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ON THE ZEROS OF A CLASS OF GENERALISED DIRICHLET SERIES-XVI

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Dedicated to Professor K. Chandrasekharan on his seventy-fourth birthday.

§1. Introduction.

In this paper we continue the investigations made in paper XV ([1]) of this series. We prove a general theorem of which we first state two special cases. We write $s = \sigma + it$ as usual.

THEOREM 1. *Let $k(\geqq 1)$ be an integer constant and let $d_k(n)$ be defined by the identity $(\zeta(s))^k = \sum_{n=1}^{\infty}(d_k(n)n^{-s})$ valid in $\sigma > 1$. Put*

$$(1)\qquad F(s) = (\zeta(s))^k + \sum_{n=1}^{\infty} d_k(n)((n+\alpha_n)^{-s} - n^{-s}), (\sigma > 0),$$

where $\{\alpha_n\}$ is any sequence of real numbers with $|\alpha_n| \leqq \frac{1}{3}$. Denote the number of zeros of $F(s)$ in $(\sigma \geqq \alpha, T \leqq t \leqq 2T)$ by $N(\alpha, T)$. Then for every $\delta > 0$, we have,

$$N(\tfrac{1}{2} + \delta, T) \ll_{\delta} T.$$

THEOREM 2. *Let K be an algebraic number field which is abelian over the rationals or over any quadratic field and let $\zeta_K(s)$ denote its zeta-function. Let $f(n)$ be the number of integral ideals of K of norm n. Put*

$$(2)\qquad F(s) = \zeta_K(s) + \sum_{n=1}^{\infty} f(n)((n+\alpha_n)^{-s} - n^{-s}), (\sigma > 0),$$

where $\{\alpha_n\}$ is as before. Denote the number of zeros of $F(s)$ in $(\sigma \geqq \alpha, T \leqq t \leqq 2T)$ by $N(\alpha, T)$. Then for eery $\delta > 0$, we have,

$$N(\tfrac{1}{2} + \delta, T) \ll_{\delta} T.$$

REMARK 1. In (1) and (2) RHS denotes the analytic continuation from $\sigma > 1$.

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REMARK 2. The restriction on $\{\alpha_n\}$ may be relaxed considerably (see the general theorem in §2 and the remark at the end of §3).

REMARK 3. For comparison note the result $N(\frac{1}{2} - \delta, T) >>_\delta T \log T$ (valid for $0 < \delta < \frac{1}{2}$) for the functions (1) and (2), established in XV ([1]). (In that paper the stronger condition $|\alpha_n| \leqq 10^{-5}$ was assumed when $k = 1$, i.e. when K is the field of rationals to prove the lower bound).

REMARK 4. The upper bounds in Theorems 1 and 2 are valid for the zeros of $F^{(l)}(s) - \beta$ where β is any complex constant and $F^{(l)}(s)$, $l \geqq 0$ are derivatives of $F(s)$. The lower bound mentioned in Remark 3 is valid for $F^{(l)}(s) - \beta$.

NOTATION. We follow the same notation regarding $<<$, $>>$ and $O(\ldots)$, as we did in XV ([1]).

§2. Definitions and the general theorem.

We begin with two definitions. We will fix two positive constants a, b with $a < b$ throughout.

GENERALISED DIRICHLET SERIES (GDS). Let $\{\lambda_n\}$ be any sequence of real numbers with $a < \lambda_1 < \lambda_2 < \ldots, \lambda_1 < b$ and $a \leqq \lambda_{n+1} - \lambda_n \leqq b$ for $n \geqq 1$. Let $\{a_n\}$ be any sequence of complex numbers such that $a_1 \neq 0$ and

$$(3) \qquad Z(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$$

is convergent for some complex number $s = s_0$. Then $Z(s)$ is called a *generalised Dirichlet series* (GDS). We remark that if $Z(s)$ is convergent at $s = s_0$, it is absolutely convergent at $s = s_0 + 2$. Note that a GDS is different from zero if the real part of s exceeds a certain constant.

