C*-ALGEBRAS ASSOCIATED WITH CELLULAR AUTOMATA

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Abstract.
We construct C*-algebras from linear cellular automata by regarding them as topological dynamical systems. We prove that some of the resulting C*-algebras become Cuntz’s algebra $\mathcal{O}_d$. We show that the limit sets of configurations of cellular automaton evolutions, one of whose examples is the Sierpinski gasket, can be obtained by using the canonical endomorphism $\Phi_d$ of $\mathcal{O}_d$. We also study some automorphisms on these C*-algebras induced by basic operations on cellular automata.

1. Introduction.
In this paper, we introduce a method to investigate cellular automata from functional analytic point of view. We regard cellular automata as topological dynamical systems on a lattice (cellular space) and construct algebras of operators, called C*-algebras, on a Hilbert space based on the lattice. Let us consider a $d$-dimensional $k$-state cellular automaton. Let $\varphi$ be its cellular automaton rule. The cellular space $\prod_{Z_k} \mathbb{Z}^d$ is a compact space in the product topology and $\varphi$ is a continuous map on $\mathcal{R}_d^d$. We identify the cellular automaton with the topological dynamical system $(\mathcal{R}_d^d, \varphi)$. Take a $\varphi$-invariant probability measure $\mu$ on $\mathcal{R}_d^d$ and consider the Hilbert space $L^2(\mathcal{R}_d^d, \mu)$ of all square integrable functions on $\mathcal{R}_d^d$. We represent the commutative C*-algebra $C(\mathcal{R}_d^d)$ of all complex valued continuous functions on $\mathcal{R}_d^d$ on $L^2(\mathcal{R}_d^d, \mu)$ by multiplication. The rule $\varphi$ induces a bounded linear operator $V_\varphi$ on $L^2(\mathcal{R}_d^d, \mu)$. We define the C*-algebra associated with the cellular automaton $\varphi$, as the C*-algebra generated by $C(\mathcal{R}_d^d)$ and $V_\varphi$. We denote it by $C_\varphi$. We notice that the isomorphism class of the C*-algebra $C_\varphi$ of course depends on the choice of the $\varphi$-invariant measure $\mu$ on $\mathcal{R}_d^d$. But as long as the Radon-Nikodým derivative with another $\varphi$-invariant measure is invertible, the resulting C*-algebras are isomorphic.

We will treat some 1-dimension 2-state 3-neighborhood linear cellular automata, numbered as 60, 90, 150 by S. Wolfram in [Wo1].

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In Section 3, we first show the C*-algebra $C_{90}$ associated with the rule 90 becomes a simple C*-algebra called the Cuntz algebra $\mathcal{O}_4$ of order 4. It is one of a series of simple C*-algebras which many operator algebraists have been interested in, cf. [Ar], [Cu1], [Cu2], [Cuk], [ETW], [Ev], [Jo], [OP]. We show that the C*-algebra $C_{150}$ constructed by the rule 150 is also isomorphic to $\mathcal{O}_4$ because the map $\varphi_{150}$ associated with the rule 150 is topologically conjugate to $\varphi_{90}$ as continuous map.

In Section 4, we show that the Sierpinski gasket as a limit set of a cellular automaton evolution can be seen in the algebraic structure of the C*-algebra $C_{90}$. In fact, in $C_{90}$, the cellular automaton rule corresponds to the canonical endomorphism $\Phi_4$ of $\mathcal{O}_4$, in the following sense: If $U_0$ is the self-adjoint unitary $S_2S_1^* + S_1S_2^* + S_4S_3^* + S_3S_4^*$, and $l^{90}(k)$ is the number of cells with value 1 after time $k$ if one starts with a state where only one cell has the value 1, using the cellular automaton rule 90, then we construct a faithful state $\tau^{90}$ on $\mathcal{O}_4$ and a number $c$ such that

$$\tau^{90}(\Phi^k_4(U_0)) = c^{l^{90}(k)}.$$ 

A similar result is established for the cellular automaton rule 150. In [Wi1], [Wi2], S. Willson has showed that the Hausdorff dimension of the limit set is equal to its growth rate dimension (cf. [Ta1]). In our language, Willson’s result is thus that

$$\lim_{m \to \infty} \frac{\log \sum_{k=0}^{m} \log_{\tau^{90}(U_0)} \tau^{90}(\Phi^k_4(U_0))}{\log m} = \begin{cases} \log_2 3 & (\ast = 90) \\ \log_2 (1 + \sqrt{5}) & (\ast = 150) \end{cases}$$

where $\tau^\ast$, $\ast = 90, 150$ are the faithful states on $\mathcal{O}_4$. The values $\log_2 3$ and $\log_2 (1 + \sqrt{5})$ are the fractal dimensions of the limit sets corresponding to the rule 90 and the rule 150 respectively.

In Section 5, we study automorphisms on cellular automaton C*-algebras induced by two basic operations on cellular space $\prod Z$. These operations are shift ($\{a_n\} \to \{a_{n+1}\}$) and conjugation ($\{a_n\} \to \{a_n + 1\}$). The C*-algebras $C_{90}$ and $C_{150}$ are both isomorphic to $\mathcal{O}_4$. However, the automorphism induced by the conjugation on $C_{90}$ is inner while the automorphism induced by the conjugation on $C_{150}$ is outer as an automorphism on $\mathcal{O}_4$. We further show that both automorphisms on $\mathcal{O}_4 (\cong C_{90} \cong C_{150})$ induced by the shift are outer.

In Section 6, we generalize our construction for 2-state cellular automata to general $k$-state cellular automata. As a consequence, we show that a C*-algebra associated with a $k$-state cellular automaton corresponding to $\varphi_{90}$ is isomorphic to the Cuntz algebra $\mathcal{O}_{k^2}$ of order $k^2$. 
In Section 7, we finally study a $C^*$-algebra associated with a 2-state cellular automaton rule numbered as 60. We show that the $C^*$-algebra becomes an inductive limit of a sequence of the Cuntz algebra $\mathcal{O}_2$ of order 2. By a recent theorem of Rørdam [Ro], the algebra itself is then isomorphic to $\mathcal{O}_2$.

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2. Preliminaries on cellular automata.

Let $\mathbb{Z}^d$ be a $d$-dimensional lattice $\mathbb{Z} \times \ldots \times \mathbb{Z}$ ($d$-times product of the integers $\mathbb{Z}$). We fix a natural number $k$. The state of the cell on each lattice point $i \in \mathbb{Z}^d$ is specified by a number $a_i \in \{0, 1, \ldots, k - 1\} = \mathbb{Z}_k$. A cellular automaton rule is a map to define the state of the next generation of each cell which depends on only a neighborhood of the lattice point $i \in \mathbb{Z}^d$. Namely, the rule is given by a $\mathbb{Z}_k$-valued map $\psi$ defined on $\mathbb{Z}_k^n$ by specifying that the state at site $i \in \mathbb{Z}^d$ for the next generation should be $\psi(a_{i-r_1}, \ldots, a_{i+r_n}) \in \mathbb{Z}_k$ where $\{a_i\}_{i \in \mathbb{Z}^d}$ is the state of the previous generation, and $r_1, \ldots, r_n$ are fixed distinct elements in $\mathbb{Z}^d$. We call such a map $\psi$ a transition function or simply a rule. Such a system is called a $d$-dimension $k$-state $n$-neighborhood cellular automaton.

For instance, the Pascal’s triangle of modulo 2 is realized as a 1-dimension 2-state 3-neighborhood cellular automaton as in the following way. Take the both side $i - 1, i + 1$ and itself $i$ for a point $i \in \mathbb{Z}$ as a neighborhood of $i$. The transition function $\psi$ is defined by $\psi(a_{i-1}, a_i, a_{i+1}) = a_{i-1} + a_{i+1} \mod 2$. If we take an initial cell configuration $\{a_i\}$ as $a_i = 1$ ($i = 0$), $a_i = 0$ ($i \neq 0$), this cellular automaton evolution is related to the Pascal’s triangle of modulo 2.

Following Wolfram [Wo1], we number all 1-dimension 2-state 3-neighborhood cellular automata as in the following way. Let $\psi$ be a transition function, which is defined on $\mathbb{Z}_2^3$. Hence there are eight possibilities of the state of the neighborhood so that we have $2^8 = 256$ possible transition functions. The state of the cell of the next generation of $a_i$ is determined by the map $\psi$, which is written by

\[
\begin{align*}
\psi(0, 0, 0) &= \psi^0, & \psi(0, 0, 1) &= \psi^1, & \psi(0, 1, 0) &= \psi^2, & \psi(0, 1, 1) &= \psi^3, \\
\psi(1, 0, 0) &= \psi^4, & \psi(1, 0, 1) &= \psi^5, & \psi(1, 1, 0) &= \psi^6, & \psi(1, 1, 1) &= \psi^7.
\end{align*}
\]

We define the number of $\psi$ by $\sum_{i=0}^{\infty} \psi^i 2^i$. 

