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K-GROUPS OF TOEPLITZ ALGEBRAS OF REINHARDT DOMAINS

ALBERT JEU-LIANG SHEU[1]

Introduction.

For D in a large class of Reinhardt domains D in $\mathbf{C}^2$, the structure of the Toeplitz C^*-algebra $\mathscr{T}(D)$ was explicitly described in [Sh1] (where $\mathscr{T}(D)$ was denoted by $\mathscr{J}(D)$) in terms of data from the boundary geometry of D (more precisely, the degrees of contact at the intersections of the boundary curve of D's logarithm domain C and its convex hull $\tilde{C}$). The key step in that paper was to use the CMR-program (initiated by Curto, Muhly and Renault [M-R, Cu-M]) to express $\mathscr{T}(D)$ as a concrete groupoid C^*-algebra. From the groupoid structure explicitly obtained in [Sh1], we can easily get that the Toeplitz C^*-algebra $\mathscr{T}(D)$ of such a Reinhardt domain contains Fredholm operators of any given indices, and then we proceed to show that the K-groups of such a C^*-algebra $\mathscr{T}(D)$ (with some restrictions on the boundary) are free abelian with ranks determined by the boundary geometry of D. More precisely, we have $K_0(\mathscr{T}(D)) \cong \mathbf{Z}^{\# + 1}$ and $K_1(\mathscr{T}(D)) \cong \mathbf{Z}^{\#}$, where $\#$ is the number of lower horizontal faces B (not including E and N) of the polygons $\tilde{P}$ (constructed in [Sh1]) corresponding to linear faces $\tilde{F}$ of $\tilde{C}$ (the logarithmic domain of the pseudoconvex hull $\tilde{D}$ of D) with rational slopes. (The question of computing these K-groups was raised to the author by professor H. Suzuki while visiting MSRI.)

Section 1.

In the CMR-program [M-R, Cu-M], interesting C^*-algebras like Wiener-Hopf C^*-algebras and Toeplitz C^*-algebras are realized as groupoid C^*-algebras [Co, R] of groupoids constructed from groups in special ways. We first make a simple observation for the following general situation. Let G be a discrete group embedded in a locally compact G-space Y as a sub-G-space such that the closure X of a positive cone P (with P generating G as a group and $P \cap P^{-1} = \{e\}$) is a regular compactification of P, and let $\mathfrak{G}$ be the reduced groupoid $(Y \times G) | X$

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(c.f. [R, M-R]). By [R, M-R], the regular representation ρ of the reduced groupoid C^*-algebra $C^*(\mathfrak{G})$ (with the canonical Haar system $\delta_x \times \delta_g$) on $l^2(P)$ is faithful. For such groupoids, we have the following lemma about indices of Fredholm operators (cf. [D2]) in $C^*(\mathfrak{G})$ (which could be suitably generalized to other groupoids).

LEMMA 1. *With P, G, Y and X as above, if there is an element $p \in P\backslash\{e\}$ such that the closure of $\{p^n \mid n = 0, 1, 2, \dots\}$ is open in X, then the operator algebra $\rho(C^*(\mathfrak{G}))$ contains Fredholm operators of any given indices.*

PROOF. Since $p \in P\backslash\{e\}$ and $P \cap P^{-1} = \{e\}$, p is not of finite order. Since the closure C of $A := \{p^n \mid n = 0, 1, 2, \dots\}$ is both open and closed, the function $f(x, g) := \chi_C(x)\delta_p(g) + \chi_{X\backslash C}(x)\delta_e(g)$ is in $C_c(\mathfrak{G})$, where χ represents characteristic functions, and its representation $\rho(f)$ under the regular representation ρ on $l^2(P)$ is clearly the direct sum of a unilateral shift on $l^2(A)$ and the identity map on $l^2(P\backslash A)$, which has index one.

Recall that in [Sh1], we considered a large class of Reinhardt domains D (containing 0) in $\mathbf{C}^2$, which includes D with the boundaries ∂C of its logarithmic domain $C = \ln(|D|)$ piecewise analytic (or piecewise smooth with a meaningful degree of contact) at the intersections with the boundary $\partial\tilde{C}$ of the convex hull $\tilde{C}$ of C and includes pseudoconvex D. An immediate application of the above lemma is that the Toeplitz C^*-algebra $\mathscr{I}(D)$ of a Reinhardt domain D considered in [Sh1] with $n_\infty = 2$ contains Fredholm operators of any given indices. In fact, for D with $n_\infty = 2$, we have $\mathscr{I}(D) \cong C^*(\mathfrak{G})$, where $\mathfrak{G}$ is the groupoid constructed in [Cu-M] from the Reinhardt domain D by reducing $Y \times \mathbf{Z}^2$ to X for some $\mathbf{Z}^2$-space Y containing $\mathbf{Z}^2$ with X the closure of the positive cone $\mathbf{Z}^2_\geq$. By the detailed study in [Sh1] of the topology of X, we know that X is a regular compactification of $\mathbf{Z}^2_\geq$ and that the closure of, say, $\{(n, 0) \mid n \in \mathbf{Z}_\geq\}$ (or $\{(0, n) \mid n \in \mathbf{Z}_\geq\}$) in X is open in X whe $n_\infty = 2$. Thus applying the above lemma, we get the following corollary.

COROLLARY. *If D is a Reinhardt domain in $\mathbf{C}^2$ considered in [Sh1] with $n_\infty = 2$, then the Toeplitz C^*-algebra $\mathscr{I}(D)$ contains Fredholm operators of any given indices.*

Note that the Toeplitz C^*-algebra $\mathscr{I}^{\alpha, \beta}$ studied in [D1, Pa] (or equivalently the Wiener-Hopf C^*-algebras $\mathscr{W}(P_{\alpha, \beta})$ in [Sh2]) can be realized as a groupoid C^*-algebra $C^*(\mathfrak{G}_{\alpha, \beta})$ by the CMR-program, and since it is easy to check that $\mathfrak{G}_{\alpha, \beta}$ satisfies the requirement in lemma 1 if $\alpha \in \mathbf{Q}$ or $\beta \in \mathbf{Q}$, lemma 1 shows that $\mathscr{I}^{\alpha, \beta}$ with $\alpha \in \mathbf{Q}$ or $\beta \in \mathbf{Q}$ contains Fredholm operators of arbitrary indices as proved in [Pa] by a different approach [D1].

