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A COMBINATORIAL PROOF OF THE EXISTENCE OF THE GENERIC HECKE ALGEBRA AND R-POLYNOMIALS

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1. Introduction.

Hecke algebras and Kazhdan-Lusztig polynomials are studied in algebraic combinatorics, although the mathematical foundation is pure algebra. Among others, Deodhar (e.g. in [3]) has worked on combinatorial aspects on these things. The aim of this paper is to help free these combinatorial concepts from their algebraic confinement by providing a construction, the *Hecke graph*, that encodes the combinatrics of Hecke algebra in such a way that an elementary analysis proves

- that the generic Hecke algebra exists, and
- that the R-polynomials (which are needed in the definition of Kazhdan-Lusztig polynomials) can be *defined* via the recurrence they satisfy, and this definition can be extended to any generic Hecke algebra.

The crux of both these things is showing that the result of a certain computation is well-defined, although the computation can be done in many different ways. To our guidance, we have the theory of *strong convergence* [4], which basically says that the computation is well-defined if it is locally well-defined. For the Hecke algebra, this boils down to the following: Let W denote a Coxeter group with generator set S, where for every generator pair $s, t \in S$ their product st has order $m(s,t)$ in the group W. Let $\{T_w\}$ ($w \in W$) be the standard basis of the associated Hecke algebra. For any pair $s, t \in S$, where $m(s,t) < \infty$, let $(T_t T_s T_t \cdots)$ denote an alternating product of $m(s,t)$ factors. Then, to prove that multiplication $T_v T_w$ is well-defined, it suffices to show that $(T_s T_t T_s \cdots) T_w = (T_t T_s T_t \cdots) T_w$. Proving this is equivalent to finding a certain pairing of alternatingly labeled paths in the Hecke graph.

Section 2 discusses the significance of the Hecke graph to the Hecke algebra and shows existence of the algebra, refering to section 3 for the canonical pairing of alternating paths. Finally, the R-polynomials and their recurrence are treated in section 4.

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2. Existence of the generic Hecke algebra.

Following Humphreys [5], we associate to the Coxeter group W the *generic Hecke algebra* over a commutative ring A as follows. The algebra has basis elements T_w (indexed by elements $w \in W$), and for a Coxeter generator $s \in S$, multiplication is defined by

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w), \\ a_s T_w + b_s T_{sw} & \text{if } l(sw) < l(w). \end{cases} \tag{1}$$

The parameters $a_s, b_s \in A$ are subject only to the condition that $a_s = a_t$ and $b_s = b_t$ if s and t are conjugate generators in W.

However, a priori it is not at all clear that such an algebra exists. We must show that multiplication of *any* two basis elements is well-defined, because in order to compute $T_v T_w$ we must first choose a reduced expression $s_1 s_2 \cdots s_k$ for v and then compute $T_{s_1} T_{s_2} \cdots T_{s_k} T_w$ according to the multiplication rule above. Thus, existence of the algebra depends on the result of this computation being independent of the choice of reduced expression for v.

The usual existence proof [5] is very algebraic, involving the algebra of all A-module endomorphisms of the free A-module. This is unsatisfactory for the combinatorialist who wants to see the Hecke algebra as an analog of the combinatorial Coxeter group, and especially since the endomorphism algebra has no further value after the proof. Instead, problems of this kind, where one must show that a result of a computation is independent of a seeming degree of freedom, have a unified theory in *strong convergence* [4]: A branching process has the strong convergence property if whenever a terminal position P can be reached in n steps, every way of going n steps will result in P. This property is equivalent to the following local property: whenever to different steps are possible, they are the first steps in two paths of equal length ending in the same position.

In the case of multiplication in the Hecke algebra, this implies that it is sufficient to prove that

$$(T_s T_t T_s \cdots) T_w = (T_t T_s T_t \cdots) T_w,$$

where both sides have an alternating product of $m(s, t) < \infty$ generators. *From now on,* $(sts\cdots)$ *will be shorthand for an alternating product of length* $m(s, t)$. The sufficiency of verifying the local property above follows in fact directly from Tits's Word Theorem, which says that every reduced expression for an element v can be obtained from any reduced expression for v by repeated substitution of factors of type $(sts\cdots) = (tst\cdots)$.

Now is the time to introduce the *Hecke graph* of the Coxeter group W: Begin with the Hasse diagram of the (left) weak order of W, or, equivalently, the Cayley graph of W rooted at 1, labeled in the natural way such that the edge between w and sw is labeled s. See Björner [1] for details about orderings of Coxeter groups. The partial order defines for every edge one direction *downwards* (towards 1), and one direction *upwards*. To get the Hecke graph of W, for every generator $s \in S$ we add a loop labeled s to every node whose s edge leads downwards.

Let $W_{s,t}$ be the parabolic subgroup generated by s and t, consisting of $2m(s,t)$ elements. We will be interested in the coset $W_{s,t}w$. This is known to have a unique minimal coset representative u, which means that in the Hasse diagram of W the elements of the coset $W_{s,t}w$ form a circuit with u at the bottom and the other elements lying on two paths that go upwards to join in $(sts\cdots)u = (tst\cdots)u$. It is important to note the following feature of the Hecke graph of W: a path from w labeled by s and t only must stay in the coset. The sketch in Fig. 1 should clarify the situation.