For instance, in the case of the preceding cellular automaton of the Pascal's triangle of modulo 2, one has number 90.

A cellular automaton is said to be symmetric if its transition function $\psi$ is symmetric, that is, $\psi^1 = \psi^4, \psi^3 = \psi^6$. It is natural to restrict cellular automata to ones with $\psi^0 = 0$. If a symmetric cellular automaton rule satisfies the condition $\psi^0 = 0$, it is said to be legal. Thus, in 1-dimension 2-state 3-neighborhood cellular automata, these restrictions leave 32 possible legal cellular automata. In general, the limit sets of linear cellular automata are fractals (cf. [Ta2]). The invariance of the Haar measure under a transition function on the cellular space $\mathcal{R}_2 = \prod_z \mathbb{Z}_2$ is an important property. In particular, the Haar measure is invariant under the transition functions 60, 90, 150 which will be considered in the sequel.

3. 1-dimension 2-state cellular automaton $C^*$-algebras.

We first treat a legal cellular automaton in the class of 1-dimension 2-state 3-neighborhood cellular automata. Hence the cellular space $\mathcal{R}_2^1 = \prod_z \{0, 1\}$ is the Cantor set $\mathcal{R}_2$. Let $\varphi_n$ be a cellular automaton rule indexed as the number $n$ $(0 \leq n \leq 256)$. We denote by $C_n$ the associated $C^*$-algebra $C_{\varphi_n}$.

The cellular automaton associated with the Pascal's triangle of mod 2, that is, $\varphi_{90}$, can be considered as the continuous map on $\mathcal{R}_2$ defined by

$$\varphi_{90}(\{a_i\}) = \{a_{i-1} + a_{i+1}\}, \quad \{a_i\} \in \mathcal{R}_2.$$ 

Let us study algebraic structure of the $C^*$-algebra $C_{90}$. It is easily seen that if $\{b_i\} \in \mathcal{R}_2$ is given, one may define $\{a_i\}$ with $\varphi_{90}(\{a_i\}) = \{b_i\}$ by choosing $a_0, a_1$ arbitrary, and then $a_2, a_3, \ldots$, and $a_{-1}, a_{-2}, \ldots$ by induction. Thus, there are 4 distinct cross sections $s_{ij}, i, j = 0, 1,$ of $\varphi_{90}$ (i.e. $\varphi_{90} \circ s_{ij} = id$) satisfying the conditions

$$P_0(s_{ij}(\{a_n\})) = i, \quad P_1(s_{ij}(\{a_n\})) = j, \quad i, j = 0, 1, \quad \{a_n\} \in \mathcal{R}_2$$

where each $P_k$ is the function on $\mathcal{R}_2$ defined by $P_k(\{a_n\}) = a_k$. We conclude:

**Lemma 3.1.** The continuous map $\varphi_{90}$ is surjective and 4-to-1.

Let $\mu_{1/2}$ be the measure on $\{0, 1\}$ defined by $\mu_{1/2}([0]) = \mu_{1/2}(\{1\}) = 1/2$. The infinite product measure $\prod_z \mu_{1/2}$ on $\mathcal{R}_2$ is called the Haar measure and is denoted by $\mu$. The lemma below is easily seen by a direct computation or as a special case of [SR; 2.4. Theorem].

**Lemma 3.2.** The measure $\mu$ is $\varphi_{90}$-invariant.

Lemma 3.2 also follows from the next easily proved lemma.
LEMMA 3.3. The Radon-Nikodým derivative \((d\mu \circ s_{ij})/d\mu\) is \(1/4\) if \(i, j = 0, 1\).

We denote by \(\mathcal{S}\) the Hilbert space \(L^2(\mathcal{S}_2, \mu)\) of all complex valued square integrable functions. We define the bounded linear operator \(V_{90}\) coming from the map \(\varphi_{90}\) on \(\mathcal{S}\) by

\[
(V_{90}\xi)(\{a_n\}) = \xi(\varphi_{90}(\{a_n\})), \quad \xi \in \mathcal{S}, \quad \{a_n\} \in \mathcal{R}_2.
\]

Then one can show the following lemma by routine computation.

LEMMA 3.4.

(i) \((V_{90}\xi\xi)(\{a_n\}) = \frac{1}{4} \sum_{i,j=0,1} \xi(s_{ij}(\{a_n\})), \quad \xi \in \mathcal{S}, \quad \{a_n\} \in \mathcal{S}_2.

(ii) \(V_{90}^* V_{90} = 1\).

We now discuss the range of the projection \(V_{90} V_{90}^*\). Let \(h_e\) and \(h_o\) be the homeomorphisms on \(\mathcal{R}_2\) exchanging the state of a cell located in even lattice points and in odd ones, respectively. That is,

\[
h_e: \begin{cases} a_n \to a_n + 1 \quad (n: \text{even}) \\
a_n \to a_n \quad (n: \text{odd})
\end{cases}, \quad h_o: \begin{cases} a_n \to a_n \quad (n: \text{even}) \\
a_n \to a_n + 1 \quad (n: \text{odd})
\end{cases}.
\]

Let \(W_e\), \(W_o\) be the unitaries on \(\mathcal{S}\) defined by

\[
(W_e^*\xi)(\{a_n\}) = \xi(h_e(\{a_n\})), \quad * = e, o, \quad \xi \in \mathcal{S}, \quad \{a_n\} \in \mathcal{R}_2.
\]

The operator \(W_e^*\) is a self-adjoint unitary. Hence, the decomposition

\[
W_e = (1 + W_e)/2 - (1 - W_e)/2
\]

is the spectral decomposition of \(W_e\). Put \(Q_* = (1 + W_e)/2, * = e, o\).

LEMMA 3.5. \(V_{90} V_{90}^* = Q_e Q_o\).

PROOF. For a vector \(\xi \in \mathcal{S}\) and an element \(\{a_n\} \in \mathcal{S}_2\), we have

\[
(V_{90} V_{90}^*\xi)(\{a_n\}) = \frac{1}{4} \sum_{i,j=0,1} \xi(s_{ij} \circ \varphi_{90}(\{a_n\}))
\]

and

\[
(Q_e Q_o \xi)(\{a_n\}) = \frac{1}{4} (1 + W_e + W_o + W_e W_o) \xi(\{a_n\}) = \frac{1}{4} \sum_{i,j=0,1} \xi(h_e^i \circ h_o^j(\{a_n\})).
\]

One easily shows that the set of the four elements \(\{s_{ij} \circ \varphi_{90}(\{a_n\})\}_{i,j=0,1}\) coincides with that of the four elements \(\{h_e^i \circ h_o^j(\{a_n\})\}_{i,j=0,1}\). Hence we get \(V_{90} V_{90}^* = Q_e Q_o\).

For each \(n \in \mathbb{Z}\), the continuous function \(P_n \in C(\mathcal{S}_2)\) defined by \(P_n(\{a_i\}) = a_n\)
satisfies the condition $P_n = P_n^2 = P_n^*$. The sequence of these projections $\{P_n\}$ gives all the information of the configuration of the states of cells. Set $U_n = 1 - 2P_n$. As $\{P_n\}$ generate the $C^*$-algebra $C(S_2)$, these unitaries $\{U_n\}$ generate it.

The proof of the next lemma is left to the reader.

**Lemma 3.6.**
(i) For an even integer $n$, $P_nW_e = W_e(1 - P_n)$, $P_nW_o = W_oP_n$.
(ii) For an odd integer $n$, $P_nW_e = W_o(1 - P_n)$, $P_nW_o = W_eP_n$.

**Corollary 3.7.**
(i) For an even integer $n$, $U_nQ_e = (1 - Q_e)U_n$, $U_nQ_o = Q_oU_n$.
(ii) For an odd integer $n$, $U_nQ_o = (1 - Q_o)U_n$, $U_nQ_e = Q_eU_n$.

Now put

(3.1) $S_1 = V_{90}$, $S_2 = U_0V_{90}$, $S_3 = U_1V_{90}$, $S_4 = U_0U_1V_{90}$.

**Proposition 3.8.** Keep the above notations. We have the following operator relations of a Cuntz algebra (cf. [Cu1])

(3.2) $S_i^*S_i = 1$ $(i = 1, 2, 3, 4)$, $\sum_{i=1}^{4} S_iS_i^* = 1$.