However lemma 1 does not apply to many interesting cases, for example $\mathscr{I}^{\alpha, \beta}$

with both α and β irrational. In [Pa], it was proved that $K_1(\mathcal{J}^{\alpha,\beta}/\mathcal{K})$ is always Z, and the interesting question of whether $\mathcal{J}^{\alpha,\beta}$ contains a Fredholm operator with index one raised there is still open. The author would like to thank Professor R. Douglas and his student for pointing out an incorrect proof in the earlier version of this paper about the existence of such operators. We would like to briefly describe the groupoid structure of $\mathfrak{G}_{\alpha,\beta}$ for irrational α and β, which shows why lemma 1 does not apply in such cases and is of some independent interest.

Let $P_{\alpha,\beta}$ be the cone $\{(m,n)\in\mathsf{Z}^2 \mid -\alpha m + n \geqq 0 \text{ and } -\beta m + n \leqq 0\}$ (with $0<\alpha<\beta$) in $G=\mathsf{Z}^2$ whose Wiener-Hopf algebra $\mathcal{W}(P_{\alpha,\beta})$ [M-R] is exactly $\mathcal{J}^{\alpha,\beta}$ studied by E. Park in [Pa]. It is not hard to see that the Muhly-Renault compactification $X_{\alpha,\beta}$ of $P_{\alpha,\beta}$ is always regular [M-R]. In fact, using Muhly-Renault's approach, we can get by direct calculation an explicit description of $X_{\alpha,\beta}$ and hence $\mathfrak{G}_{\alpha,\beta}$ as follows. We only describe it here for the case where both α and β are irrational. Let $\iota_\alpha(\mu) := -\alpha\mu_1 + \mu_2$ and $\iota_\beta(\mu) := -\beta\mu_1 + \mu_2$ for $\mu\in\mathsf{Z}^2$. We have

$$X_{\alpha,\beta} = P_{\alpha,\beta} \cup R_\alpha \cup R_\beta \cup \{\infty\}$$

as sets, where R_α and R_β are R as sets, and $X_{\alpha,\beta}$ is a compact subset of a Z^2-space $Y_{\alpha,\beta}$ [M-R]. We shall say that a sequence r_n of real numbers converges to r from the right (or left) if for n sufficiently large, r_n is close to and no less than (or no greater than) r. The topology on $X_{\alpha,\beta}$ is characterized by the following. For any non-negative real number r, a sequence $z(n)\in P_{\alpha,\beta}$ with $\lim \|z(n)\| = \infty$ converges to r (respectively, $-r$) in R_α if and only if $\iota_\alpha(z(n))$ converges to r from the right (respectively, from the left). Similar description applies to R_β, while $z(n)$ converges to ∞ if and only if $\lim(\iota_\alpha(z(n))) = \lim(\iota_\beta(z(n))) = \infty$. The most interesting part is the topology on R_α and R_β. A basis of the topology on R_α or R_β is

$$\{[r, r+\varepsilon) \cup [-r-\varepsilon, -r) \mid r \geqq 0 \text{ and } \varepsilon > 0\}$$

(note that this is different from $\{[r, r+\varepsilon) \mid r\in\mathsf{R} \text{ and } \varepsilon>0\}$). Now R_α and R_β are not invariant under the Z^2-action on $Y_{\alpha,\beta}$ and the partial Z^2-action, say, on R_α is described by that

$$\mu\cdot(r) = r + \iota_\alpha(\mu)$$

for $r\geqq 0$ and $\mu\in\mathsf{Z}^2$ such that $r+\iota_\alpha(\mu)\geqq 0$, and that

$$\mu\cdot(-r) = -r - \iota_\alpha(\mu)$$

for $r>0$ and $\mu\in\mathsf{Z}^2$ such that $r+\iota_\alpha(\mu)>0$. Now $\mathfrak{G}_{\alpha,\beta}$ is the transformation group groupoid $Y_{\alpha,\beta}\times\mathsf{Z}^2$ reduced to the subset $X_{\alpha,\beta}$, and by the general theory of groupoid C^*-algebras [R2], we get a composition sequence of $\mathcal{W}(P_{\alpha,\beta})$, namely

$$\mathcal{K} = C^*(\mathfrak{G}_{\alpha,\beta} \mid P_{\alpha,\beta}) \subseteq \mathcal{J} = C^*(\mathfrak{G}_{\alpha,\beta} \mid P_{\alpha,\beta}\cup R_\alpha\cup R_\beta) \subseteq C^*(\mathfrak{G}_{\alpha,\beta}) = \mathcal{W}(P_{\alpha,\beta})$$

such that

$$\mathscr{I}/\mathscr{K} = C^*(\mathfrak{G}_{\alpha,\beta} \mid R_\alpha \cup R_\beta) \cong C(\mathfrak{G}_{\alpha,\beta} \mid R_\alpha) \oplus C^*(\mathfrak{G}_{\alpha,\beta} \mid R_\beta)$$

and

$$\mathscr{W}(P_{\alpha,\beta})/\mathscr{I} \cong C^*(\mathbb{Z}^2) \cong C(\mathbb{T}^2).$$

