Omega} = \Omega + C_\alpha$ satisfies $|\tilde{\Omega}(t)| \leqq Ct^n$. In [NS] they also showed that for any curve approaching the boundary tangentially, there is a region Ω, satisfying (2) and (3), which contains points on the curve arbitrarily close to the boundary. This shows that this is really an extension of the classical Fatou theorem.

The original proof of Nagel and Stein has been simplified by Sueiro [Su] and by Andersson and Carlsson [AC]. In [AC] it is proved that the distribution functions of $M_\Omega u$ and $M_{C_\alpha} u$ are equivalent whenever Ω satisfy (2) and (3), i.e.

$$|\{M_\Omega u > \lambda\}| \leqq C\,|\{M_{C_\alpha} u > \lambda\}|, \tag{4}$$

for all measurable functions u in R^{n+1}_+. (Of course $M_\Omega u(x) = \sup_{\Omega^x} |u|$.) The L^p estimates follows from this,

$$\|M_\Omega u\|_p \leqq C\,\|M_{C_\alpha} u\|_p. \tag{5}$$

These estimates can be applied to Poisson integrals, convolutions with other approximative identities, estimates for H^p-spaces (even when $p < 1$), etc. Whenever we have estimates for $M_{C_\alpha} u$ we immediately get the same for $M_\Omega u$.

One can also use (4) to deduce the local Fatou theorem of Mair, Philipp and Singman [MPS], see §3.

We now consider the multi-parameter case, so instead of having t in R_+ we let t be in $(\mathsf{R}_+)^m$, and Ω a region in $\mathsf{R}^{n+m}_+ = \mathsf{R}^n \times (\mathsf{R}_+)^m$, with $0 \in \bar{\Omega}$, and $\Omega^x = x + \Omega$. If $x \in \mathsf{R}^n$ we can write $x = (x_1, \ldots, x_m)$, where $x_i \in \mathsf{R}^{n_i}$, $\sum_{i=1}^m n_i = n$, and let $C_\alpha = \{(x, t) \in \mathsf{R}^{n+m}_+ : |x_i| \leqq \alpha t_i\}$.

We consider the product Poisson kernel in R^{n+m}_+, $P_t(x) = \prod_{i=1}^m P_{t_i}(x_i)$, where $P_{t_i}(x_i)$ is the n_i-dimensional Poisson kernel, and let $Pf(x, t) = P_t * f(x)$. The maximal function M_Ω is given by $M_\Omega u(x) = \sup_{(y,t) \in \Omega^x} |u(y, t)|$.

Our aim is to give a necessary and sufficient condition on the region Ω for $f \to M_\Omega Pf$ to be of weak type $(1, 1)$ and bounded on L^p, $p > 1$.

On L^p this turns out well, as we can prove

THEOREM 1. *If Ω satisfies*

$$|\Omega(t)| = |x \in \mathsf{R}^n : (x, t) \in \Omega\}| \leqq Ct_1^{n_1} \cdots t_m^{n_m} = Ct^n, \tag{6}$$

and

(7) $$\Omega + C_\alpha \subset \Omega,$$

then we get for $1 < p \leqq +\infty$

(8) $$\|M_\Omega u\|_p \leqq C \|M_{C_\alpha} u\|_p,$$

for all measurable functions u.

If we apply this to $u = Pf$, and use the well-known fact $\|M_{C_\alpha} Pf\|_p \leqq C \|f\|_p$, see e.g. [Z], we obtain $\|M_\Omega Pf\|_p \leqq C \|f\|_p$.

To prove Theorem 1, we first show that (6) and (7) imply that $\Omega \subset \Omega_1 \times \cdots \times \Omega_m$, where Ω_i satisfy the one-parameter conditions (2) and (3). Hence, $M_\Omega u \leqq M_{\Omega_1} \circ \cdots \circ M_{\Omega_m} u$, and by repeated use of Fubini's theorem and (5) we obtain (8).

Conversely, we have

THEOREM 2. *If, for some* $p \geqq 1$,

$$|\{M_\Omega u > \lambda\}| \leqq C \left[\frac{\|M_{C_\alpha} u\|_p}{\lambda} \right]^p,$$

or

$$|\{M_\Omega Pf > \lambda\}| \leqq C \left[\frac{\|f\|_p}{\lambda} \right]^p,$$

then $\tilde{\Omega} = \Omega + C_\alpha$ *satisfies* $|\tilde{\Omega}(t)| \leqq Ct^n$.

In brief, if $\Omega + C_\alpha \subset \Omega$, then $M_\Omega Pf$ is bounded on L^p, $p > 1$, if and only if (6) holds.