**Proof.** Following the direct sum decomposition of the Hilbert space $S$: $Q_eQ_o + (1 - Q_e)Q_o + Q_e(1 - Q_o) + (1 - Q_e)(1 - Q_o) = 1$.

one obtains the relation $\sum_{i=1}^{4} S_iS_i^* = 1$ by Corollary 3.7.

Let $C^*(S_i, 1 \leq i \leq 4)$ be the $C^*$-algebra generated by $S_i, i = 1, 2, 3, 4$. Since the generators $S_i, i = 1, 2, 3, 4$ satisfy the relation (3.2), we know, by [Cu1], that $C^*(S_i, 1 \leq i \leq 4)$ is uniquely determined up to isomorphism and is simple. It is denoted by $C_4$.

**Lemma 3.9.** Both unitaries $U_0$ and $U_1$ belong to the $C^*$-algebra $C^*(S_i, 1 \leq i \leq 4)$.

**Proof.** By $U_0^2 = 1$, it follows that $U_0S_1 = S_2$, $U_0S_2 = S_1$, $U_0S_3 = S_4$, $U_0S_4 = S_3$.

Hence from the identity $\sum_{i=1}^{4} S_iS_i^* = 1$, we have

$U_0 = S_2S_1^* + S_1S_2^* + S_4S_3^* + S_3S_4^*$. 
Similarly one sees
\[ U_1 = S_3 S_1^* + S_4 S_2^* + S_1 S_3^* + S_2 S_4^*. \]

Therefore one obtains the following:

**Proposition 3.10.** The C*-algebra \( C^*(U_0, U_1, V_{90}) \) generated by the three operators \( U_0, U_1, V_{90} \) coincides with \( C^*(S_i, 1 \leq i \leq 4) \), that is, the Cuntz algebra \( \mathcal{O}_4 \).

The C*-algebra \( C^*(U_0, U_1, V_{90}) \) is a subalgebra of \( C_{90} \), but we will see that they actually coincide by the further discussion. The rule \( \varphi_{90} \) satisfies the condition
\[ \varphi_{90}(\{a_n\}) = \{a_{n-1} + a_{n+1}\} \pmod{2}. \]

As \( (a_{n-1} - a_{n+1})^2 = a_{n-1} + a_{n+1} \pmod{2} \), the next lemma and the corollary are immediate.

**Lemma 3.11.** \( V_{90} P_n = (P_{n-1} - P_{n+1})^2 V_{90}, \quad n \in \mathbb{Z}. \)

**Corollary 3.12.**
\[ (3.3) \quad V_{90} U_n = U_{n-1} U_{n+1} V_{90}, \quad n \in \mathbb{Z}. \]

Hence we obtain

**Lemma 3.13.** For every \( n \in \mathbb{Z} \), the unitary \( U_n \) belongs to the C*-algebra \( C^*(U_0, U_1, V_{90}) \).

**Proof.** By induction, it suffices to show that for an arbitrary but fixed integer \( k \), both operators \( U_{k+2} \) and \( U_{k-1} \) belong to the C*-algebra \( C^*(U_k, U_{k+1}, V_{90}) \). By using a similar discussion to the previous one, we can show the identity below from Corollary 3.7:
\[ 1 = V_{90} V_{90}^* + U_k V_{90} V_{90}^* U_k + U_{k+1} V_{90} V_{90}^* U_{k+1} + U_k U_{k+1} V_{90} V_{90}^* U_{k+1} U_k. \]

Hence, one has, by (3.3):
\[ (3.4) \quad U_{k+2} = U_k V_{90} U_{k+1} V_{90}^* + V_{90} U_{k+1} V_{90}^* U_k + U_k U_{k+1} V_{90} U_{k+1} V_{90}^* U_{k+1} U_k + U_{k+1} V_{90} U_{k+1} V_{90}^* U_{k+1} U_k. \]

This implies \( U_{k+2} \) belongs to \( C^*(U_k, U_{k+1}, V_{90}) \). Similarly, we see that \( U_{k-1} \) does to it.

Consequently, we arrive at the theorem below.

**Theorem 3.14.** The C*-algebra \( C_{90} \) (= \( C^*(C(S_2), V_{90}) \)) associated to the cellular automaton \( \varphi_{90} \) is isomorphic to the Cuntz algebra \( \mathcal{O}_4 \) (= \( C^*(S_i, 1 \leq i \leq 4) \)) under the following correspondence:
\[ S_1 = V_{90}, \quad S_2 = U_0 V_{90}, \quad S_3 = U_1 V_{90}, \quad S_4 = U_0 U_1 V_{90}. \]

**Proof.** This is immediate from Proposition 3.10 and Lemma 3.13.

**Remark 3.15.** Let \( S_{ij}, i,j = 0,1 \) be the bounded linear operators on \( \mathfrak{H} \) defined by

\[ (S_{ij}\xi)(\{a_n\}) = \xi(s_{ij}(\{a_n\})), \quad \xi \in \mathfrak{H}, \quad \{a_n\} \in \mathfrak{K}_2 \]

where \( s_{ij}, i,j = 0,1 \) are the four sections for \( \varphi_{90} \) cited before. We then see the following relations:

\[ S_{00} + S_{01} + S_{10} + S_{11} = 4S_1^*, \quad S_{00} + S_{01} - S_{10} - S_{11} = 4S_2^*, \]
\[ S_{00} - S_{01} + S_{10} - S_{11} = 4S_3^*, \quad S_{00} - S_{01} - S_{10} + S_{11} = 4S_4^*, \]

and we have

\[ S_{ij}S_{ij}^* = 4, \quad \sum_{i,j=0,1} S_{ij}^*S_{ij} = 4. \]

Namely, the four operators \( \frac{1}{2}S_{ij}^*, i,j = 0,1 \) generate the C*-algebra \( C_{90} \), and they satisfy the Cuntz relations for \( \mathcal{O}_4 \). Hence we have another proof of the result that \( C_{90} \) is isomorphic to \( \mathcal{O}_4 \).

There is another interesting legal cellular automaton rule numbered as 150, which is defined by

\[ \varphi_{150}(\{a_n\}) = \{a_{n-1} + a_n + a_{n+1}\} \pmod{2}. \]

Corresponding to the relation (3.3), we have

\[ V_{150}U_n = U_{n-1}U_nU_{n+1}V_{150}, \quad n \in \mathbb{Z}. \]

By a similar discussion to the previous one or the argument below, one has

**Proposition 3.16.** The C*-algebra \( C_{150} (= C^*(\mathfrak{K}_2), V_{150}) \) associated with the cellular automaton \( \varphi_{150} \) is isomorphic to the Cuntz algebra \( \mathcal{O}_4 (= C^*\{(S_i, 1 \leq i \leq 4)\}) \) under the following correspondence:

\[ S_1 = V_{150}, \quad S_2 = U_0 V_{150}, \quad S_3 = U_1 V_{150}, \quad S_4 = U_0 U_1 V_{150}. \]

Once one knows Theorem 3.14, one automatically obtains Proposition 3.16, because it is known that there is a homeomorphism \( h \) on the Cantor set \( \mathfrak{K}_2 \) satisfying \( h \circ \varphi_{90} = \varphi_{150} \circ h \). In fact, take a homeomorphism on \( \mathfrak{K}_2 \) induced by the following automorphism on the algebra \( C(\mathfrak{K}_2) \) defined by the correspondence: \( i = 0,1 \).
\[ \begin{align*}
U_i & \rightarrow U_i \\
U_{i-1}U_{i+1} & \rightarrow U_{i-1}U_iU_{i+1} \\
U_{i-2}U_{i+2} & \rightarrow U_{i-2}U_iU_{i+2} \\
U_{i-3}U_{i-1}U_{i+1}U_{i+3} & \rightarrow U_{i-3}U_{i-2}U_iU_{i+2}U_{i+3} \\
\vdots & \vdots
\end{align*} \]

Since \( C(\mathcal{R}_2) \) is the universal \( C^* \)-algebra generated by countable infinite mutually commuting self-adjoint unitaries, it is easy to see that the above correspondence gives rise to a well-defined automorphism on it. Hence, by taking a unitary operator \( W \) on \( \mathcal{S} \) induced by the homeomorphism \( h \) one knows that

\[ W^*V_{90}W = V_{150}, \quad W^*C(\mathcal{R}_2)W = C(\mathcal{R}_2) \]

so that both \( C^* \)-algebras \( C_{90} \) and \( C_{150} \) are isomorphic each other.