DIRICHLET SERIES. A generalised Dirichlet series $Z(s)$ is said to be a *Dirichlet series* if $\lambda_1, \lambda_2, \cdots$ is a subsequence of the sequence of natural numbers. It is said to be *normalised* if $\sum_{n \leqq x} |a_n|^2 <<_\varepsilon x^{1+\varepsilon}$ for every $\varepsilon > 0$ and all $x \geqq 1$.

GENERAL THEOREM. *Let $r \geqq 1$ be any integer constant and let $\varphi_1(s)$, $\varphi_2(s), \ldots, \varphi_r(s)$ be r Dirichlet series each continuable analytically in $(\sigma \geqq \frac{1}{2}, t \geqq t_0)$ and there* $\max_j |\varphi_j(s)| < t^A$ *where t_0 and A are some positive constants. Suppose that for $T \geqq t_0$, we have,*

$$(4) \qquad \max_j \left(\frac{1}{T} \int_T^{2T} |\varphi_j(\tfrac{1}{2} + it)|^2 \, dt \right) <<_\varepsilon T^\varepsilon$$

for every $\varepsilon > 0$, and further $Z(s)$ defined by

$$Z(s) = \prod_{j=1}^{r} \varphi_j(s) = \sum_{n=n_0}^{\infty} b_n n^{-s}, (n_0 \geqq 1, b_{n_0} \neq 0), \tag{5}$$

is a normalised Dirichlet series. Suppose that $\{\alpha_n\}$ *is any sequence of real numbers with* $|\alpha_n| \ll_\varepsilon n^\varepsilon$ *for every* $\varepsilon > 0$ *and that*

$$F(s) = Z(s) + \sum_{n=n_0}^{\infty} b_n((n+\alpha_n)^{-s} - n^{-s}), (\sigma \geqq \tfrac{1}{2}, t \geqq t_0), \tag{6}$$

is a GDS. *Define* $N(\alpha, T)$ *to be the number of zeros of* $F(s)$ *in* $(\sigma \geqq \alpha, T \leqq t \leqq 2T)$. *Then for every* $\delta > 0$, *we have*

$$N(\tfrac{1}{2} + \delta, T) \ll_\delta T. \tag{7}$$

REMARK 1. Note that in $t \geqq t_0$ we have $|F(s)| \leqq t^{r(A+2)}$.

REMARK 2. We can state a suitable analogue in case $\varphi_1(s), \ldots, \varphi_r(s)$ are Dirichlet polynomials i.e. when the coefficients of the Dirichlet series are allowed to depend on T and $a_n = 0$ for all large n depending on T. But we do not do it here.

REMARK 3. An assertion similar to (7) holds good for the derivatives $F^{(l)}(s)$ (or even for $F^{(l)}(s) - \beta$ where $l \geqq 0$, and β is a complex constant) provided these derivatives (or $F^{(l)}(s) - \beta$) are GDS.

§3. Proof of the general theorem.

The constants ε, δ, $(0 < \varepsilon < \frac{1}{100}, 0 < \delta < \frac{1}{100})$ will be chosen at the end of the proof. All the constants implied by $\gg$ and $\ll$ depend on ε and δ. We begin with a lemma which is well-known (see for example the proof of Theorem 9.2 on page 211 of [2]).

LEMMA 1. *Let* $n \geqq n_0(\delta)$ *and* I_0 *denote the interval* $n \leqq t \leqq n+1$. *Then the number of zeros of* $F(s)$ *in* $(\sigma \geqq \frac{1}{2} + 2\delta, t \in I_0)$ *is* $\ll \log n$.