The L^1-case is more complicated. To get a weak type estimate we have to restrict the parameters. To see why, we consider the strong maximal function in R^2

$$M_s f(x) = \sup_{x \in R} \frac{1}{|R|} \int_R |f|.$$

The supremum is taken over all rectangles containing x and having sides parallel to the axis. It is well-known that the strong maximal function is not of weak type (1, 1). Thus $M_\Omega Pf$ cannot be of weak type (1, 1), since clearly $M_\Omega Pf \geqq M_s f$, for positive f. If we take $f = \delta_0$, the point mass at the origin, and let R be a rectangle containing the origin with area $|R| = 1$, then $M_s \delta_0 \geqq 1$ on R. Hence, if $R_k = [0, 2^k] \times [0, 2^{-k}]$ then

$$\{M_s \delta_0 \geqq 1\} \supset \bigcup_{k=-\infty}^{\infty} R_k.$$

So a weak type estimate implies that we may only have a bounded number of the rectangles R_k, i.e. for a fixed area we cannot allow rectangles of all shapes. What we could possibly hope for is that it would be suffiient to allow a bounded number of different shapes for rectangels of a given area. But this is not the case; in § 2 we prove that if $R_k = [0, 2^k] \times [0, 2^{-2k}]$, then $f \to \sup_{x+R_k} \frac{1}{|R_k|} \int_{x+R_k} |f|$ is not of weak type (1, 1).

To describe our positive results, we let Λ be a region in R_+^m, and set

$$M_\Omega^\Lambda u(x) = \sup_{\substack{(y,t)\in\Omega^x \\ t\in\Lambda}} |u(y,t)|.$$

We want to find conditions on Λ (and Ω), so that the distribution functions of M_Ω^Λ and $M_{C_\alpha}^\Lambda$ are equivalent. What we need is a covering lemma for rectangles $R_t(x) = \{y \in \mathrm{R}^n\colon |x_i - y_i| \leqq t_i\}$, where $t \in \Lambda$. We say that Λ *has the covering property* if for every family of rectangles $\{R_t(x)\}$ with $|\cup R_t(x)|$ bounded, $t \in \Lambda$, there exists a subfamily $\{\tilde{R}_t(x)\}$ so that

$$\sum \chi_{\tilde{R}_t(x)} \leqq C, \tag{9}$$

and each rectangle is contained in a multiple of a rectangle from the subfamily, i.e.

$$R_t(x) \subset \tilde{R}_{ct'}(x'). \tag{10}$$

By an argument similar to that in [AC], we prove in Section 2 that if Λ has the covering property, and if Ω satisfies (6) and (7), then $|\{M_\Omega^\Lambda u > \lambda\}| \leqq C|\{M_{C_\alpha}^\Lambda u > \lambda\}|$. The covering property also guarantees that $f \to M_{C_\alpha}^\Lambda Pf$ is of weak type (1, 1), and hence $|\{M_\Omega^\Lambda Pf > \}| \leqq \frac{C}{\lambda} \|f\|_1$. If we take Λ to be the set where the parameters are of comparable size, the standard covering lemma for cubes shows that $\Lambda = \{t \in \mathrm{R}_+^m\colon t_i \sim t_j\}$ has the covering property.

We can extend this to larger regions $\Lambda = \bigcap_{i=1}^m \Lambda_i$, where Λ_i satisfies the same conditions as the regions Ω. Let Λ_i be subsets of $\mathrm{R}^m \cap \{t_i > 0\}$ with $0 \in \bar{\Lambda}_i$, and set $\Lambda_i(t_i) = \{(t_1, \ldots, \hat{t}_i, \ldots, t_m) \in \mathrm{R}^{m-1}\colon (t_1, \ldots, t_m) \in \Lambda_i\}$. The conditions we need are

$$|\Lambda_i(t_i)| \leqq C t_i^{m-1}, \tag{11}$$

and

$$\Lambda_i + C_\alpha^i \subset \Lambda_i, \tag{12}$$

where C_α^i is the cone $C_\alpha^i = \{t \in \mathrm{R}^m\colon t_i > 0, |t_j| \leqq \alpha t_i\}$. For such $\Lambda = \bigcap_{i=1}^m \Lambda_i$, where

Λ_i satisfies the above conditions, we can prove that Λ has the covering property, and we obtain our main result

THEOREM 3. *If* Ω *and* Λ *satisfy* (6), (7), (11) *and* (12) *then*

$$|\{M_\Omega^\Lambda u > \lambda\}| \leqq C\,|\{M_{C_\alpha}^\Lambda u > \lambda\}|,$$

and

$$|\{M_\Omega^\Lambda Pf > \lambda\}| \leqq C\frac{\|f\|_1}{\lambda}.$$

We prove this theorem in §2.

1. The L^p-theory, $p > 1$.

To prove Theorem 1 we need the following lemma.

LEMMA 1. *If* $\Omega \subset \mathsf{R}_+^{n+m}$ *satisfies* $|\Omega(t)| \leqq Ct^n$, *and* $\Omega + C_\alpha \subset \Omega$, *then there exist* $\Omega_i \subset \mathsf{R}_+^{n_i+1}$ *so that* $\Omega \subset \Omega_1 \times \cdots \times \Omega_m$, $|\Omega_i(t_i)| \leqq Ct_i^{n_i}$, *and* $\Omega_i + C_\alpha \subset \Omega_i$.