4. The Sierpinski gasket in \( \mathcal{O}_4 \) and the growth rate dimension.

In this section, we first show that a cellular automaton evolution may be identified with the canonical endomorphism on the Cuntz algebra \( \mathcal{O}_4 \). Thus we represent the Sierpinski gasket, as a limit set of a cellular automaton evolution by \( \varphi_{90} \), in the \( C^* \)-algebra \( C_{90} \) by using an endomorphism on \( C_{90} \). Hence it is possible to describe the growth rate dimension of the evolution by using a certain state on \( \mathcal{O}_4 \). We construct a faithful state on \( \mathcal{O}_4 \) which counts the number of cells with value 1 in the evolution at each stage. Then we describe the growth rate dimension of the limit set. A similar discussion works for the cellular automaton \( C^* \)-algebra \( C_{150} \).

We first explain notations following [Cu1]. Let \( S_i, 1 \leq i \leq 4 \) be the generators of the algebra \( C_{90} (= \mathcal{O}_4) \) defined by (3.1), which satisfy the relation (3.2). Let \( W^k_4, k = 1, 2, \ldots \) be the set of all \( k \)-tuples \((\mu(1), \ldots, \mu(k))\) with \( 1 \leq \mu(i) \leq 4 \). We denote by \( S_\mu \) the isometry \( S_\mu = S_{\mu(1)} \cdots S_{\mu(k)} \) for \( \mu \in W^k_4 \). Let \( \mathcal{F}^k_4 \) be the \( C^* \)-algebra generated by \( \{ S_\mu S^*_v; \mu, v \in W^k_4 \} \) and \( \mathcal{F}_4 \) be the \( C^* \)-algebra generated by \( \bigcup_{k=1}^{\infty} \mathcal{F}^k_4 \).

As in [Cu1], \( \mathcal{F}^k_4 \) is isomorphic to the \( 4^k \times 4^k \) complex full matrix algebra \( M_{4^k} \) because \( \{ S_\mu S^*_v; \mu, v \in W^k_4 \} \) become a system of matrix units of \( M_{4^k} \). The identity \( S_\mu S^*_v = \sum_{i=1}^{4} S_\mu S_i S^*_i S^*_v \) defines an inclusion \( \mathcal{F}^k_4 \subset \mathcal{F}_{(k+1)} = M_4 \otimes \mathcal{F}^k_4 \) so that \( \mathcal{F}_4 \) becomes a UHF-algebra of type \( 4^\infty \) ([Cu1; 1.4. Proposition]).

Consider the two sequences of unitaries \( \{(\varphi_{90}^n(U_i))_{m \in \mathbb{N}}; i = 0, 1 \) obtained by iterating the morphism \( \varphi_{90} \) defined by

\[ \varphi_{90}(U_{n_1}U_{n_2} \cdots U_{n_k}) = U_{n_1-1}U_{n_1+1} \cdots U_{n_2-1}U_{n_2+1} \cdots U_{n_k-1}U_{n_k+1} \]
We will show in a moment that \( \hat{\phi}_{90} \) is well-defined as a morphism, and extends to the one-sided shift on \( \mathcal{F}_4 = \bigotimes_1^\infty M_4 \). We have
\[
\begin{align*}
\hat{\phi}_{90}^0(U_0) &= U_0, & \hat{\phi}_{90}^0(U_1) &= U_1, \\
\hat{\phi}_{90}^1(U_0) &= U_{-1} U_1, & \hat{\phi}_{90}^1(U_1) &= U_0 U_2, \\
\hat{\phi}_{90}^2(U_0) &= U_{-2} U_2, & \hat{\phi}_{90}^2(U_1) &= U_{-1} U_3, \\
\hat{\phi}_{90}^3(U_0) &= U_{-3} U_{-1} U_1 U_3, & \hat{\phi}_{90}^3(U_1) &= U_{-2} U_0 U_2 U_4, \\
\hat{\phi}_{90}^4(U_0) &= U_{-4} U_4, & \hat{\phi}_{90}^4(U_1) &= U_{-3} U_5. \\
& \vdots & & \vdots
\end{align*}
\]
Namely, each of two sequences \( \{ \hat{\phi}_{90}^m(U_i) \}_{m \in \mathbb{N}}, i = 0, 1 \) shows the cellular automaton evolution starting from a state containing a single cell with value 1.

**Lemma 4.1.** Each unitary \( U_n, n \in \mathbb{Z} \) belongs to the algebra \( \bigcup_{k=1}^\infty \mathcal{F}_{4^k} \). Hence the two sequences \( \{ \hat{\phi}_{90}^m(U_i) \}_{m \in \mathbb{N}}, i = 0, 1 \) belong to the UHF-algebra \( \mathcal{F}_4 \).

**Proof.** We already know that both unitaries \( U_0, U_1 \) belong to \( \bigcup_{k=1}^\infty \mathcal{F}_{4^k} \) as in the proof of Lemma 3.9. Under the assumption that two unitaries \( U_k, U_{k+1} \) belong to \( \bigcup_{k=1}^\infty \mathcal{F}_{4^k} \), we see that \( U_{k+2}, U_{k-1} \) also belong to it by the identity (3.4).

We now show that \( \hat{\phi}_{90} \) extends to a morphism of \( \mathcal{F}_4 \).

**Proposition 4.2.** For \( i = 0, 1 \), \( \hat{\phi}_{90}^m(U_i) = 1 \otimes \cdots \otimes 1 \otimes U_i \in M_4 \otimes \cdots \otimes M_4 \otimes M_4 \): \((m + 1)\)-times tensor product of \( M_4 \).

To prove Proposition 4.2, we provide notations and a lemma.

Let \( W_S \) be a unitary operator on \( S \) induced by the forward shift \((S: \{a_n\} \rightarrow \{a_{n+1}\})\) on \( R_2 \). Put \( \sigma^S = \text{Ad} W_S \). Since the shift commutes with the rule \( \varphi_{90} \), we have
\[
\sigma^S(V_{90}) = V_{90}, \quad \sigma^S(U_n) = U_{n+1}, \quad n \in \mathbb{Z}.
\]
Thus \( \sigma^S \) gives rise to an automorphism on the \( C^* \)-algebra \( C_{90} \). Since one has, by the relation (3.4),
\[
(4.1) \quad U_2 = U_0 V_{90} U_1 V_{90}^* + V_{90} U_1 V_{90}^* U_0 + U_0 U_1 V_{90} U_1 V_{90}^* U_1 + U_1 V_{90} U_1 V_{90}^* U_1 U_0
\]
\[
= S_2 U_1 S_1^* + S_1 U_1 S_2^* + S_4 U_1 S_3^* + S_3 U_1 S_4^*,
\]
on one obtains, by (3.1),
$$\sigma^2(S_1) = S_1, \quad \sigma^2(S_2) = S_3, \quad \sigma^2(S_3) = S_2 U_1, \quad \sigma^2(S_4) = S_4 U_1$$

(cf. [MT]).

**Lemma 4.3.** \( U_{n-1} U_{n+1} = \sum_{i=1}^{4} S_i U_n S_i^*, \quad n \in \mathbb{Z}. \)

**Proof.** By (4.1), one sees,

\[
U_0 U_2 \\
= (S_2 S_1^* + S_1 S_2^* + S_4 S_3^* + S_3 S_4^*) (S_2 U_1 S_1^* + S_1 U_1 S_2^* + S_4 U_1 S_3^* + S_3 U_1 S_4^*) \\
= \sum_{i=1}^{4} S_i U_1 S_i^*.
\]

Since \( \sigma^2(U_n) = U_{n+1}, \) one can show the identity \( U_{n-1} U_{n+1} = \sum_{i=1}^{4} S_i U_n S_i^* \) for all \( n \in \mathbb{Z} \) by applying the map \( \sigma^2 (n - 1) \)-times to the identity \( U_0 U_2 = \sum_{i=1}^{4} S_i U_1 S_i^*. \)

It is clear that Lemma 4.3 implies Proposition 4.2, and, moreover:

**Corollary 4.4.** \( \phi_{90} \) is realized as the canonical endomorphism \( \Phi_4 \) on \( \mathcal{O}_4 \) defined by \( \Phi_4(X) = \sum_{i=1}^{4} S_i X S_i^*, \) \( X \in \mathcal{O}_4 ([C_4]), \) that is

\[
\phi_{90}(U_n) = U_{n-1} U_{n+1} = \Phi_4(U_n), \quad n \in \mathbb{Z}.
\]

Note that the commutative \( C^* \)-algebra \( C(\mathbb{A}_2) \) coincides with the \( C^* \)-algebra \( C^*(U_i, 1 \otimes U_i, 1 \otimes 1 \otimes U_i, \ldots, i = 0, 1) \) generated by the following two sequences of unitaries in \( \mathcal{F}_4 \)

\[
U_0, \quad 1 \otimes U_0, \quad 1 \otimes 1 \otimes U_0, \quad 1 \otimes 1 \otimes 1 \otimes U_0, \ldots \\
U_1, \quad 1 \otimes U_1, \quad 1 \otimes 1 \otimes U_1, \quad 1 \otimes 1 \otimes 1 \otimes U_1, \ldots
\]

Hence \( C(\mathbb{A}_2) \) is a maximal abelian \( C^* \)-subalgebra of \( \mathcal{F}_4 \). This is because the algebra generated by \( U_0, U_1 \) is unitarily equivalent to the algebra consisting of all diagonal elements of \( M_4 \), see also [Cu1, CuK].