It can be checked that $R_\alpha \cup \{\infty\}$ is the Muhly-Renault compactification X_α of the cone $P_\alpha := \{(m,n) \in \mathbb{Z}^2 \mid -\alpha m + n \geqq 0\}$ in $\mathbb{Z}^2$ and the (partial) $\mathbb{Z}^2$-action on X_α is compatible with that on $R_\alpha \cup \{\infty\}$. Thus we have

$$\mathscr{W}(P_{\alpha,\beta})/C^*(\mathfrak{G} \mid P_{\alpha,\beta} \cup R_\beta) \cong C^*(\mathfrak{G}_{\alpha,\beta} \mid R_\alpha \cup \{\infty\}) \cong C^*(\mathfrak{G}_\alpha) \cong \mathscr{W}(P_\alpha)$$

where $\mathscr{W}(P_\alpha)$ is the algebra studied by R. Douglas in [D1]. We can get a similar statement for $P_\beta = \{(m,n) \in \mathbb{Z}^2 \mid -\beta m + n \leqq 0\}$. Putting these together, we get $\mathscr{W}(P_{\alpha,\beta})/\mathscr{K}$ isomorphic to the pull-back of the quotient maps

$$\sigma_i \colon \mathscr{W}(P_i) = C^*(\mathfrak{G}_i) \to C^*(\mathfrak{G}_i \mid \{\infty\}) \cong C(\mathbb{T}^2),$$

$i = \alpha, \beta$, i.e.

$$\mathscr{W}(P_{\alpha,\beta}) \cong \{(a,b) \in \mathscr{W}(P_\alpha) \times \mathscr{W}(P_\beta) \mid \sigma_\alpha(a) = \sigma_\beta(b)\}.$$

In [Pa], this result was derived by working directly on the Toeplitz operators in a tricky way.

Section 2.

Now we proceed to compute the K-groups of $\mathscr{I}(D)$ and in the following, D shall always denote a Reinhardt domain in $\mathbb{C}^2$ considered in [Sh1] with $n_\infty = 2$. For simplicity, we shall also assume that X_2 (identified with $\partial_0 C$ [Sh1]) has only finitely many, say n, connected components. For the properties about $\mathfrak{G}$ used in the rest of this paper, we refer to [Sh1]. First, we state a simple lemma.

LEMMA 2. *If a C^*-algebra $\mathscr{A} \subseteq \mathscr{B}(\mathscr{H})$ contains the compact operators $\mathscr{K}$ and a Fredholm operator of index one, then $K_0(\mathscr{A}) \cong K_0(\mathscr{A}/\mathscr{K})$ and $K_1(\mathscr{A}) \oplus \mathbb{Z} \cong K_1(\mathscr{A}/\mathscr{K})$.*

PROOF. Since $\mathscr{A}$ contains $\mathscr{K}$ and a Fredholm operator of index one, there is a unilateral shift (with respect to a suitable orthonormal basis) in $\mathscr{A}$. Thus any finite rank projection in $\mathscr{K}$ is stably equivalent to 0 over $\mathscr{A}$ and hence the homonorphism from $K_0(\mathscr{K})$ to $K_0(\mathscr{A})$ induced by the inclusion map is the zero map. So by the six term exact sequence of K-groups [B] for $0 \to \mathscr{K} \to \mathscr{A} \to \mathscr{A}/\mathscr{K} \to 0$, we get the statement.

By lemma 2 and the above corollary, we get $K_0(\mathscr{I}(D)) \cong K_0(\mathscr{I}(D)/\mathscr{K})$ and

$K_1(\mathscr{I}(D)) \cong K_1(\mathscr{I}(D)/\mathscr{K}) \ominus \mathsf{Z}$. Let $\partial X = X_1 \cup X_2$. By the result of [Sh1], we have $\mathscr{I}(D)/\mathscr{K} \cong C^*(\mathfrak{G} \mid \partial X)$ and a short exact sequence

$$0 \to C^*(\mathfrak{G} \mid X_1) \to C^*(\mathfrak{G} \mid \partial X) \to C^*(\mathfrak{G} \mid X_2) \to 0. \tag{1}$$

Our strategy is to determine the index map ind and the exponential map exp in the 6-term exact sequence corresponding to this short exact sequence, and then by the known results on the K-groups of the ideal $C^*(\mathfrak{G} \mid X_1)$ and the quotient $C^*(\mathfrak{G} \mid X_2)$, we can easily compute the K-groups of $C^*(\mathfrak{G} \mid \partial X)$.

Recall that $C^*(\mathfrak{G} \mid X_1)$ is the direct sum of $C^*(\mathfrak{G} \mid X_B)$'s where B's are 1-dimensional lower faces (including $B = E, N$) of polygons $\tilde{P}$ corresponding to 1-dimensional faces $\tilde{F}$ of $\partial\tilde{C}$, so $K_*(C^*(\mathfrak{G} \mid X_1)) = \Sigma^{\oplus} K_*(C^*(\mathfrak{G} \mid X_B))$. On the other hand, $C^*(\mathfrak{G} \mid X_2)$ is the direct sum of $C^*(\mathfrak{G} \mid X_I) \cong C(I \times \mathsf{T}^2)$ where I's are connected components of X_2 and $X_I = \{\chi_p \mid p \in I\}$. Since I's are homeomorphic to closed intervals (including points), the K-groups of $C^*(\mathfrak{G} \mid X_2)$ are direct sum of those of $C(\{p\} \times \mathsf{T}^2)$'s, one p for each I, i.e. $K_*(C^*(\mathfrak{G} \mid X_1)) = \Sigma^{\oplus} K_*(C(\{p\} \times \mathsf{T}^2)) = n\mathsf{Z}^2$. In order to determine ind and exp, we only need to know how they act on the direct summands.

First we claim that the image of $K_*(C(I \times \mathsf{T}^2))$ under ind or exp is contained in the sum of those $K_*(C^*(\mathfrak{G} \mid X_B))$'s with one of the end points b, c of $B\,(\subseteq P)$ corresponding to one of the end points of $I\,(\subseteq \partial_0 C)$. In fact, if the end points b, c of B do not correspond to any end point of I, then $X_{\bar{B}}$ and X_I are disjoint closed subsets of X, where $X_{\bar{B}} = X_B \cup \{\chi_b, \chi_c\}$. By modding out the ideal $C^*(\mathfrak{G} \mid \partial C \backslash (X_{\bar{B}} \cup X_I))$ from the sequence (1), we get a commuting diagram

$$\begin{array}{ccccc}
0 \to C^*(\mathfrak{G} \mid X_1) \to & C^*(\mathfrak{G} \mid \partial X) & \to & C^*(\mathfrak{G} \mid X_2) & \to 0 \\
\downarrow & \downarrow & & \downarrow & \\
0 \to C^*(\mathfrak{G} \mid X_B) \to & C^*(\mathfrak{G} \mid X_{\bar{B}} \cup X_I) & \xrightarrow{\pi} & C^*(\mathfrak{G} \mid \{\chi_b, \chi_c\} \cup X_I) & \to 0 \\
 & \| & & \| & \\
 & C^*(\mathfrak{G} \mid X_{\bar{B}}) \oplus C(I \times \mathsf{T}^2) & \to & 2C(\mathsf{T}^2) \oplus C(I \times \mathsf{T}^2) &
\end{array} \tag{2}$$