PROOF. For t in R_+^m we let $R_t = \{x \in \mathsf{R}^n : |x_i| \leqq t_i\}$, and set $\tilde\Omega(t) = \bigcup_{x \in \Omega(t)} (x + R_t)$. The cone condition on Ω clearly implies $\tilde\Omega(t) \subset \Omega\left(\left(1 + \dfrac{1}{\alpha}\right)t\right)$, and hence $|\tilde\Omega(t)| \leqq Ct^n$. Let $\tilde\Omega_i(t)$ be the projection of $\tilde\Omega(t)$ onto R^{n_i}. We claim that

$$|\tilde\Omega_i(t)| \leqq Ct_i^{n_i}. \tag{13}$$

for all t in R_+^m. To see this, observe that for each point x in $\tilde\Omega(t)$ we can find a rectangle $R_t(x')$ so that $x \in R_t(x') \subset \tilde\Omega(t)$. Thus,

$$\begin{aligned}\frac{t^n}{t_i^{n_i}}|\tilde\Omega_i(t)| &= \int_{\tilde\Omega_i(t)} \frac{t^n}{t_i^{n_i}}\,dx_i \\ &\leqq \int_{\tilde\Omega_i(t)} |\{(x_1,\ldots,\hat x_i,\ldots,x_m) \in \mathsf{R}^{n-n_i} : x \in \tilde\Omega(t)\}|\,dx_i = |\tilde\Omega(t)| \leqq Ct^n,\end{aligned}$$

and hence $|\tilde\Omega_i(t)| \leqq Ct_i^{n_i}$ as desired.

Now set $\Omega_i(t_i) = \bigcup_{s:\, s_i = t_i} \tilde\Omega_i(s)$. Since $\tilde\Omega_i(s)$ increases with $s_j, j \neq i$, we have $\Omega_i(t_i) = \lim_{\substack{s_j \to \infty,\, j \neq i \\ s_i = t_i}} \tilde\Omega_i(s)$, and (13) implies $|\Omega_i(t_i)| \leqq Ct_i^{n_i}$.

Furtermore, $\Omega_i = \{(x_i, t_i) \in \mathsf{R}_+^{n_i+1} : x_i \in \Omega_i(t_i)\}$ satisfies the cone condition since the region built from $\tilde\Omega_i(s)$, for fixed $s_j, j \neq i$, does.

With this Lemma, and the one-parameter result, Theorem 1 follows easily by repeated integration.

PROOF OF THEOREM 2. The ideas in this proof are from [NS]. Assume first that

$$|\{M_\Omega u > \lambda\}| \leqq C\left[\frac{\|M_{C_\alpha}u\|_p}{\lambda}\right]^p. \tag{14}$$

For t in R^m_+ set $R_t = \{x \in \mathsf{R}^n : |x_i| \leqq t_i\}$, and $Q_t = \{(x,y) \in \mathsf{R}^{n+m}_+ : |x_i| \leqq t_i, y_i \leqq t_i\}$, and let $u = \chi_{Q_t}$ the characteristic function of Q_t. Then $M_{C_\alpha}u = 1$ on $R_{(1+\alpha)t}$, and $M_{C_\alpha}u = 0$ otherwise, and hence

$$\|M_{C_\alpha}u\|_p^p = |R_{(1+\alpha)t}| = (1+\alpha)^n \cdot t^n.$$

If we take x in R_t, then $u(x,t) = 1$, and $x - \tilde{\Omega}(t) \subset \{M_{\tilde{\Omega}}u \geqq 1\}$ (Recall that $\tilde{\Omega} = \Omega + C_\alpha$). Hence

$$|\tilde{\Omega}(t)| \leqq |\{M_{\tilde{\Omega}}u \geqq 1\}|. \tag{15}$$

Now we claim that

$$\{M_{\tilde{\Omega}}u \geqq 1\} \subset \{M_\Omega \chi_{Q_{(1+\alpha)t}} \geqq 1\}. \tag{16}$$

For, if $x \in \{M_{\tilde{\Omega}}u \geqq 1\}$, then there exist $(x',t') \in \Omega^x$ such that $C_\alpha(x',t') \cap Q_t \neq \emptyset$, where $C_\alpha(x',t')$ is the cone with vertex at (x',t'). Take $(x'',t'') \in C_\alpha(x',t') \cap Q_t$. Then $t'' < t$ and $|x_i'' - x_i'| \leqq \alpha(t''-t') \leqq \alpha t$, and thus $(x',t') \in Q_{(1+\alpha)t}$.

By combining (14), (15) and (16) we have

$$|\tilde{\Omega}(t)| \leqq |\{M_{\tilde{\Omega}}u \geqq 1\}| \leqq |\{M_\Omega \chi_{Q(1+\alpha)t} \geqq 1\}| \leqq C\,\|M_{C_\alpha}\chi_{Q(1+\alpha)t}\|_p^p \leqq Ct^n.$$

And the first part of Theorem 2 is established.