We will next express the fractal (Hausdorff) dimension (see [Fa]) of the Sierpinski gasket in \( C^* \)-algebra language by using the above discussion. In [Wi1], [Wi2], Willson has showed that the fractal dimension of the limit set of the cellular automaton evolution starting from a state containing a single cell with value 1 is equal to its growth rate dimension \( D_g \), defined by

\[
D_g = \lim_{t \to \infty} \log N(t)/\log t
\]
where \( N(t) \) is the number of cells with value 1 until time \( t \).

We express the number \( N(t) \) in terms of the \( C^* \)-algebra.

Let \( F_0 \) be the conditional expectation from \( \mathcal{O}_4 \) to the UHF-subalgebra \( \mathcal{F}_4 \) defined by

\[
F_0(X) = \int_{\mathbb{T}} \rho_z(X) \, dz, \quad X \in \mathcal{O}_4
\]

where \( \rho \) is the action of the circle \( \mathbb{T} \) defined by \( \rho_z : S_1 \to z S_1, z \in \mathbb{C}, |z| = 1 \). We next construct a conditional expectation from \( \mathcal{F}_4 \) to \( C(\mathbb{R}_2) \) in regarding \( C(\mathbb{R}_2) \) as a maximal abelian \( C^* \)-subalgebra of \( \mathcal{F}_4 \). One easily sees that the map \( e \) below gives rise to an expectation from \( M_4 \) to the subalgebra \( C^*(U_0, U_1) \) of \( M_4 \):

\[
e(A) = \frac{1}{4} (A + U_0 A U_0 + U_1 A U_1 + U_0 U_1 A U_0 U_1), \quad A \in M_4.
\]

The map \( e^{90} = \prod_1^{\infty} \otimes e \) yields an expectation from \( F_4 (= \prod_1^{\infty} \otimes M_4) \) to \( C(\mathbb{R}_2) \)

\( (\prod_1^{\infty} \otimes C^*(U_0, U_1)) \), under the identification between \( C(\mathbb{R}_2) \) and \( \prod_1^{\infty} \otimes C^*(U_0, U_1) \). Let us consider the faithful tracial state \( \psi_\lambda \) for \( 0 < \lambda < 1 \) on \( C(\mathbb{R}_2) \) defined by the integral induced by the measure \( \prod_{-\infty}^{\infty} \otimes \mu_\lambda \), where

\[
\mu_\lambda(\{0\}) = \frac{1}{1 + \lambda}, \quad \mu_\lambda(\{1\}) = \frac{\lambda}{1 + \lambda}.
\]

By composing these maps, one has a faithful state \( \tau^{90}_\lambda \) on \( C_{90} \) for each \( 0 < \lambda < 1 \), namely,

\[
\tau^{90}_\lambda = \psi_\lambda \circ e^{90} \circ F_0 : C^{90} \xrightarrow{F_0} \mathcal{F}_4 \xrightarrow{e^{90}} C(\mathbb{R}_2) \xrightarrow{\psi_\lambda} C.
\]

One now easily proves:

**Lemma 4.5.**

\[
\psi_\lambda(U_{i_1} U_{i_2} \ldots U_{i_k}) = \left( \frac{1 - \lambda}{1 + \lambda} \right)^k \text{ for distinct numbers } i_1, i_2, \ldots, i_k.
\]

Let \( l^{90}(k) \) be the number of cells with value 1 at time \( k \) starting from a state containing a single cell with value 1. The sequence \( \{l^{90}(k)\}_{k=0}^{\infty} \) is inductively determined by the following relations:

\[
l^{90}(0) = 1, \quad l^{90}(1) = 2, \quad l^{90}(2^n + k) = 2 l^{90}(k) \quad 0 \leq k \leq 2^n.
\]

Put \( c_\lambda = \frac{1 - \lambda}{1 + \lambda} \). By Corollary 4.4, one sees

\[
\tau^{90}_\lambda(\Phi^k_4(U_n)) = c^{90}(k), \quad n \in \mathbb{Z}, \quad k = 0, 1, 2, \ldots
\]
Since \( N(m) = \sum_{k=0}^{m} l^{90}(k) \) and the fractal dimension of the associated limit set is \( \log_2 3 \), we reach

\[
\lim_{m \to \infty} \log \left[ \sum_{k=0}^{m} \log \tau^{90}_\lambda(U_0) \right] / \log m = \log_2 3.
\]

We may analogously construct a state \( \tau^{150}_\lambda \) for the rule 150 which has similar properties. As a result, we have

\[
\lim_{m \to \infty} \log \left[ \sum_{k=0}^{m} \log \tau^{150}_\lambda(U_0) \right] / \log m = \log_2 (1 + \sqrt{5}).
\]

5. Automorphisms induced by operations on cellular automata.

In this section, we study automorphisms on cellular automaton \( C^* \)-algebras induced by some basic homeomorphisms on cellular space. Our main purpose to study these automorphisms is to make clear the differences between cellular automaton \( C^* \)-algebras associated with different cellular automaton rules. For instance, as we have seen in Section 3, the \( C^* \)-algebras \( C_{90} \), \( C_{150} \) are mutually isomorphic to \( \mathcal{O}_\lambda \) as \( C^* \)-algebra. Hence there are no difference between them as \( C^* \)-algebras. However, the two rules \( \varphi_{90}, \varphi_{150} \) are different. We look at some operations on cellular space \( \mathcal{R}_2 \) and consider automorphisms on \( C_{90}, C_{150} \) induced by them. We see that their behavior on \( C_{90} \) are different from those on \( C_{150} \). Namely, we show that the difference between \( \varphi_{90} \) and \( \varphi_{150} \) appears as a difference of a property of certain automorphisms on the two \( C^* \)-algebras \( C_{90} \) and \( C_{150} \). These automorphisms seem to belong a new class of automorphisms on the Cuntz algebra, which has not been treated in [Ar], [ETW], [Vo], \ldots, etc.

The following lemma proved in [MT] is used in the sequel.

**Lemma 5.1.** ([MT; Corollary B]). Let \( \alpha \) be an automorphism on the Cuntz algebra \( \mathcal{O}_n \) with \( \alpha(S_1) = S_1 \), where \( S_1 \) is a generator of isometries satisfying

\[
\sum_{i=1}^{n} S_i S_i^* = 1.
\]

If \( \alpha \) is not trivial, it is outer.

Let \( \gamma \) be a homeomorphism on \( \mathcal{R}_2 \) satisfying the condition

\[
\gamma \circ \varphi = \varphi \circ \gamma, \quad * = 90 \text{ or } 150.
\]

Let \( W_\gamma \) be the unitary on \( \mathcal{S} = L^2(\mathcal{R}_2, \mu) \) induced by \( \gamma \). Put \( \sigma = \text{Ad } W_\gamma \). Lemma 5.1 is used in our situation as:

**Lemma 5.2.** If \( \gamma \) is not trivial, \( \sigma \) gives rise to an outer automorphism on \( C_{90} \) and \( C_{150} \).
PROOF. Under the usual correspondence

\( (5.1) \quad S_1 = V_*, \quad S_2 = U_0 V_*, \quad S_3 = U_1 V_*, \quad S_4 = U_0 U_1 V_*, \quad * = 90, 150, \)

the condition \( \gamma \circ \varphi_* = \varphi_* \circ \gamma \) implies \( \sigma(S_1) = S_1. \)