Since the lower sequence splits on $C(I \times \mathsf{T}^2)$, i.e. there is a homomorphism s from $C(I \times \mathsf{T}^2)$ to $C^*(\mathfrak{G} \mid X_{\bar{B}} \cup X_I)$ such that $\pi \circ s$ is the identity map, the index and exponential maps corresponding to the lower sequence vanish on the summand $K_*(C(I \times \mathsf{T}^2))$. Now note that the left and right vertical arrows in (2) induce direct summand projection on the K-group level and so by the functoriality of the 6-term exact sequence, we get our claim.

Now consider a similar commuting diagram gotten by modding out $C^*(\mathfrak{G} \mid \partial X \backslash X_{\bar{B}})$ from (1)

(3)
$$\begin{array}{ccccccccc}
0 \to & C^*(\mathfrak{G} \mid X_1) & \to & C^*(\mathfrak{G} \mid \partial X) & \to & C^*(\mathfrak{G} \mid X_2) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & C^*(\mathfrak{G} \mid X_B) & \to & C^*(\mathfrak{G} \mid X_{\bar{B}}) & \xrightarrow{\pi} & C^*(\mathfrak{G} \mid \{\chi_b, \chi_c\}) & \to 0 \\
& & & & & \| & \\
& & & & & 2C(\mathsf{T}^2) &
\end{array}$$

As above the left and right vertical arrows induce direct summand projections on the K-group level, so we only need to determine the exponential map and the index map corresponding to the lower sequence. More precisely, we shall determine the maps η and ι in the 6-term exact sequence

(4)
$$\begin{array}{ccccc}
& & & & \mathsf{Z}^2 \oplus \mathsf{Z}^2 \\
& & & & \| \\
K_0(C^*(\mathfrak{G} \mid X_B)) & \to & K_0(C^*(\mathfrak{G} \mid X_{\bar{B}})) & \to & K_0(C^*(\mathfrak{G} \mid \{\chi_b, \chi_c\})) \\
\uparrow \iota & & & & \downarrow \eta \\
K_1(C^*(\mathfrak{G} \mid \{\chi_b, \chi_c\})) & \leftarrow & K_1(C^*(\mathfrak{G} \mid X_{\bar{B}})) & \leftarrow & K_1(C^*(\mathfrak{G} \mid X_B)) \\
\| & & & & \\
\mathsf{Z}^2 \oplus \mathsf{Z}^2 & & & &
\end{array}$$

where the first (respectively, the second) copy of Z^2 in $K_*(C^*(\mathfrak{G} \mid \{\chi_b, \chi_c\})) \cong 2K_*(C(\mathsf{T}^2)) \cong \mathsf{Z}^2 \oplus \mathsf{Z}^2$ corresponds to the direct summand $K_*(C^*(\mathfrak{G} \mid \{\chi_b\})) = \mathsf{Z}^2$ (respectively, $K_*(C^*(\mathfrak{G} \mid \{\chi_c\})) \cong \mathsf{Z}^2$). Accordingly, we shall decompose η and ι into $\eta_b \oplus \eta_c$ and $\iota_b \oplus \iota_c$ respectively by restricting η and ι to the first or the second copy of Z^2 in $K_*(C^*(\mathfrak{G} \mid \{\chi_b, \chi_c\}))$. Since the same point b may be the end point of two different one-dimensional faces B and B', we must fix the generators of $\mathsf{Z}^2 \cong K_*(C^*(\mathfrak{G} \mid \{\chi_b\}))$ for fixed b, and describe η and ι (or more precisely, η_b and ι_b) in terms of these fixed generators when $K_*(C^*(\mathfrak{G} \mid X_B))$ are identified with suitable concrete abelian groups. By Künneth formula [Sc], we have

$$K_0(C(\mathsf{T}^2)) \cong K_0(C(\mathsf{T})) \otimes K_0(C(\mathsf{T})) \oplus K_1(C(\mathsf{T})) \otimes K_1(C(\mathsf{T}))$$

and

$$K_1(C(\mathsf{T}^2)) \cong K_1(C(\mathsf{T})) \otimes K_0(C(\mathsf{T})) \oplus K_0(C(\mathsf{T})) \otimes K_1(C(\mathsf{T}))$$

So the canonical generators of $K_*(C(\mathsf{T})) \cong \mathsf{Z}$ give rise to canonical generators of $K_*(C(\mathsf{T}^2))$ and we shall take these to be the generators of $K_*(C^*(\mathfrak{G} \mid \{\chi_b\})) \cong \mathsf{Z}^2$. Note that an orientation preserving change of basis for Z^2, i.e. an isomorphism of Z^2 represented by an element ϕ of $\mathrm{SL}(2, \mathsf{Z})$, induces the identity map on the K_0-group of $C^*(\mathsf{Z}^2) \cong C(\mathsf{T}^2)$ and the isomorphism ϕ on the K_1-group of $C^*(\mathsf{Z}^2) \cong C(\mathsf{T}^2)$ with respect to the above canonical generators. We shall study (4) in three different cases.