Assume next that

$$|\{M_\Omega Pf > \lambda\}| \leqq C\left[\frac{\|f\|_p}{\lambda}\right]^p.$$

Let $u(x,y) = P_y * \chi_{R_t}(x) = \prod_{i=1}^m P_{y_i} * \chi_{R_{t_i}}(x_i)$. We have $P_1(x_i) \geqq C_i \chi_{R_1}(x_i)$, and $P_{y_i}(x_i) \geqq \dfrac{C_i}{y_i^{n_i}} \chi_{R_{y_i}}(x_i)$. If we take x_i in R_{t_i}, and let $y_i \leqq t_i$ then

$$P_{y_i} * \chi_{R_{t_i}}(x_i) \geqq \frac{C_i}{y_i^{n_i}} \int \chi_{R_{y_i}}(x_i - z)\chi_{R_{t_i}}(z)\,dz \geqq \frac{C_i}{y_i^{n_i}} \cdot \frac{y_i^{n_i}}{2^{n_i}} = \frac{C_i}{2^{n_i}}.$$

Thus we have $P\chi_{R_t}(x,y) \geqq \prod_{i=1}^m \dfrac{C_i}{2^{n_i}} = C$, if $(x,y) \in Q_t$, and hence

$$|\{M_\Omega \chi_{Q_t} > \tfrac{1}{2}\}| \leqq |\{M_\Omega P\chi_{R_t} > C\}| \leqq C\,\|\chi_{R_t}\|_p^p \leqq Ct^n.$$

If we combine this with (15) and (16), we obtain the desired result

$$|\tilde{\Omega}(t)| \leqq Ct^n.$$

2. The L^1-case.

We first prove that the covering property implies the equivalence of the distribution functions of M_Ω^A and $M_{C_\alpha}^A$ when Ω satisfies condition (6) and (7). To prove this, following the ideas in [AC], we define the outer measure μ_Ω by $\mu_\Omega(E) = |\{x \in \mathrm{R}^n \colon \Omega^x \cap E \neq \emptyset\}|$. Then clearly $\mu_\Omega(\{z \in \mathrm{R}^n \times \Lambda \colon |u(z)| > \lambda\}) = |\{M_\Omega^A u > \lambda\}|$, and we also have

LEMMA 2. *If Ω satisfies* (6) *and* (7) *then* $\mu_\Omega(Q_t(x)) \leqq C\,|R_t(x)|$.

Let us assume this for a minute and prove

PROPOSITION 1. *If Λ has the covering property, and Ω satisfies* (6) *and* (7), *then*

$$|\{M_\Omega^A u > \lambda\}| \leqq C\,|\{M_{C_\alpha}^A u > \lambda\}|.$$

PROOF. $M_\Omega^A u$ only depends on the values of u in $\mathrm{R}^n \times \Lambda$ so we set $E_\lambda = \{z \in \mathrm{R}^n \times \Lambda \colon |u(z)| > \lambda\}$. Clearly, $E_\lambda \subset \bigcup_{(x,t)\in E_\lambda} Q_t(x)$. We want to apply the cvering property to the corresponding family, $\{R_t(x)\}$, $(x,t) \in E_\lambda$, to do this it is required that $|\cup R_t(x)| < +\infty$, we can assume this since otherwise is $|\{M_{C_\alpha}^A u > \lambda\}| = +\infty$, and there is nothing to prove. Hence, we have a subfamily $\{\tilde{R}_t(x)\}$ satisfying (9) and (10). Let $\{\tilde{Q}_t(x)\}$ be the corresponding subfamily of $\{Q_t(x)\}$. It is clear that each $Q_t(x)$ is included in some $\tilde{Q}_{ct'}(x')$. Hence

$$\begin{aligned}
|\{M_\Omega^A u > \lambda\}| &= \mu_\Omega(E_\lambda) \leqq \mu_\Omega(\cup Q_t(x)) \leqq \mu_\Omega(\cup \tilde{Q}_{ct}(x)) \\
&\leqq \sum \mu_\Omega(\tilde{Q}_{ct}(x)) \leqq \sum C\,|\tilde{R}_{ct}(x)| \leqq C \sum |\tilde{R}_t(x)| \\
&\leqq C\,|\cup R_t(x)| \leqq C\,|\{M_{C_\alpha}^A u > \lambda\}|,
\end{aligned}$$

as desired.

PROOF OF LEMMMA 2. We first observe that $\Omega(t)$ increases in t and if $R_t(x) \cap \Omega^y(t) \neq \emptyset$, then $x \in \Omega^y(Ct)$. Thus,

$$\begin{aligned}
\mu_\Omega(Q_t(x)) &= |\{y \in \mathrm{R}^n \colon \Omega^y \cap Q_t(x) \neq \emptyset\}| \\
&= |\{y \in \mathrm{R}^n \colon \Omega^y(t) \cap R_t(x) \neq \emptyset\}| \leqq |\{y \in \mathrm{R}^n \colon x \in \Omega^y(Ct)\}| \\
&= |\{y \in \mathrm{R}^n \colon -y \in \Omega^{-x}(Ct)\}| \leqq C\,|R_t(x)|.
\end{aligned}$$

COROLLARY 1. *If Λ has the covering property, and Ω satisfies conditions* (6) *and* (7) *then* $|\{M_\Omega^A Pf > \lambda\}| \leqq C \dfrac{\|f\|_1}{\lambda}$.