Let us consider some automorphisms of the \( C^* \)-algebras \( C_* \) induced by basic homeomorphisms on \( \Omega_2. \) We first deal with automorphisms induced by shift on \( \Omega_2. \) Let \( w_0 \) be the unitary on \( S_4 \) induced by the forward shift: \( S \) on \( \Omega_2 = \prod \mathbb{Z}_2. \) Put \( \sigma^S = \text{Ad } w_0. \) Since one has \( S \circ \varphi_* = \varphi_* \circ S, * = 90, 150, \) \( \sigma^S \) yields an outer automorphism on \( C_{90} \) and on \( C_{150} \) by Lemma 5.2. We write these outer automorphisms as \( \sigma^S_{90} \) and \( \sigma^S_{150} \) respectively. Under the correspondence \( (5.1), \) one can easily write automorphisms \( \sigma^S_* \) by using the generators \( S_i, 1 \leq i \leq 4, \) in the following way (cf. [MT])

\[
\begin{align*}
\sigma^S_{90}(S_1) &= S_1, & \sigma^S_{90}(S_3) &= S_2(S_1 S_3^* + S_3 S_1^* + S_2 S_4^* + S_4 S_2^*), \\
\sigma^S_{90}(S_2) &= S_3, & \sigma^S_{90}(S_4) &= S_4(S_1 S_3^* + S_3 S_1^* + S_2 S_4^* + S_4 S_2^*), \\
\sigma^S_{150}(S_1) &= S_1, & \sigma^S_{150}(S_3) &= S_4(S_1 S_3^* + S_3 S_1^* + S_2 S_4^* + S_4 S_2^*), \\
\sigma^S_{150}(S_2) &= S_3, & \sigma^S_{150}(S_4) &= S_2(S_1 S_3^* + S_3 S_1^* + S_2 S_4^* + S_4 S_2^*).
\end{align*}
\]

Since each automorphism \( \sigma^S_*, * = 90, 150, \) shifts \( U_i \) to \( U_{i+1}, i \in \mathbb{Z}, (\sigma^S_*)^n, n(\neq 0) \in \mathbb{Z} \) is not trivial and satisfies \( (\sigma^S_*)^n(S_1) = S_1, n \in \mathbb{Z}. \) Therefore we have:

**PROPOSITION 5.3.** Both the automorphisms \( \sigma^S_{90}, \sigma^S_{150} \) on \( \mathcal{O}_4 \) induced by the shift on \( \Omega_2 \) give rise to outer automorphisms. Moreover each of them yields an outer action of the infinite cyclic group \( \mathbb{Z} \) on \( \mathcal{O}_4. \)

**REMARK 5.4.** Let \( \rho_{(24)}, \rho_{(34)} \) be the automorphisms on \( \mathcal{O}_4 \) induced by the permutations \( (24), (34) \) on the generators \( S_1, S_2, S_3, S_4 \) respectively. Then we have the relations:

\[
\sigma^S_{150} = \sigma^S_{90} \circ \rho_{(34)} = \rho_{(24)} \circ \sigma^S_{90}.
\]

Now we refer a compatibility the automorphisms \( \sigma_*^S \) with the states \( \tau_*^S, * = 90, 150. \)

**PROPOSITION 5.5.** \( \tau_*^{90} \) (resp. \( \tau_*^{150} \)) is invariant under \( \sigma^S_{90} \) (resp. \( \sigma^S_{150} \)). However, it is not invariant under \( \sigma^S_{150} \) (resp. \( \sigma^S_{90} \)).

**PROOF.** The invariance of \( \tau_*^{90} \) (resp. \( \tau_*^{150} \)) under \( \sigma^S_{90} \) (resp. \( \sigma^S_{150} \)) is easy from their definition. We show \( \tau_*^{90} \circ \sigma^S_{150} \neq \tau_*^{90} \). As we have \( \rho_{(34)}(U_1) = U_0 U_1, \) we get

\[
\tau_*^{90} \circ \rho_{(34)}(U_1) = \left( \frac{1 - \lambda}{1 + \lambda} \right)^2 \text{ and hence } \tau_*^{90} \circ \rho_{(34)}(U_1) \neq \tau_*^{90}(U_1).
\]

Since \( \sigma^S_{150} = \sigma^S_{90} \circ \rho_{(34)}, \) we conclude \( \tau_*^{90} \circ \sigma^S_{150}(U_1) \neq \tau_*^{90}(U_1) \) so that \( \tau_*^{90} \circ \sigma^S_{150} \neq \tau_*^{90}. \) Similarly, we have \( \tau_*^{150} \circ \sigma^S_{90} \neq \tau_*^{150}. \)
We do not know whether or not the two automorphisms $\sigma_{90}^5$, $\sigma_{150}^5$ are conjugate on $\mathcal{C}_4$. However the following automorphisms on $\mathcal{C}_4$ make a clear distinction between the two rules $\varphi_{90}$ and $\varphi_{150}$. They are induced by a homeomorphism $J$ on $\mathcal{R}_2$ called the conjugation, defined by

$$J(\{a_n\}) = \{a_n + 1\} \pmod{2}, \quad \{a_n\} \in \mathcal{R}_2.$$

Let $W_J$ be the unitary on $\mathcal{S}$ induced by $J$. Hence we have $W_J = W_e W_o$ where $W_e$, $W_o$ are unitaries defined in Section 3. Put $\sigma^c = \text{Ad } W_J$ so that one has $\sigma^c(U_n) = -U_n$. We first notice:

$$\varphi_{90} \circ J = \varphi_{90}, \quad \varphi_{150} \circ J = J \circ \varphi_{150}.$$

By lemma 5.2, we have

**Lemma 5.6.** $\sigma^c$ gives rise to an automorphism of period 2 on $C_{150}$, which is outer.

We denote by $\sigma_{150}^c$ the above automorphism on $C_{150}$.

On the other hand, we obtain

**Lemma 5.7.** $\sigma^c$ gives rise to an automorphism of period 2 on $C_{90}$, which is inner.

**Proof.** It suffices to show that $W_J$ belongs to $C_{90}$. We notice that $W_J = W_e W_o$, $W_e = 2Q_e - 1$, $*, = e, o$. As in Section 3, we know that

$$Q_e Q_o = V_{90} V_{90}^*, \quad Q_e (1 - Q_o) = U_1 V_{90} V_{90} U_1^*, \quad (1 - Q_e) Q_o = U_0 V_{90} V_{90} U_0^*$$

so that $Q_*$ and hence $W_*$, $* = e, o$ belong to $C_{90}$.

We denote by $\sigma_{90}^c$ the above automorphism on $C_{90}$.

Thus we conclude the following:

**Theorem 5.8.** The two pairs $(C_{90}, \sigma_{90}^c)$ and $(C_{150}, \sigma_{150}^c)$ of cellular automaton $C^*$-algebras with automorphisms induced by the conjugation on $\mathcal{R}_2$ are not conjugate each other. In fact, $\sigma_{90}^c$ is inner but $\sigma_{150}^c$ is outer.

We can explicitly write the implementing unitary $W_J$ of the inner automorphism $\sigma_{90}^c$ on $\mathcal{C}_4$ as

$$W_J = (2Q_e - 1)(2Q_o - 1)$$

$$= 4Q_e Q_o - 2Q_e - 2Q_o + 1$$

$$= 4S_1 S_1^* - 2(S_1 S_1^* + S_3 S_3^*) - 2(S_1 S_1^* + S_2 S_2^*) + 1$$

$$= S_1 S_1^* - S_2 S_2^* - S_3 S_3^* + S_4 S_4^*.$$
Hence it follows that
\[ \sigma_{90}(S_1) = S_1 W_f, \quad \sigma_{90}(S_2) = -S_2 W_f, \quad \sigma_{90}(S_3) = -S_3 W_f, \quad \sigma_{90}(S_4) = S_4 W_f. \]

On the other hand, as we have \( \sigma_{150}(V_{150}) = V_{150}, \sigma_{150}(U_n) = -U_n \), it follows that
\[ \sigma_{150}(S_1) = S_1, \quad \sigma_{150}(S_2) = -S_2, \quad \sigma_{150}(S_3) = -S_3, \quad \sigma_{150}(S_4) = S_4. \]

6. Generalization to \( k \)-state cellular automata.

There is no essential obstruction to generalizing our preceding discussions for 2-state to \( k \)-state \( (k \geq 3) \). We consider the 3-state version of \( \varphi_{90} \). It is the Pascal’s triangle of modulo 3. Let \( \mathcal{S}_3 \) be the infinite product \( \prod_{z} \mathbb{Z}_3 \) of \( \mathbb{Z}_3 = \{0, 1, 2\} \).

Consider the cellular automaton rule
\[ \psi(\{a_n\}) = \{a_{n-1} + a_{n+1}\} \pmod{3} \quad \{a_n\} \in \mathcal{S}_3. \]

Take a probability measure \( \mu \) on \( \mathcal{S}_3 \) which is the infinite product of the measure \( \mu_{1/3} \) on \( \mathbb{Z}_3 \) defined by
\[ \mu_{1/3}(\{0\}) = \mu_{1/3}(\{1\}) = \mu_{1/3}(\{2\}) = \frac{1}{3}. \]

It is easy to see that \( \psi \) is a 9-to-1 onto map on \( \mathcal{S}_3 \) and \( \mu \) is \( \psi \)-invariant. Let \( V_\psi \) be the linear operator on the Hilbert space \( \mathfrak{H}_3 = L^2(\mathcal{S}_3, \mu) \) induced by the map \( \psi \).