PROOF. We first prove an auxiliary result. Let τ be fixed in $n \leqq \tau \leqq n+1$, $s_0 = \sigma_0 + i\tau$, where σ_0 is a constant large enough so as to ensure that $G(s)$ defined by

$$G(s) = \mu^{-1} F(s), \text{ where } \mu = b_{n_0}(n_0 + \alpha_{n_0})^{-s}$$

is such that $G(s) - 1$ has absolute value $\leqq \frac{1}{10}$ for all $\sigma \geqq \sigma_0$. Put $R = \sigma_0 - \frac{1}{2}$, $R_1 = \sigma_0 - \frac{1}{2} - \delta$. Note that the circle $|s - s_0| = R$ touches $\sigma = \frac{1}{2}$ and the circle $|s - s_0| = R_1$ touches $\sigma = \frac{1}{2} + \delta$. By Jensen's theorem, we have,

$$\int_0^R \frac{n(r)}{r} dr = \frac{1}{2\pi i} \int_{|s-s_0|=R} \log|G(s)| \frac{ds}{s} - \log|G(s_0)|, \tag{8}$$

where $n(r)$ is the number of zeros of $F(s)$ in $|s - s_0| \leqq r \leqq R$. In (8) we can replace $G(s)$ by $F(s)$ on RHS, and LHS is

$$\geqq \int_{R_1}^{R} \frac{n(r)}{r} dr \geqq n(R_1) \log \frac{R}{R_1}.$$

We now remark that $|s - s_0| \leqq R_1$ includes the rectangle $(\frac{1}{2} + 2\delta \leqq \sigma \leqq \sigma_0,$ $|t - \tau| \leqq (R_1^2 - (R_1 - \delta)^2)^{\frac{1}{2}})$ and that $F(s)$ has no zeros in $\sigma \geqq \sigma_0$. But the RHS of (8) does not exceed a constant times $\log n$. This proves the following auxiliary result:

Number of zeros of $F(s)$ in the rectangle $(\sigma \geqq \frac{1}{2} + 2\delta, |t - \tau| \leqq (R_1^2 - (R_1 - \delta)^2)^{\frac{1}{2}}$ is $\ll_\delta \log n$, provided $R_1 = \sigma_0 - \frac{1}{2} - \delta$.

The lemma follows from this in an obvious way.

LEMMA 2. *Divide the interval $[T, 2T]$ into abutting equal intervals I of length $H = T^\varepsilon$ ignoring a bit at one end. Then we have the following results.*

(a) *The number of zeros of $F(s)$ in $(\sigma \geqq \frac{1}{2} + 2\delta, t \in I)$ or $(\sigma \geqq \frac{1}{2} + 2\delta, t \in$ ignored interval) is $\ll H \log T$.*

(b) *Let $M(I)$ denote the maximum over j of the maximum of $|\varphi_j(s)|$ in $(\sigma \geqq \frac{1}{2} + 3\delta,$ $t \in I)$. Then*

$$\sum_I (M(I))^2 < T^{1+2\varepsilon} \text{ for all } T \geqq T_0(\varepsilon, \delta). \tag{9}$$

PROOF. The part (a) follows from Lemma 1. W now sketch the proof of part (b). Note that if $\sigma \geqq B$ where B is a large positive constant the maximum over j of $|\varphi_j(s)|$ is bounded above by a constant M_0. Hence by standard convexity arguments

$$\max_j \left(\int_T^{2T} |\varphi_j(\sigma + it)|^2 \, dt \right) \ll T^{1+\varepsilon} \tag{10}$$

uniformly in σ for $\sigma \geqq \frac{1}{2} + \delta$. Next if $M(I)$ is attained at $s = z$ (z depending on I) and the maximum comes from $\varphi_j(z)$, we have,

$$|\varphi_j(z)|^2 \leqq \frac{1}{\pi\delta^2} \iint_{|s-z| \leqq \delta} |\varphi_j(s)|^2 \, d\sigma \, dt, \tag{11}$$

where the integral is over the disc $|s - z| \leqq \delta$. (This is a well-known consequence of Cauchy's Theorem). Hence by integrating over a suitable rectangle (bigger than the disc) and summing up over all intervals I for which $M(I)$ exceeds $2M_0$, we have,

$$\sum_I (M(I))^2 \leqq 2(\pi\delta^2)^{-1} \sum_{j=1}^{r} \iint_{\substack{\frac{1}{2}+\delta \leqq \sigma \leqq B+2 \\ T-1 \leqq t \leqq 2T+1}} |\varphi_j(s)|^2 \, d\sigma \, dt$$

and the result (b) follows in view of (10) of course using the fact that in LHS of the last inequality the sum over I for which $M(I) \ll 1$ contributes a small quantity.