PROOF. First note that the covering property implies the weak type (1, 1) of the Hardy-Littlewood maximal function $Hf(x) = \sup_{t\in A} \frac{1}{|R_t(x)|} \int_{R_t(x)} |f| = \sup_{t\in A} \frac{1}{|R_t|} \chi_{R_t} * f(x)$.

The Poisson kernels $P_{t_i}(x_i)$ can be estimated by

$$P_{t_i}(x_i) \leqq \sum_{j=0}^{\infty} 2^{-j} \frac{1}{|R_{2^j t_i}|} \chi_{R_{2^j t_i}}(x_i),$$

hence

$$P_t(x) \leqq \sum_{j_1,\ldots,j_m} 2^{-j_1} \cdots 2^{-j_m} \frac{1}{|R_{2^{j_1}t_1}|} \chi_{R_{2^{j_1}t_1}}(x_1) \cdots \frac{1}{|R_{2^{j_m}t_m}|} \cdot \chi_{R_{2^{j_m}t_m}}(x_m)$$

$$= \sum_j 2^{-j} \frac{1}{|R_{2^j t}|} \chi_{R_{2^j t}}(x).$$

Thus we have

$$\sup_{t\in A} P_t * f(x) \leqq \sum_j 2^{-j} \sup_{t\in A} \frac{1}{|R_{2^j t}|} \chi_{R_{2^j t}} * f(x) = \sum_j 2^{-j} H^j f(x). \tag{17}$$

It is enough to prove that H^j is uniformly of weak type (1, 1) to be able to sum in weak L^1. Let T_j be the map defined by $T_j(y) = 2^{-j}y$. Then $H^j f(x) = H(f \circ T_j^{-1})(T_j(x))$. Hence H^j are uniformly of weak type (1, 1) and, by (17), so is $\sup_{t\in A} P_t * f(x)$. From this and the inequality $P_t(x + z) \leqq C_\alpha P_t(x)$, $|z| \leqq \alpha t$, the weak type (1, 1) for $M_{C_\alpha}^A Pf$ follows. By letting $u = Pf$ in Proposition 1 the Corollary follows.

To prove Theorem 3, it remains to show that A has the covering property. This is a consequence of the following lemma and its corollary.

LEMMA 3. *If A satisfies conditions* (11) *and* (12), *then for each* $D < \infty$, $A \cap \{t; t_j \leqq D\}$ *is included in the union of finitely many sets* T_k^i, $i = 1,\ldots,m$, $k \in \mathbf{Z}^{m-1}$, *such that if* $s, t \in T_k^i$ *then* $s_i \leqq t_i$ *impliees* $s_j \leqq Ct_j$, $1 \leqq j \leqq m$. *The number of sets* T_k^i *and the constant* C *depend only on the constant in* (11) *and on* α *in* (12).

More precisely, we will construct T_k^i such that for each i, $A_i(t_i) \cap \{(t_1,\ldots,\hat{t}_i,\ldots,t_m) \in \mathbf{R}^{m-1}; t_i \leqq t_j\} \subseteq \bigcup_k T_k^i(t_i)$, if $t_i \leqq D$.

COROLLARY 2. *If A satisfies* (11) *and* (12), *then A has the covering property.*

PROOF OF LEMMA 3. First assume that $m = 2$, and for simplicity we assume $\alpha = 1$. We want to show that $A_2(t_2) \cap \{t_1 : t_2 \leqq t_1\}$ is contained in a finite union of intervals and that the number of intervals depend only on the constant in (11).

Let

$$\tilde{A}_2(t_2) = \bigcup_{t_1 \in A(t_2)}]t_1 - t_2, t_1 + t_2[\cap \{t_1 : t_1 > t_2\} = \bigcup_{k=1}^{N} I_k(t_2),$$

where $I_k(t_2)$ are pairwise disjoint intervals, and $|I_k(t_2)| \geqq t_2$. (12) gives $\tilde{A}_2(t_2) \subset A_2(2t_2)$, and by (11), $|\tilde{A}_2(t_2)| \leqq |A_2(2t_2)| \leqq 2Ct_2$. Thus $Nt_2 \leqq |\tilde{A}_2(t_2)| \leqq 2Ct_2$, and hence $N \leqq 2C$. We arrange $I_k(t_2)$ so that if $s_k \in I_k(t_2)$ then $s_1 > s_2 > \cdots > s_N$. We want to construct sets $T_k(t_2)$ so that if $s_1, t_1 \in T_k(t_2)$ and $s_1, t_1 \in A_2(t_2')$, for some $t_2' < t_2$, then $s_1, t_1 \in T_{k'}(t_2')$ for some k'. This is in general not true for the intervals $I_k(t_2)$. The problem is that an interval can split into several intervals as t_2 decreases, and we want to group these intervals together. Let M be the set of "splitting points", i.e.

$$M = \{(t_1, t_2) : t_2 \leqq D, t_1 \notin \tilde{A}(t_2), t_1 \in \operatorname{Int}(\overline{\tilde{A}_2(t_2)})\}.$$

We observe that for fixed t_2 there are finitely many (t_1', t_2') in M with, $t_2' \in]\frac{1}{2}t_2, t_2[$, since if $(t_1', t_2') \in M$ then $]t_1' - \frac{1}{4}t_2, t_1' + \frac{1}{4}t_2[\times \mathsf{R}_+ \cap M = (t_1', t_2')$, and $]t_1' - \frac{1}{4}t_2, t_1' + \frac{1}{4}t_2[\subset \tilde{A}_2(t_2) \cup [\frac{1}{4}t_2, t_2]$. So the first coordinate of the points in $M \cap (\mathsf{R}_+ \times]\frac{1}{2}t_2, t_2[)$ are contained in disjoint intervals in $\tilde{A}_2(t_2)$ of length $\frac{1}{2}t_2$, and since $|\tilde{A}_2(t_2)| \leqq Ct_2$ there can only be finitely many such points.