We define two unitaries \( W_e, W_o \) on \( \mathfrak{H}_3 \) induced by similar homeomorphisms \( h_e, h_o \) on \( \mathcal{S}_3 \) to the previous ones respectively. Let \( \omega \) be the principal 3-rd root of unity. Put
\[ Q^0_\ast = \frac{1}{2}(1 + W_\ast + W_\ast^2), \quad Q^1_\ast = \frac{1}{2}(1 + \omega^2 W_\ast + \omega W_\ast^2), \]
\[ Q^2_\ast = \frac{1}{2}(1 + \omega W_\ast + \omega^2 W_\ast^2) \ast = e, o. \]

Hence we have
\[ W_\ast = Q^0_\ast + \omega Q^1_\ast + \omega^2 Q^2_\ast \ast = e, o. \]

Corresponding to Lemma 3.5, one has

**Lemma 6.1.** \( V_\psi^* V_\psi = 1, \quad V_\psi^* V_\psi^* = Q^0_\psi Q^0_\psi. \)

Let \( E^i \in C(\mathcal{S}_3) = C\{0\} \oplus C\{1\} \oplus C\{2\} \) \( i = 0, 1, 2 \) be projections defined by
\[ E^i(x) = \begin{cases} 1 & (x = i) \\ 0 & (x \neq i) \end{cases} \quad i, x = 0, 1, 2. \]

Three sequences \( \{E^i_n\}_{n \in \mathbb{Z}}, \quad i = 0, 1, 2 \) of projections in \( C(\mathcal{S}_3) \) are defined by
\[ E^i_n(\{a_k\}) = E^i(a_n), \quad i = 0, 1, 2, \quad \{a_n\} \in \mathcal{S}_3. \]
Put unitary \( U_n = E_n^0 + \omega E_n^1 + \omega^2 E_n^2, n \in \mathbb{Z} \). Similarly to Lemma 3.6 and Corollary 3.7, one has

**Lemma 6.2.** For \( i = 0, 1, 2 \) (mod 3),

(i) For an even integer \( n \), \( E_n^i W_e = W_e E_n^{i+1} \), \( E_n^i W_o = W_o E_n^i \).

(ii) For an odd integer \( n \), \( E_n^i W_o = W_o E_n^{i+1} \), \( E_n^i W_e = W_e E_n^i \).

**Corollary 6.3.** For \( j = 0, 1, 2 \) (mod 3),

(i) For an even integer \( n \), \( U_n Q_e^j = Q_e^{j+1} U_n \), \( U_n Q_o^j = Q_o^j U_n \).

(ii) For an odd integer \( n \), \( U_n Q_o^j = Q_o^{j+1} U_n \), \( U_n Q_e^j = Q_e^j U_n \).

Put \( S_1 = V_\psi, S_2 = U_0 V_\psi, S_3 = U_1 V_\psi, S_4 = U_0^2 V_\psi, S_5 = U_1^2 V_\psi, S_6 = U_0 U_1 V_\psi, S_7 = U_0 U_1 V_\psi, S_8 = U_0^2 U_1 V_\psi, S_9 = U_0^2 U_1^2 V_\psi \).

It is obvious that \( S_i^* S_i = 1, 1 \leq i \leq 9 \). By the decomposition of the Hilbert space below

\[
1 = (Q_e^0 + Q_e^1 + Q_e^2)(Q_o^0 + Q_o^1 + Q_o^2) = \sum_{i,j = 0, 1, 2} Q_e^i Q_o^j,
\]

one has \( \sum_{i=1}^9 S_i S_i^* = 1 \). As we see the identity \( V_\psi U_n = U_{n-1} U_{n+1} V_\psi, n \in \mathbb{Z} \), we consequently have the next theorem by a similar argument to the previous one.

**Theorem 6.4.** The C*-algebra \( C^*(C(\mathcal{R}_3), V_\psi) \) generated by the commutative C*-algebra \( C(\mathcal{R}_3) \) and the isometry \( V_\psi \) coincides with the Cuntz algebra \( \mathcal{O}_9 (= C^*(S_i, 1 \leq i \leq 9)) \) generated by 9 isometries.

More generally, for a \( k \)-state cellular automaton \( \Psi \) defined by

\[
\Psi(\{a_i\}) = \{a_{i-1} + a_{i+1}\}, \quad \{a_i\} \in \mathcal{R}_k = \prod_{i} \mathbb{Z},
\]

we can summarize our discussion as the following theorem:

**Theorem 6.5.** Let \( C^*(C(\mathcal{R}_k), V_\psi) \) be the C*-algebra generated by the commutative C*-algebra \( C(\mathcal{R}_k) \) and the isometry \( V_\psi \) induced by the cellular automaton rule \( \Psi \). Then \( C^*(C(\mathcal{R}_k), V_\psi) \) is isomorphic to the Cuntz algebra \( \mathcal{O}_{k^2} (= C^*(S_i, 1 \leq i \leq k^2)) \) generated by \( k^2 \) mutually orthogonal isometries \( U_0^i U_1^j V_\psi \), \( i, j = 0, 1, \ldots, k - 1 \) satisfying

\[
\sum_{i,j = 0, 1, \ldots, k - 1} (U_0^i U_1^j V_\psi)(U_0^i U_1^j V_\psi)^* = 1
\]

where \( U_0 = \sum_{i=0}^{k-1} \omega^i E_0^i, U_1 = \sum_{i=0}^{k-1} \omega^i E_1^i \) and \( \omega \) is the principal \( k \)-th root of unity and \( \{E_n^i\} \) are projections defined in a similar way to the previous ones.
7. C*-algebras associated with illegal cellular automata.

Finally, we treat an example of a non-symmetric and hence illegal cellular automaton. It is the 1-dimension 2-state 3-neighborhood cellular automaton numbered as 60 which is defined by

\[ \varphi_{60}(\{a_n\}) = \{a_{n-1} + a_n\} \pmod{2}, \quad \{a_n\} \in \mathcal{A}_2. \]

It is easy to see that the map \( \varphi_{60} \) is surjective and 2-to-1. As the measure \( \mu \) cited in Section 3 is also \( \varphi_{60} \)-invariant, our previous discussions basically work for \( \varphi_{60} \). We denote by \( V_{60} \) the operator on the Hilbert space \( \mathcal{H} = L^2(\mathcal{A}_2, \mu) \) induced by \( \varphi_{60} \) as usual. Let \( s_i, i = 0, 1 \) be the two cross sections for \( \varphi_{60} \) satisfying \( P_0(s_i(\{a_n\})) = i, \quad i = 0, 1, \quad \{a_n\} \in \mathcal{A}_2 \). Since the Radon-Nikodým derivative \( (d\mu \circ s_i)/d\mu = 1/2 \), \( i = 0, 1 \), one has

**Lemma 7.1.**

(i) \( (V_{60}^* \xi)(\{a_n\}) = \frac{1}{2} \sum_{i=0,1} \xi(s_i(\{a_n\})), \quad \xi \in \mathcal{H}, \quad \{a_n\} \in \mathcal{A}_2. \)

(ii) \( V_{60}^* V_{60} = 1. \)

Let \( h \) be the homeomorphism on \( \mathcal{A}_2 \) defined by \( h(\{a_n\}) = \{a_n + 1\} \) and \( W \) the unitary on \( \mathcal{H} \) induced by \( h \). Put \( Q = (W + 1)/2. \)

**Lemma 7.2.** \( V_{60} V_{60}^* = Q. \)

Let \( U_n, n \in \mathbb{Z} \) be the self-adjoint unitaries defined in Section 3.

**Lemma 7.3.** \( U_n Q = (1 - Q) U_n, \quad n \in \mathbb{Z}. \)

**Lemma 7.4.**

\[
(7.1) \quad V_{60} U_n = U_{n-1} U_n V_{60}, \quad n \in \mathbb{Z}.
\]

We fix an arbitrary integer \( N \) henceforth. Put

\[
S_1^N = V_{60}, \quad S_2^N = U_N V_{60}.
\]

By Lemma 7.3, we have the following relations

\[
S_i^N S_i^N = 1 \quad (i = 1, 2), \quad \sum_{i=1}^{2} S_i^N S_i^N = 1.
\]

As we have the identity

\[
U_N = S_2^N S_1^N + S_1^N S_2^N,
\]

we know the following lemmas.
LEMMA 7.5. Under fixing an integer \( N \), the C*-algebra \( C^*(U_N, V_{60}) \) generated by the operators \( U_N \) and \( V_{60} \) coincides with the C*-algebra \( C^*(S_1^N, S_2^N) \) generated by \( S_1^N \), \( S_2^N \), which is the Cuntz algebra \( \mathcal{O}_2 \) of order 2.