DEFINITION. Consider an interval I contained in $[T+H, 2T-H]$ for which $M(I) > T^{4\varepsilon}$. Let us augment such an interval I by $(\log T)^2$ on both sides. Denote such an augmented interval by J. Such intervals will be called *bad intervals*. The complement of these bad intervals in $[T+H, 2T-H]$ will be called *good intervals*. These will be portions of the interval I introduced in Lemma 2. The good interals will be denoted by G.

LEMMA 3. *We have the following results.*
(a) *The number of bad intervals is* $\ll T^{1-6\varepsilon}$
(b) *The total number of zeros of* $F(s)$ *in* $(\sigma \geqq \frac{1}{2} + 2\delta, t \in U)$ *where* U *is the union of bad intervals, is* $\ll T^{1-\varepsilon}$.

PROOF. The part (a) follows from the second part of Lemma 2. The part (b) follows from the fact that for each bad interval J the number of zeros of $F(s)$ in $(\sigma \geqq \frac{1}{2} + 2\delta, t \in J)$ is $\ll H \log T$ and so the total contribution from all the bad intervals is $\ll HT^{1-6\varepsilon}\log T \ll T^{1-\varepsilon}$.

LEMMA 4. *Let* $M(\alpha, G)$ *denote the number of zeros of* $F(s)$ *in* $(\sigma \geqq \alpha, t \in G)$ *and let* $\psi(t) = 1$ *if* t *belongs to a good interval and* 0 *otherwise. Then*

$$\int_{\frac{1}{2}+4\delta}^{\infty}\left(\sum_{I} M(\alpha, G)\right) d\alpha \leqq \int_{T}^{2T} (\log|F(\tfrac{1}{2}+4\delta+it)|)\psi(t)\,dt + O(T). \tag{12}$$

PROOF. The proof is similar to the proof due to J. E. Littlewood ([2]) of Theorem 9.15 (A) on page 230. Here use is made of §9.4 on page 213 of [2]. We can apply this method to obtain an upper bound for $M(\alpha, G)$ for each G. We have only to remark that $q \ll v$ where v is the number of zeros of

$$g(z) = \tfrac{1}{2}\{f(z+iT) + \bar{f}(z-iT)\}$$

and Jensen's theorem does the rest. In these statements in our indication of the proof we have followed the notation of Titchmarsh's book [2]. Each interval G gives rise to an error term $\ll \log T$ and so the total of the error terms is $\ll T^{1-\varepsilon}\log T$. However to get the lower bound for the real part of $F(s)$ for large σ we replace $F(s)$ by $\mu^{-1}F(s)$ where μ is as in the proof of Lemma 1. This gives Lemma 4 since we have the error term $O(T)$ when we come back to $F(s)$.

LEMMA 5. *We have the following results.*
(a) LHS *of* (12) *is*

$$\geqq \sum_{G} M(\tfrac{1}{2}+5\delta, G)\int_{\frac{1}{2}+4\delta}^{\frac{1}{2}+5\delta} d\alpha \tag{13}$$

$$\geqq \delta N(\tfrac{1}{2} + 5\delta, T) + O(T^{1-\varepsilon}). \tag{14}$$

(b) RHS *of* (12) *is*

$$\leqq T \log\left(\frac{1}{T}\int_T^{2T} |F(\tfrac{1}{2} + 4\delta + it)|^{\psi(t)}\, dt\right) + O(T). \tag{15}$$

PROOF. The part (a) follows from the second part of Lemma 3. The part (b) follows from a limiting form of the arithmetico-geometric inequality (see for example the proof of Theorem 9.15 (A) on page 230 of [2]. There the integrand is required to be continuous; but plainly Riemann integrability is enough).