Let $\mathscr{D} = \{t_2 : t_2 \leqq D, (t_1, t_2) \in M \text{ for some } t_1\} = \{D_j\}$ and $D_0 = D$. Assume D_j is ordered so that $D_0 > D_1 > D_2 > \cdots$. The only possible limit point of $\mathscr{D}$ is 0. If $\mathscr{D}$ is finite and $D_N = \min D_j$, then set $D_{N+1} = 0$.

For each D_j, there is a t_1 and an $I_k(D_j)$ such that $I_{k+1}(D_j) \cup \{t_1\} \cup I_k(D_j)$ is an interval and we want to consider this as one interval $I_k(D_j)$. To achieve this, we modify the definition of the sets $I_k(D_j)$ with $\tilde{A}_2(D_j)$ replaced by $\tilde{A}_2(D_j) \cup \{t_1 : (t_1, D_j) \in M\}$, and replace the intervals $I_k(D_j)$, with these new ones. Now we can start the construction of the sets $T_k(t_2)$.

The sets $T_k(D_0)$ are defined as

$$T_k(D_0) = I_k(D_0), \ 1 \leqq k \leqq N.$$

Assume now that $T_k(D_j)$ is defined and that $D_{j+1} \leqq t_2 < D_j$. Then we define $T_1(t_2)$ as

$$T_1(t_2) = I_1(t_2) \cup (B \cap \tilde{A}_2(t_2)),$$

where

$$B = \bigcup_i \{T_i(D_j) : I_1(t_2) \cap T_i(D_j) \neq \emptyset\}.$$

If $T_1(t_2), \ldots, T_{k-1}(t_2)$ are defined, then we set

$$T_k(t_2) = A \cup \left(B \cap \left(\tilde{\Lambda}_2(t_2) \Big\backslash \bigcup_{i=1}^{k-1} T_i(t_2)\right)\right)$$

where $A = I_l \Big\backslash \bigcup_{i=1}^{k-1} T_i(t_2)$, and l is the smallest integer with $A \neq \emptyset$, and

$$B = \bigcup_i \{T_i(D_j): A \cap T_i(D_j) \neq \emptyset\}.$$

This ends the construction of the sets $T_k(t_2)$. Let $T_k^2 = \{(t_1, t_2): t_1 \in T_k(t_2)\}$. To prove that the sets T_k^2 has the desired property we first check if $t_1, s_1 \in T_k(t_2)$ implies $|t_1 - s_1| \leqq Ct_1$. Assume first that t_1 and s_1 are in the A part of $T_k(t_2)$ (if $k = 1$ then the A part means $I_1(t_2)$). Then clearly $|t_1 - s_1| \leqq 2Ct_2 \leqq 2Ct_1$. If we take t_1 in the A part and s_1 from the B part of $T_k(t_2)$, then the construction of T_k gives that there exists a $D_j, t_2 \leqq D_j$, with $s_1, t_1 \in I_{k'}(D_j)$, and this gives

$$|s_1 - t_1| \leqq 2CD_j \leqq 2Cs_1.$$

If $s_1 \leqq t_1$ are in both the B part, and t_1' in A part then $s_1 \leqq t_1 \leqq t_1'$, and the former case gives $|s_1 - t_1| \leqq |s_1 - t_1'| \leqq 2Cs_1$. So we have that if $s_1, t_1 \in T_k(t_2)$, then $|s_1 - t_1| \leqq 2Ct_1$.

Now take $t_1 \in T_k(t_2)$, and $s_1 \in T_k(s_2)$ with $s_2 \leqq t_2$. We want to show that $|s_1 - t_1| \leqq 2Ct_1$. If $s_1 \leqq t_1$, then the conclusion of the Lemma is clearly true. If $s_1 > t_1$, then we must have that $s_1 \in T_k(t_2)$, and hence $|t_1 - s_1| \leqq 2Ct_1$. This completes the proof in the case $m = 2$.

If $m > 2$, it follows from Lemma 1 that $\Lambda_i(t_i)$ is contained in a product of $m - 1$ sets $\Lambda_i^j(t_i)$,

$$\Lambda_i(t_i) \subset \prod_{\substack{j=1 \\ j \neq i}}^{m} \Lambda_i^j(t_i),$$

where each Λ_i^j satisfies

$$|\Lambda_i^j(t_i)| \leqq Ct_i, \text{ and } \Lambda_i^j + \mathrm{C}_\alpha \subset \Lambda_i^j.$$

From the case $m = 2$, we know that $\Lambda_i^j(t_i) \cap \{t_j: t_i \leqq t_j\} \subseteq \bigcup_{k=1}^{N_j} T_k^{i,j}(t_i)$, and if $s_j \in T_k^{i,j}(s_i)$, $t_j \in T_k^{i,j}(t_i)$, then $s_i \leqq t_i$ implies $s_j \leqq Ct_j$.