(i) If the slope α of $\tilde{F}$ is irrational, then $\mathfrak{G} \mid X_{\bar{B}}$ is the transformation groupoid

$(\mathrm{R} \cup \{\pm\infty\}) \times_\alpha \mathrm{Z}^2$ where Z^2 acts by $(m, n)\cdot t = t - (m\alpha + n)$, and the above short exact sequence becomes

$$0 \to \mathscr{A}_\alpha \otimes \mathscr{K} \to C(\mathrm{R} \cup \{\pm\infty\}) \times_\alpha \mathrm{Z}^2 \to 2C(\mathrm{T}^2) \to 0,$$

the 6-term exact sequence corresponding to which can be easily computed using results from [Ji-Ka] (note that the symmetry maps $x \mapsto -x$ on $\mathrm{R} \cup \{\pm\infty\}$ and $\mu \mapsto -\mu$ on Z^2 induce a symmetry of the above short exact sequence). In fact, we have

$$\begin{array}{ccccccc}
\mathrm{Z}^2 & = K_0(\mathscr{A}_\alpha \otimes \mathscr{K}) & \to & K_0(C(\mathrm{R} \cup \{\pm\infty\}) \times_\alpha \mathrm{Z}^2) & \to & 2K_0(C(\mathrm{T}^2)) & = \mathrm{Z}^2 \oplus \mathrm{Z}^2 \\
 & \uparrow \iota & & & & \downarrow \eta & \\
\mathrm{Z}^2 \oplus \mathrm{Z}^2 = & 2K_1(C(\mathrm{T}^2)) & \leftarrow & K_1(C(\mathrm{R} \cup \{\pm\infty\}) \times_\alpha \mathrm{Z}^2) & \leftarrow & K_1(\mathscr{A}_\alpha \otimes \mathscr{K}) = & \mathrm{Z}^2
\end{array}$$

where $\eta(\mu, \nu) = \mu - \nu$ and $\iota(\mu, \nu) = \mu - \nu$ for $\mu, \nu \in \mathrm{Z}^2$, when $K_*(\mathscr{A}_\alpha \otimes \mathscr{K})$ are suitably identified with Z^2. So it is easy to see that the K-groups of $C(\mathrm{R} \cup \{\pm\infty\}) \times_\alpha \mathrm{Z}^2$ are both Z^2.

(ii) If the slope α of $\tilde{F}$ is rational, say $\alpha = p/q$ with p, q relatively prime natural numbers, and the slope of B is 0, then $\mathfrak{G} \mid X_{\bar{B}}$ is the transformation groupoid $(\mathrm{Z} \cup \{\pm\infty\}) \times_{\alpha'} \mathrm{Z}^2$ with Z^2 acting by $\alpha'(\mu)\cdot n = n - (p\mu_1 + q\mu_2)$. Under a change of basis for Z^2, we have only the first component of Z^2 acting on Z non-trivially (by translation). More precisely, we have $C^*((\mathrm{Z} \cup \{\pm\infty\}) \times_{\alpha'} \mathrm{Z}^2) \cong C^*((\mathrm{Z} \cup \{\pm\infty\}) \times_{\tau'} \mathrm{Z}^2) \cong (C(\mathrm{Z} \cup \{\pm\infty\}) \times_\tau \mathrm{Z}) \otimes C(\mathrm{T})$, where τ is the Z action by translation and $\tau'(\mu)\cdot n = n - \mu_1$ for $\mu \in \mathrm{Z}^2$. Note that the first isomorphism is implemented by the groupoid isomorphism ψ sending $(n, \mu) \in (\mathrm{Z} \cup \{\pm\infty\}) \times_{\alpha'} \mathrm{Z}^2$ to $(n, \phi(\mu)) \in (\mathrm{Z} \cup \{\pm\infty\}) \times_{\tau'} \mathrm{Z}^2$, where ϕ is the isomorphism sending $\mu \in \mathrm{Z}^2$ to $(p\mu_1 + q\mu_2,\ p'\mu_1 + q'\mu_2) \in \mathrm{Z}^2$ with $pq' - p'q = 1$, and hence $(\tau' \circ \phi)(\mu)\cdot n = n - p\mu_1 - q\mu_2 = \alpha'(\mu)\cdot n$, and that ψ induces the following commuting diagram

$$\begin{array}{ccccccc}
0 \to \mathscr{K} \otimes C(\mathrm{T}) & \to & C(\mathrm{Z} \cup \{\pm\infty\}) \times_{\alpha'} \mathrm{Z}^2 & \to & 2C^*(\mathrm{Z}^2) \cong 2C(\mathrm{T}^2) \to 0 \\
\| & & \downarrow \psi & & \downarrow \\
0 \to \mathscr{K} \otimes C(\mathrm{T}) & \to & C(\mathrm{Z} \cup \{\pm\infty\}) \times_{\alpha' \circ \phi} \mathrm{Z}^2 & \to & 2C^*(\mathrm{Z}^2) \cong 2C(\mathrm{T}^2) \to 0
\end{array}$$

where the ideals on the left have ben identified with $\mathscr{K} \otimes C(\mathrm{T})$ in a suitable way for each fixed face B, while the right vertical arrow is induced by ϕ. Note that the right arrow induces the identity map and the isomorphism ϕ on the K_0-group and the K_1-group respectively, and that the lower short exact sequence can be obtained by tensoring

$$0 \to \mathscr{K} \to C(\mathrm{Z} \cup \{\pm\infty\}) \times_\tau \mathrm{Z} \to 2C(\mathrm{T}) \to 0$$

with $C(\mathrm{T})$. Since the regular representation of $\chi_{\mathrm{Z}_<} + \chi_{\mathrm{Z}_\geq}\delta_1$ is a Fredholm operator of index one, we get by lemma 2 the 6-term exact sequence

$$\begin{array}{ccccc} \mathsf{Z} & = K_0(\mathscr{K}) & \xrightarrow{0} K_0(C(\mathsf{Z}\cup\{\pm\infty\})\times_\tau \mathsf{Z}) \to & 2K_0(C(\mathsf{T})) & = \mathsf{Z}\oplus\mathsf{Z} \\ & \uparrow \imath' & & \downarrow \eta' & \\ \mathsf{Z}\oplus\mathsf{Z} & = 2K_1(C(\mathsf{T})) & \leftarrow K_1(C(\mathsf{Z}\cup\{\pm\infty\})\times_\tau \mathsf{Z}) \leftarrow & K_1(\mathscr{K}) & = 0 \end{array}$$