LEMMA 7.6. The C*-algebra \( C^*(U_k; k \leq N, V_{60}) \) generated by the sequence \( U_k \), \( k \leq N \) and \( V_{60} \) coincides with \( C^*(U_N, V_{60}) \) and hence with \( C^*(S_1^N, S_2^N) \) (\( \cong \mathcal{O}_2 \)).

PROOF OF LEMMA 7.6. It suffices to show that the unitary \( U_{N-1} \) belongs to \( C^*(U_N, V_{60}) \) by induction. As we have

\[
1 = V_{60} V_{60}^* + U_N V_{60} V_{60}^* U_N,
\]

it follows that, by (7.1),

\[
U_{N-1} = U_N V_{60} U_N V_{60}^* + V_{60} U_N V_{60}^* U_N.
\]

Hence one sees that \( U_{N-1} \) belongs to \( C^*(U_N, V_{60}) \).

We denote by \( C_{60}^N \) the C*-algebra \( C^*(U_k; k \leq N, V_{60}) \). Thus we have a sequence of natural inclusions of C*-algebras \( \{C_{60}^N\}_{N \in \mathbb{Z}} \).

\[
\cdots \subset C_{60}^{N-2} \subset C_{60}^{N-1} \subset C_{60}^N \subset C_{60}^{N+1} \subset C_{60}^{N+2} \subset \cdots
\]

Each of C*-algebras \( \{C_{60}^k\}_{k \in \mathbb{Z}} \) is isomorphic to \( \mathcal{O}_2 \). We study the inclusion \( C_{60}^N \subset C_{60}^{N+1} \) by a C*-algebra technique.

Put \( \alpha_N = \text{Ad } U_{N+1} \). By the relation (7.1), we have

\[
\alpha_N(S_1^N) = S_2^N, \quad \alpha_N(S_2^N) = S_1^N.
\]

Namely, \( \alpha_N \) yields the "flip-flop" automorphism on \( \mathcal{O}_2 \) \((\cong C_{60}^N)\) studied by R. Archbold. His result in [Ar] says \( \alpha_N \) is outer on \( \mathcal{O}_2 \). Let \( C^*(U_{N+1}, C_{60}^N) \) be the C*-algebra generated by the unitary \( U_{N+1} \) and the algebra \( C_{60}^N \). It is nothing but \( C_{60}^{N+1} \). Obviously, there is a canonical surjective homomorphism \( \pi_{N+1} \) from the crossed product \( C_{60}^N \rtimes \mathbb{Z}_2 \) (\( = \mathcal{O}_2 \rtimes \mathbb{Z}_2 \)) of \( C_{60}^N \) by the action \( \alpha_N \) of the group \( \mathbb{Z}_2 \) (\( = \{0, 1\}\)) to the algebra \( C^*(U_{N+1}, C_{60}^N) \). By [Ki], \( C_{60}^N \rtimes \mathbb{Z}_2 \) is simple so that \( \pi_{N+1} \) is injective. Thus we have

LEMMA 7.7. The C*-algebra \( C_{60}^{N+1} \) is isomorphic to the crossed product \( C_{60}^N \rtimes \mathbb{Z}_2 \) through the map \( \pi_{N+1} \). The isomorphism is compatible with two natural inclusions \( i_N^*: C_{60}^N \to C_{60}^{N+1} \) and \( j_N^*: C_{60}^N \to C_{60}^N \rtimes \mathbb{Z}_2 \). Namely the following sequence of diagrams is commutative.

\[
\begin{array}{ccc}
\cdots & \rightarrow & C_{60}^N \\
\text{ } & \| & \text{ } \\
\cdots & \rightarrow & C_{60}^N \rtimes \mathbb{Z}_2
\end{array}
\]

Although the following corollary is a special case of the theorem in [CuE], our approach to the result is completely different from Cuntz-Evans's one.
**Corollary 7.8** ([CuE; Theorem]). The crossed product $\mathcal{O}_2 \times_{\alpha} \mathbb{Z}_2$ of $\mathcal{O}_2$ by the flip-flop automorphism is isomorphic to the original Cuntz algebra $\mathcal{O}_2$.

We identify $C^{N+1}_{60}$ with $C_N^{60} \times_{\alpha} \mathbb{Z}_2$ in the previous way. Therefore we conclude

**Theorem 7.9.** The $C^*$-algebra $C_{60}$ (\(= C^*(\mathcal{R}_2, V_{60})\)) generated by the commutative $C^*$-algebra $C(\mathcal{R}_2)$ and the isometry $V_{60}$ is isomorphic to the inductive limit $C^*$-algebra $\lim_{i\to\infty} C^{N}_{60}$. Hence $C_{60}$ is also simple.

**Proof.** Since $C(\mathcal{R}_2)$ is an inductive limit $C^*$-algebra of the sequence of the $C^*$-algebras \(\{C^*(U_k; k \leq N)\}_{N \in \mathbb{N}}\), $C_{60}$ is also an inductive limit of the sequence of the $C^*$-algebras \(\{C^N_{60}\}_{N \in \mathbb{N}}\). It is well known that an inductive limit of simple $C^*$-algebras is also simple.

By a recent result of Rørdam, [Rø], it follows that $C_{60}$ is isomorphic to $\mathcal{O}_2$.

**Remark 7.10.** The morphism $\phi_{60}$ given by $\phi_{60}(U_n) = U_{n-1}U_n$ is also represented as the canonical endomorphism $\Phi_2$ on $\mathcal{O}_2$ defined by $\Phi_2(X) = \sum_{i=1}^{2} S_iXS_i^*$, because we have

\[
\sum_{i=1}^{2} S_iU_nS_i^* = U_{n-1}U_n \quad (= \phi_{60}(U_n)).
\]

We easily see that the sequence \(\{\Phi_2^n\}\) of the endomorphism at each $C^*$-algebra $C^N_{60}$ is compatible with the inclusions $i_N: C^N_{60} \to C^{N+1}_{60}$ so that $\{\Phi_2^n\}$ define an endomorphism on $\lim_{i\to\infty} C^{N}_{60}$. Hence we can continue to discuss on the $C^*$-algebra $C_{60}$ in a similar fashion to the previous ones $C_{90}$ and $C_{150}$ as in Section 4.

**Remark 7.11.** It is easy to generalize our discussions to a general $k$-state cellular automaton rule corresponding to the rule $\varphi_{60}$. Consequently, we have an inductive limit $C^*$-algebra $\lim_{i\to\infty} \mathcal{O}_k$ of the sequence of the Cuntz algebra $\mathcal{O}_k$ of order $k$ under the inclusion $i_N: \mathcal{O}_k \to \mathcal{O}_k \times_{\sigma_k} \mathbb{Z}_k \cong \mathcal{O}_k$ where $\sigma_k$ is the action induced by the cyclic permutation of generators of isometries $S_1, S_2, \ldots, S_k$.

As a generalization of the above fact $\mathcal{O}_k \times_{\sigma_k} \mathbb{Z}_k \cong \mathcal{O}_k$, M. Izumi privately informed the author about the following fact:

For a finite group $G$ of order $n$, consider the action $\alpha$ of it on $\mathcal{O}_n$ by $\alpha_g(S_h) = S_{gh}$, $g, h \in G$, where $\{S_g\}_{g \in G}$ are generators of isometries of $\mathcal{O}_n$ with $\sum_{g \in G} S_gS_g^* = 1$. Then the crossed product $\mathcal{O}_n \times_{\alpha} G$ is isomorphic to $\mathcal{O}_n$.

We notice that this fact may be similarly proved if we start with the cellular space $\prod G$ in place of $\prod \mathbb{Z}_k$ and consider the corresponding map $\varphi_{60}^G$ on $\prod G$, $z$.
defined by \( \varphi_{60}^G(\{g_i\}) = \{g_{i-1}g_i\} \), \( \{g_i\} \in \prod \mathbb{Z} G \). In fact, the resulting C*-algebra is an inductive limit C*-algebra \( \varprojlim \mathcal{O}_n \) of \( \mathcal{O}_n \) under the inclusion \( i_N : \mathcal{O}_n \to \mathcal{O}_n \times \mathbb{Z} G \cong \mathcal{O}_n \).

REFERENCES


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