We define T_k^i for $k \in \mathbf{Z}^{m-1}$, $1 \leqq k_j \leqq N_j, j \neq i$ by

$$t \in T_k^i(t_i),\ k = (k_1, \ldots, \hat{k}_i, \ldots, k_m), \text{ iff } t_j \in T_{k_j}^{i,j}(t_i).$$

Then it is obvious that if $t, s \in T_k^i$, then $s_i \leqq t_i$ implies $s_j \leqq Ct_j$, and Lemma 3 is proved.

PROOF OF COROLLARY 2. Let $R = \{R_t(x)\}$ be a given family of rectangles where $t \in \Lambda$, and with $|\cup R_t(x)|$ bounded. If $|R_t(x)| \leqq N$ and $t \in \Lambda \cap \{t_i \leqq t_j\}$, then $t_i \leqq N^{1/n}$. The sets $\Lambda_i^j(N^{1/n})$ are contained in finitely many intervals, and hence there is a D_i such that $\sup \Lambda_i^j(N^{1/n}) \leqq D_i$. Thus, if $t \in \Lambda \cap \{t_i \leqq t_j\}$ then $t_i \leqq N^{1/n}$ implies $t_j \leqq D_i$. Let $D = \max D_i$. So, we can assume $t_i \leqq D$, $1 \leqq i \leqq m$, and choose the subfamily by the usual selection principle. We divide $\{R_t(x)\}$ into different groups where $t \in T_k^i$ and fix $k \in \mathbf{Z}^{m-1}$ and i. We order $R_t(x)$ according to decreasing t_i value and successively add rectangles to the subfamily $\tilde{R}_t(x)$, when they are disjoint from the ones already chosen.

If $R_t(x)$ is not chosen, then there is a $\tilde{R}_{t'}(x')$, with $t'_j \geqq t_i$, $\tilde{R}_{t'}(x') \cap R_t(x) \neq \emptyset$. Lemma 3 gives that $t_j \leqq Ct'_j$, since $t_i \leqq t'_i$, and we get

$$R_t(x) \subset \tilde{R}_{5Ct'}(x'),$$

which is (10). Since $\tilde{R}_t(x)$ are disjoint $\sum \chi_{\tilde{R}_t(x)} \leqq 1, t \in T_k^i$, for fixed i and k. Hence (9) follows since we have a finite number of T_k^i: s.

A COUNTEREXAMPLE. If $\Lambda = \bigcup_{k=0}^{\infty} (2^k, 2^{-2k})$, then $M_{C_1}^{\Lambda}$ is not of weak type (1, 1). To see this, we construct u_k, $\|u_k\| = 1$, so that $|\{M_{C_1}^{\Lambda} u_k \geqq 1\}| \geqq 1 + \frac{k}{2}$. We let $E_0 = [0, 1]$, and E_k will be a union of 2^k intervals with length 2^{-2k}. If E_{k-1} is defined, then for each component $I = [a, a + 4 \cdot 2^{-2k}]$ of E_{k-1}, take two subintervals of I, $I_1 = [a, a + 2^{-2k}]$, $I_2 = [a + 3 \cdot 2^{-2k}, a + 4 \cdot 2^{-2k}]$, and let E_k be the union of these intervals. Then we have $E_k \subset E_{k-1} \cdots \subset E_0$, $|E_k| = 2^{-k}$. We set $f_k = 2^k \cdot \chi_{[0,1] \times E_k}$. Then $\|f_k\|_1 = 1$. Set $u_k(x, t) = \frac{1}{|R_t|} \int_{R_t(x)} |f_k|$.

We will show that $\{M_{C_1}^{\Lambda} u_k \geqq 1\} \supset \bigcup_{i=0}^{k} [0, 2^i] \times E_i$, and $\left| \bigcup_{i=0}^{k} [0, 2^i] \times \mathrm{E}_i \right| = 1 + \frac{k}{2}$. Fix k and let I be one of the components of $E_i, i \leqq k$, let $R = [0, 2^i] \times I$, $|R| = 2^{-i}$.

$$\frac{1}{|R|} \int_R |f_k| = 2^i \cdot 2^k \int_I \chi_{E_k} = 2^i \cdot 2^k \cdot 2^{k-i} \cdot 2^{-2k} = 1.$$

Hence $M_{C_1}^{\Lambda} u_k \geqq 1$ on $[0, 2^i] \times \mathrm{E}_i$, $i \leqq k$. Thus,

$$
\begin{aligned}
|\{M^{\Lambda}_{C_1}u_k \geqq 1\}| &\geqq \left|\bigcup_{i=0}^{k} [0,2^i] \times E_i\right| \\
&= \left|\bigcup_{i=0}^{k-1} [0,2^i] \times E_i\right| + \left|[0,2^k] \times E_k \Big\backslash \left(\bigcup_{i=0}^{k-1} [0,2^i] \times E_i\right)\right| \\
&= \left|\bigcup_{i=0}^{k-1} [0,2^i] \times E_i\right| + |[2^{k-1},2^k] \times E_k| \\
&= \left|\bigcup_{i=0}^{k-1} [0,2^i] \times E_i\right| + \tfrac{1}{2} = \cdots = 1 + \frac{k}{2}.
\end{aligned}
$$

Hence M^{Λ}_{Ω} is not of weak type $(1,1)$.