in which $\eta' = 0$ and $\imath'(m,n) = m - n$. So we get $K_0(C(\mathsf{Z}\cup\{\pm\infty\})\times_\tau \mathsf{Z}) = \mathsf{Z}^2$ and $K_1(C(\mathsf{Z}\cup\{\pm\infty\})\times_\tau \mathsf{Z}) = \mathsf{Z}$. With the help of Künneth formula for K-groups [Sc], we get

$$\begin{array}{ccccc} \mathsf{Z} & = K_0(\mathscr{K}\otimes C(\mathsf{T})) & \xrightarrow{0} K_0(C(\mathsf{Z}\cup\{\pm\infty\})\times_{\alpha'} \mathsf{Z}^2) \to & 2K_0(C(\mathsf{T}^2)) & = \mathsf{Z}^2\oplus\mathsf{Z}^2 \\ & \uparrow \imath & & \downarrow \eta & \\ \mathsf{Z}^2\oplus\mathsf{Z}^2 & = 2K_1(C(\mathsf{T}^2)) & \leftarrow K_1(C(\mathsf{Z}\cup\{\pm\infty\})\times_{\alpha'} \mathsf{Z}^2) \leftarrow & K_1(\mathscr{K}\otimes C(\mathsf{T})) & = \mathsf{Z} \end{array}$$

where $\eta(\mu,\nu) = \mu_2 - \nu_2$ and $\imath(\mu,\nu) = \phi(\mu)_1 - \phi(\nu)_1 = p(\mu_1 - \nu_1) + q(\mu_2 - \nu_2)$. (Note that if $B = E$ or N, then the result is the same except that we have now only one end point and hence only one copy of $K_*(C(\mathsf{T}^2))$ in the above diagram with $(p,q) = (1,0)$ or $(0,-1)$.)

(iii) If the slope α of $\tilde{F}$ is rational and the slope of B is not 0, then, under a change of basis for Z^2 as we did in (ii), $\mathfrak{G}\,|\,X_{\bar{B}}$ is the transformation groupoid $(\mathsf{R}\cup\{\pm\infty\})\times_{\tau'}\mathsf{Z}^2$ with only the first component of Z^2 acting non-trivially on R (by translation) and so the lower short exact sequence of (3) can be obtained by tensoring

$$0 \to C(\mathsf{T})\otimes\mathscr{K} \to C(\mathsf{R}\cup\{\pm\infty\})\times_\tau \mathsf{Z} \to 2C(\mathsf{T}) \to 0$$

with $C(\mathsf{T})$. By the results of [Pi-Vo, B], it is easy to show that the 6-term exact sequence corresponding to the above is

$$\begin{array}{ccccc} \mathsf{Z} & = K_0(C(\mathsf{T})\otimes\mathscr{K}) & \xrightarrow{0} K_0(C(\mathsf{R}\cup\{\pm\infty\})\times_\tau \mathsf{Z}) \to & 2K_0(C(\mathsf{T})) & = \mathsf{Z}\oplus\mathsf{Z} \\ & \uparrow \imath' & & \downarrow \eta' & \\ \mathsf{Z}\oplus\mathsf{Z} & = 2K_1(C(\mathsf{T})) & \leftarrow K_1(C(\mathsf{R}\cup\{\pm\infty\})\times_\tau \mathsf{Z}) \leftarrow & K_1(C(\mathsf{T})\otimes\mathscr{K}) & = \mathsf{Z} \end{array}$$

in which $\imath'(m,n) = m - n = \eta'(m,n)$, $K_0(C(\mathsf{R}\cup\{\pm\infty\})\times_\tau \mathsf{Z}) = \mathsf{Z}$ and $K_1(C(\mathsf{R}\cup\{\pm\infty\})\times_\tau \mathsf{Z}) = \mathsf{Z}$. Again, with the help of Künneth formula for K-groups, we get

$$\begin{array}{ccccc} \mathsf{Z}\oplus\mathsf{Z} & = K_0(C(\mathsf{T}^2)\otimes\mathscr{K}) & \xrightarrow{0} K_0(C(\mathsf{R}\cup\{\pm\infty\})\times_{\alpha'} \mathsf{Z}^2) \to & 2K_0(C(\mathsf{T}^2)) & = \mathsf{Z}^2\oplus\mathsf{Z}^2 \\ & \uparrow \imath & & \downarrow \eta & \\ \mathsf{Z}^2\oplus\mathsf{Z}^2 & = 2K_1(C(\mathsf{T}^2)) & \leftarrow K_1(C(\mathsf{R}\cup\{\pm\infty\})\times_{\alpha'} \mathsf{Z}^2) \leftarrow & K_1(C(\mathsf{T}^2)\otimes\mathscr{K}) & = \mathsf{Z}\oplus\mathsf{Z} \end{array}$$

where $\eta(\mu,\nu) = \mu - \nu$ and $\imath(\mu,\nu) = \phi(\mu) - \phi(\nu)$ with ϕ as defined in (ii).

Piecing together the data obtained in the above discussion, we can describe the 6-term exact sequence corresponding to (1) in the following diagram

$$\begin{array}{ccccc}
\Sigma^{\oplus}K_0(C^*(\mathfrak{G}\mid X_B)) & \to & K_0(C^*(\mathfrak{G}\mid \partial X)) & \to & \Sigma^{\oplus}K_0(C(\{p\}\times \mathrm{T}^2)) \\
\uparrow \Sigma^{\oplus}i_p & & & & \downarrow \Sigma^{\oplus}\eta_p \\
\Sigma^{\oplus}K_1(C(\{p\}\times \mathrm{T}^2)) & \leftarrow & K_1(C^*(\mathfrak{G}\mid \partial X)) & \leftarrow & \Sigma^{\oplus}K_1(C^*(\mathfrak{G}\mid X_B))
\end{array}$$

where, for $p \in I$, the map η_p takes value in the sum of (two) $K_*(C^*(\mathfrak{G}\mid X_B))$'s with B and I having a common end point and is determined according to the slope of B and the slope α of the corresponding $\tilde{F}$ as we have seen in the above three cases. Now since $K_*(C(\{p\}\times \mathrm{T}^2))$ and $K_*(C^*(\mathfrak{G}\mid X_B))$ are free abelian, we have

$$K_0(C^*(\mathfrak{G}\mid \partial X)) \cong \ker(\Sigma^{\oplus}\eta_p) \oplus \operatorname{coker}(\Sigma^{\oplus}\iota_p)$$

and

$$K_1(C^*(\mathfrak{G}\mid \partial X)) \cong \ker(\Sigma^{\oplus}\iota_p) \oplus \operatorname{coker}(\Sigma^{\oplus}\eta_p)$$

where the right hand sides can be computed once the data about the boundary geometry of D are given.