From now on till the end of this section we write $s = \frac{1}{2} + 4\delta + it$.

LEMMA 6. *We have,*

$$\int_T^{2T} |F(s)|^{\psi(t)}\, dt \leqq \sum_G \int_G (|F(s) - Z(s)| + |Z(s)|)\, dt + T. \tag{16}$$

PROOF. Trivial since $\psi(t) = 0$ on intervals which are not good.

LEMMA 7. *We have,*

$$\frac{1}{T}\int_T^{2T} |F(s) - Z(s)|^2 \ll 1. \tag{17}$$

PROOF. Same as the proof of equation (15) of XV ([1]).

LEMMA 8. *We have,*

$$\frac{1}{T}\int_T^{2T} |Z(s)|^2 \psi(t)\, dt \ll 1 \tag{18}$$

PROOF. Let t belong to a good interval G and let $X = T^{\frac{1}{4}}$. We start with

$$\sum_{n=n_0}^{\infty} a_n n^{-s} \exp\left(-\frac{n}{X}\right) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} Z(s+w) X^w \Gamma(w)\, dw, \tag{19}$$

where $w = u + iv$ is a complex variable. We cut off $|v| \geqq (\log T)^2$ with a small error and in the rest move the line of integration to $u = -\delta$ and choose ε very small. Thus it is not hard to see that LHS of (19) is an approximation to $Z(s)$ and so Lemma 8 follows by a simple application of the Montgomery-Vaughan Theorem (see Theorem 3 of XV ([1])).

The result $N(\frac{1}{2} + 5\delta, T) \ll T$ for every $\delta > 0$ follows in a simple way from Lemmas 5 to 8. Now the assertion $N(\frac{1}{2} + \delta, T) \ll T$ of the general theorem, follows from this since we can replace δ by $\dfrac{\delta}{5}$.

REMARK. The proliferation in §6 of XV ([1]) has relevence to our general theorem also. Thus in place of $|\alpha_n| \ll_\varepsilon n^\varepsilon$ we can manage with the simultaneous conditions

$$\sum_{n \leqq x} |b_n \alpha_n|^2 \ll_\varepsilon x^{1+\varepsilon} \text{ and } |\alpha_n| \leqq (1-\eta)n$$

(where $\varepsilon > 0$ is arbitrary and η is a fixed positive constant less than 1), the requirement of the first condition being for all real $x \geqq 1$ and that of the second being for all integers $n \geqq 1$.

§4. Examples.

Theorem 1 follows by taking $r = k$, $\varphi_j(s) = \zeta(s)$ for $j = 1$ to k. Theorem 2 can be deduced as follows. If K is an abelian extension of rationals, $\zeta_K(s)$ is the product of d L-functions $L_1(s), \ldots, L_d(s)$ where d is the degree of K. We can take $r = d$ and $\varphi_j(s) = L_j(s)$ ($j = 1, 2, \ldots, d$). If K is abelian over a quadratic field, then $\zeta_K(s)$ is the product of d abelian L-functions $L_1(s), \ldots, L_d(s)$ of the quadratic field where $2d$ is the degree of K. We can take $r = d$ and $\varphi_j(s) = L_j(s)$ ($j = 1, 2, \ldots, d$). We have only to check the validity of condition (4) for $\varphi_j(s)$. For this purpose we apply equation (8) of paper XV ([1]) with $Z(s) = \varphi_j(s)$ for any fixed j. (The $Z(s)$ of that paper should not be confused with $Z(s)$ of the present paper. Also k of that paper should not be confused with k of Theorem 1 of the present paper). In the notation of that paper $\varphi_j(s)$ satisfy FE with $k = 1$ (for Theorem 1 and also the first case of Theorem 2) and PFE with $k = 2$ (for the second case of Theorem 2). Hence equation (8) of that paper is valid and hence condition (4) of the present paper is satisfied. This completes the proof of Theorems 1 and 2.

REFERENCES

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