In this example, the non-boundedness of $M^{\Lambda}_{C_\alpha}$ is due to the fact that Λ contains t, with t_1 arbitrary large. Since we are interested in boundary convergence, we would like a counterexample where all t_i are arbitrary small. This also follows with the same method, if we rescale Λ, i.e. let $\tilde{\Lambda} = \{4^{-2^l}(2^k, 2^{-2k});\ 2^{l-1} \leqq k < 2^l,\ l > l_0\}$. Then as above, we obtain $\|M^{\tilde{\Lambda}}_{C_\alpha}\|_{L^1 \to L^{1,\infty}} \geqq C \cdot 2^{l_0}$ for each l_0, and hence $M^{\tilde{\Lambda}}_{C_\alpha}$ is not of weak type $(1,1)$.

Also by a slight change in the argument we can generalize the example to sets $\Lambda_f = \{(2^k, f(2^k))\}$, for any f where $xf(x)$ is decreasing. (In the example $f(x) = x^{-2}$). The difference is that in the construction of the sets E_k, we subdivide so that E_k consist of 2^{n_k} intervals, where n_k satisfies $\frac{1}{2} < 2^{n_k} 2^k \cdot f(2^k) \leqq 1$.

3. A local Fatou Theorem.

Let u be defined in R^{n+1}_+. A function u, define in R^{n+1}_+, is said to be non-tangentially bounded a.e., if for a.a. x_0 in R^n there is a cone $C^{x_0}_\alpha$, such that u is bounded in $C^{x_0}_\alpha$. If u is harmonic and non-tangentially bounded a.e. then the classical local Fatou theorem [C] asserts that u has non-tangential limits a.e. This has been extended by Mair, Philipp and Singman [MPS] to our approach regions Ω.

THEOREM. *If Ω satisfies $|\Omega(t)| \leqq Ct^n$, and $\Omega + C_\alpha \subset \Omega$, and if u has non-tangential limits* a.e. *then* $\displaystyle\lim_{\substack{(x,t)\to(x_0,0)\\ (x,t)\in\Omega^{x_0}}} u(x,t)$ *exists for* a.a. x_0 *in* R^n.

This follows easily from the inequality (4), $|\{M_\Omega u > \lambda\}| \leqq C|\{M_{C_\alpha} u > \lambda\}|$.

Let $u^*(x) = \displaystyle\lim_{C_\alpha} u$. We first assume that $u^* \equiv 0$, and that $u(x,t) = 0$ if $|x| > N$, set

$$
u^\varepsilon(x,t) = \begin{cases} u(x,t) & \text{if } t < \varepsilon \\ 0 & \text{otherwise} \end{cases}
$$

That $u^\varepsilon(x,t) \to 0$ in C_α as $\varepsilon \to 0$ implies $|\{M_{C_\alpha}u^\varepsilon > \lambda\}| \to 0$, $\varepsilon \to 0$, and hence $|\{M_\Omega u^\varepsilon > \lambda\}| \to 0$, $\varepsilon \to 0$. Since $|\{\limsup_{\Omega} |u| > \lambda\}| \leqq |\{M_\Omega u^\varepsilon > \lambda\}| \to 0$, $\varepsilon \to 0$, $u \xrightarrow{\Omega} 0$ a.e.

If $u^* \not\equiv 0$, choose a sequence $\lambda_k \nearrow + \infty$, such that $|\{x\colon |u^*(x)| = \lambda_k\}| = 0$.

Let

$$u_k(x,t) = \begin{cases} u(x,t) & \text{if } |u(x,t)| \leqq \lambda_k \text{ and } |x| \leqq \lambda_k \\ 0 & \text{otherwise} \end{cases}$$

and

$$u_k^*(x) = \begin{cases} u^*(x) & \text{if } |u^*(x)| \leqq \lambda_k \text{ and } |x| \leqq \lambda_k \\ 0 & \text{otherwise} \end{cases}.$$

Then clearly $u_k \xrightarrow{C_\alpha} u_k^*$. From the Nagel-Stein theorem $Pu_k^* \xrightarrow{\Omega} u_k^*$ a.e. Hence $\tilde{u}_k = u_k - Pu_k^* \xrightarrow{C_\alpha} 0$, and from the previous case we have $\tilde{u}_k \xrightarrow{\Omega} 0$. Thus, $u_k = \tilde{u}_k + Pu_k^* \xrightarrow{\Omega} 0 + u_k^*$, and from this we obtain the desired result $u \xrightarrow{\Omega} u^*$.

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