We summarize in the following.

PROPOSITION. *If D is a Reinhardt domain in* C^2 *considered in* [Sh1] *with* $n_\infty = 2$ *and with only finitely many connected components of* $\partial_0 C$, *then*

$$K_0(\mathscr{I}(D)) \cong \ker(\Sigma^{\oplus}\eta_p) \oplus \operatorname{coker}(\Sigma^{\oplus}\iota_p)$$

and

$$K_1(\mathscr{I}(D)) \cong \ker(\Sigma^{\oplus}\iota_p) \oplus \operatorname{coker}(\Sigma^{\oplus}\eta_p) \ominus \mathrm{Z}.$$

Note that $\Sigma^{\oplus}\iota_p$ and $\Sigma^{\oplus}\eta_p$ can be represented by integral matrices. In the next section, we compute their kernels and cokernels, and get a more explicit relation *between the* K-groups and the boundary geometry.

Section 3.

In this section, we shall compute the kernel and cokernel of $\Sigma^{\oplus}\iota_p$ and $\Sigma^{\oplus}\eta_p$ in terms of the boundary geometry of D.

Now let $I_1, I_2, \ldots, I_n$ and $E = B_0, B_1, \ldots, B_n = N$ be the enumerations of I's and B's used in section 2 such that B_i has common end points with I_i and I_{i+1}. So the homomorphisms $\Sigma^{\oplus}\eta_p$ and $\Sigma^{\oplus}\iota_p$ in section 2 can be written as $\Sigma^{\oplus}\eta_i$ and $\Sigma^{\oplus}\iota_i$ respectively, where i ranges from 1 to n. We group B's into the following three categories: (i) the slope $\alpha(\tilde{F})$ of the corresponding face $\tilde{F}$ is not rational, (ii) $\alpha(\tilde{F})$ is rational and $\alpha(B)$ is zero (including $B = E$ and $B = N$), (iii) $\alpha(\tilde{F})$ is rational and $\alpha(B)$ is not zero. Since $K_*(C^*(\mathfrak{G}\mid X_B))$ is isomorphic to Z if B belongs to the category (ii) and to Z^2 otherwise, we get $\Sigma^{\oplus}\eta_i$ and $\Sigma^{\oplus}\iota_i$ homomorphisms from Z^{2n} to $\mathrm{Z}^{2n-\#}$, where $\#$ is the number of B's in category (ii) not equal to E or N.

We first prove that $\ker(\Sigma^{\oplus}\eta_i) \cong \mathbf{Z}^{\#+1}$ and $\operatorname{coker}(\Sigma^{\oplus}\eta_i) \cong \mathbf{Z}$. According to the result of section 2, $\Sigma^{\oplus}\eta_i$ has a matrix representation of the form

$$\begin{bmatrix} \beta_{01} & 0 & 0 & \cdot\ \cdot & & 0 \\ \beta_{11} & \beta_{12} & 0 & \cdot\ \cdot & & 0 \\ 0 & \beta_{22} & \beta_{23} & \cdot\ \cdot & & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot\ \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \beta_{(n-1)(n-1)} & \beta_{(n-1)n} \\ 0 & 0 & 0 & \cdot & 0 & \beta_{nn} \end{bmatrix}$$

where $\beta_{01} = (0 \quad 1)$, $\beta_{nn} = (0 \quad -1)$, $\beta_{ii} = (0 \quad -1)$ and $\beta_{i(i+1)} = (0 \quad 1)$ if B_i is in category (ii), and $\beta_{ii} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\beta_{i(i+1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ if B_i is in categories (i) and (iii). We rearrange the columns and rows of this integral matrix in the following way. Each B_i determines a minor $(0 \ \ldots \ 0 \ \ \beta_{ii} \ \ \beta_{i(i+1)} \ \ 0 \ \ldots \ 0)$ which consists of either one or two rows denoted by ρ_{i1} and ρ_{i2} where ρ_{i1} (not ρ_{i2}) is void when there is only one row in the minor. (So ρ_{01} and ρ_{n1} are always void.) We permute the rows so that the new rows listed from top to bottom are $\rho_{02}, \rho_{12}, \rho_{22}, \ldots, \rho_{n2}, \rho_{01}, \rho_{11}, \rho_{21}, \ldots, \rho_{n1}$. Now let $\gamma_1, \gamma_2, \ldots, \gamma_{2n}$ be the columns (not the block columns) in the new matrix listed from left to right. We permute the columns so that the new columns from left to right are $\gamma_2, \gamma_4, \ldots, \gamma_{2n}, \gamma_1, \gamma_3, \ldots, \gamma_{2n-1}$. After the above two permutations, we get a matrix of the form

$$\begin{bmatrix} 1 & 0 & 0 & \cdot & \cdot & 0 & 0 & \cdot & 0 \\ -1 & 1 & 0 & \cdot & \cdot & 0 & 0 & \cdot & 0 \\ 0 & -1 & 1 & \cdot & \cdot & 0 & 0 & \cdot & 0 \\ 0 & 0 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & -1 & 1 & 0 & \cdot & 0 \\ 0 & 0 & 0 & \cdot & 0 & -1 & 0 & \cdot & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & & & \\ & & & & & & & * & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ 0 & 0 & 0 & \cdot & 0 & 0 & & & \end{bmatrix}$$

where $*$ is a $n - \# - 1$ by n matrix with rows of the form $(0 \ \ . \ \ . \ \ 0 \ \ -1 \ \ 1 \ \ 0 \ \ . \ \ . \ \ 0)$ such that the leading -1's in all these rows appear in distinct columns (in fact, the column number of such leading -1 increases as its row number increases). Thus by some elementary row reductions on the first $n + 1$ rows and some elementary column reductions on last n columns, this matrix can be reduced to a (similar) matrix with exactly $n + (n - \# - 1) = 2n - \# - 1$ non-zero entries (in fact, -1) sitting in distinct rows and distinct columns. Thus the kernel and cokernel of this new matrix (and hence the original matrix) are free abelian groups. Furthermore. the rank and the corank of this new matrix (and hence the original matrix) are $2n - \# - 1$ and $(2n - \#) - (2n - \# - 1) = 1$. So we get $\ker(\Sigma^{\oplus}\eta_i) \cong \mathbf{Z}^{\# + 1}$ and $\operatorname{coker}(\Sigma^{\oplus}\eta_i) \cong \mathbf{Z}$ as we claimed.

Next we prove that $\ker(\Sigma^{\oplus}\iota_i) \cong \mathbf{Z}^{\#}$ and $\operatorname{coker}(\Sigma^{\oplus}i_i) = 0$. According to the result of section 2, $\Sigma^{\oplus}\iota_i$ has a matrix representation of the form

$$\begin{bmatrix} \beta_{01} & 0 & 0 & \cdot\ \cdot & & 0 \\ \beta_{11} & \beta_{12} & 0 & \cdot\ \cdot & & 0 \\ 0 & \beta_{22} & \beta_{23} & \cdot\ \cdot & & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot\ \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \beta_{(n-1)(n-1)} & \beta_{(n-1)n} \\ 0 & 0 & 0 & \cdot & 0 & \beta_{nn} \end{bmatrix}$$

where $\beta_{01} = (1 \quad 0)$, $\beta_{nn} = (0 \quad 1)$, $\beta_{ii} = (p_i \quad q_i)$ and $\beta_{i(i+1)} = -\beta_{ii}$ for some relatively prime integers p_i and q_i if B_i is in category (ii), and $\beta_{ii} = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ and $\beta_{i(i+1)} = -\beta_{ii}$ for some $\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ in $SL(2, \mathbf{Z})$ if B_i is in categories (i) and (iii). Now let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be the block columns in this matrix listed from left to right. By elementary row reductions, we can replace each γ_i by $\gamma_1 + \gamma_2 + \ldots + \gamma_i$ and get an equivalent matrix

$$\begin{bmatrix} \beta_{01} & \beta_{01} & \beta_{01} & \cdot & \beta_{01} & 1 & 0 \\ \beta_{11} & 0 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & \beta_{22} & 0 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \beta_{(n-1)(n-1)} & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 1 \end{bmatrix}$$

which can be further reduced, by elementary column reductions, to

$$\begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 1 & 0 \\ \beta_{11} & 0 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & \beta_{22} & 0 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \beta_{(n-1)(n-1)} & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 1 \end{bmatrix}$$

Since β_{ii}'s are either elements in $\mathbf{Z}^2$ with relatively prime components or elements in $\mathrm{SL}(2, \mathbf{Z})$, it is easy to see that the kernel and cokernel of the last matrix (and hence the original matrix) are free abelian groups, and that the rank and the corank of this new matrix (and hence the original matrix) are $2n - \#$ and 0. So we get $\ker(\Sigma^{\oplus}\iota_i) \cong \mathbf{Z}^{\#}$ and $\operatorname{coker}(\Sigma^{\oplus}\iota_i) \cong 0$ as we claimed.

THEOREM. *If D is a Reinhardt domain in $\mathbf{C}^2$ considered in* [Sh1] *with $n_\infty = 2$ and with only finitely many connected components of $\partial_0 C$, then*

$$K_0(\mathscr{T}(D)) \cong \mathbf{Z}^{\# + 1}$$

and

$$K_1(\mathscr{T}(D)) \cong \mathbf{Z}^{\#},$$

where $\#$ is (*as defined above*) *the number of lower horizontal faces B* (*not including E and N*) *of the polygons $\tilde{P}$* (*constructed in* [Sh1]) *corresponding to linear faces $\tilde{F}$ of $\tilde{C}$, the logarithmic domain of the pseudoconvex hull $\tilde{D}$ of D, with rational slopes.*

PROOF. From the proposition in section 2 and the above computation, we have

$$K_0(\mathscr{T}(D)) \cong \ker(\Sigma^{\oplus}\eta_i) \oplus \operatorname{coker}(\Sigma^{\oplus}\iota_i) \cong \mathbf{Z}^{\# + 1} \oplus 0 = \mathbf{Z}^{\# + 1}$$

and

$$K_1(\mathscr{T}(D)) \cong \ker(\Sigma^{\oplus}\iota_i) \oplus \operatorname{coker}(\Sigma^{\oplus}\eta_i) \ominus \mathbf{Z} \cong \mathbf{Z}^{\#} \oplus \mathbf{Z} \ominus \mathbf{Z} \cong \mathbf{Z}^{\#}.$$

When D is already pseudoconvex (and hence $\tilde{D} = D$) [Sa-Sh-U], each face $\tilde{F} (= F)$ of $\tilde{C} (= C)$ has corresponding polygon $\tilde{P}$ degenerated into a horizontal segment B and so we get the following corollary.

COROLLARY. *If D is a pseudoconvex Reinhardt domain in $\mathbf{C}^2$ with $n_\infty = 2$ and with only finitely many connected components of $\partial_0 C$, then*

$$K_0(\mathscr{T}(D)) \cong \mathbf{Z}^{\# + 1}$$

and

$$K_1(\mathscr{T}(D)) \cong \mathbf{Z}^{\#},$$

where $\#$ is the number of non-horizontal and non-vertical linear faces F (*of the logarithmic domain C*) *with rational slopes.*

It was conjectured by some people that the K_1-group of $\mathscr{T}(D)$ for a pseudoconvex D is always trivial. But this corollary gives a negative answer.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KANSAS
LAWRENCE, KS 66045
U.S.A.