The entire reason for introducing the Hecke graph is that it encodes the combinatorics of multiplication in the Hecke algebra: every path from w labeled by some word $s_1 s_2 \cdots s_k$ corresponds bijectively to a term in the expansion of the product $T_{s_1} T_{s_2} \cdots T_{s_k} T_w$. The correspondence is easy and works as follows:

- The group element w' at the end point of the path gives an algebra element $T_{w'}$.
- Every step *upwards* contributes a factor 1 (i.e. no contribution).
- Every step *downwards*, with label s, contributes a factor b_s.
- Every step in a *loop*, with label s, contributes a factor a_s.

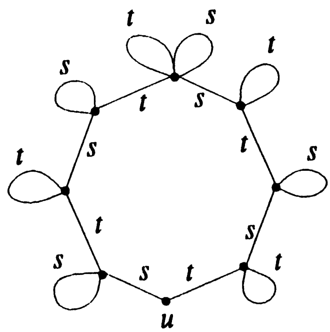

Fig. 1. The Hecke graph of the coset $W_{s,t}w$ when $m(s,t) = 4$.

The reader is urged to check that this precisely encodes the multiplication rule (1). Now the property that we want to prove follows from interpreting, via this correspondence, the properties of the Hecke graph stated in Proposition 3.2 and 3.3. We postpone the proofs to the next section in order to here pursue our current line of thought.

LEMMA 2.1. *If s and t are conjugate in W, then*

$$(T_sT_tT_s\cdots)T_w = (T_tT_sT_t\cdots)T_w$$

PROOF. Since s and t are conjugate, we have $a_s = a_t(= a)$ and $b_s = b_t(= b)$. The terms of the expansion of the two sides correspond respectively to the $sts\cdots$-paths and $tst\cdots$-paths (of length $m(s,t)$) from w in the Hecke graph. By Proposition 3.2 there is a bijection between these sets of paths that preserve the endpoint w', the number l of loops and the number d of edges traveled downwards. Hence, the bijection preserves the corresponding term $a^lb^dT_{w'}$.

Before treating the other case, observe that $m(s,t)$ odd implies that s and t are conjugate, because then $s(ts)^{(m-1)/2} = (sts\cdots) = (tst\cdots) = (ts)^{(m-1)/2}t$.

LEMMA 2.2. *If s and t are not conjugate in W, then*

$$(T_sT_tT_s\cdots)T_w = (T_tT_sT_t\cdots)T_w$$

PROOF. Since s and t are not conjugate, $m(s,t)$ must be even. A path ending in w' with l_s loops labeled s, d_s edges traveled downwards labeled s (and l_t and d_t analogously) corresponds to a term

$$a_s^{l_s}a_t^{l_t}b_s^{d_s}b_t^{d_t}T_{w'}$$

By Proposition 3.3, there is a bijection between $sts\cdots$-paths and $tst\cdots$-paths that preserves all these parameters.

Now recall the discussion in the beginning of this section: The two lemmas above imply that T_vT_w is always well-defined, which in turn implies the existence of the generic Hecke algebra.

REMARK 2.3. Obviously the proof allows for one additional parameter: we could define multiplication more generally by

$$T_sT_w = \begin{cases} c_sT_{sw} & \text{if } l(sw) > l(w), \\ a_sT_w + b_sT_{sw} & \text{if } l(sw) < l(w). \end{cases}$$

where of course also $c_s = c_t$ must hold when s and t are conjugate. This possibility can be said to reflect the fact (which follows directly from Tits's Word Theorem) that all reduced expressions for a Coxeter group element w have the same distribution of letters in conjugacy classes: If $s_1s_2\cdots s_k = t_1t_2\cdots t_k$ are two reduced expressions, then

$$c_{s_1}c_{s_2}\cdots c_{s_k} = c_{t_1}c_{t_2}\cdots c_{t_k}$$

3. Paths in the Hecke graph.

A path is *alternatingly labeled* if the word defined by the labels of the edges visited is alternating, i.e. either $stst\cdots$ or $tsts\cdots$. We shall now investigate what an alternatingly labeled path in the Hecke graph can look like. Recall the sketch in Fig. 1.

LEMMA 3.1. *An alternatingly labeled path in the Hecke graph must visit the top node between every pair of visited loops.*

PROOF. Follows by inspection from the fact that every loop that is not at the top node has the same label as the incident edge that goes downwards, so after a loop is visited an alternating path must continue upwards to the top node.

What we would like is a matching of the alternatingly labeled paths of length $m(s,t)$, such that every such $stst\ldots$-path is matched with a unique $tsts\ldots$-path with the same endpoint and the same number of visited loops. Further, if $m(s,t)$ is even, then in every pair the $tsts\ldots$-path shall have as many loops labeled s and t respectively as the $stst\ldots$-path.

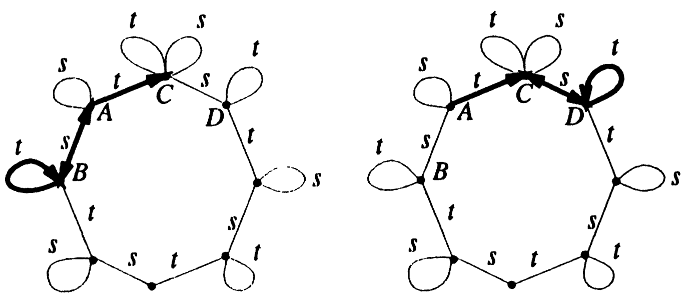

Fig. 2. Complementing path $P: A \to^s B \to^t B \to^s A \to^t C$, to the left, gives $P': A \to^t C \to^s D \to^t D \to^s C$, to the right.

The fundamental operation in this construction will be *complementing* a path segment P that contains precisely one loop. The complement of P is defined as the path P' (from the same point) that starts with the complementary label, and then goes as many edges *before* looping as P does *after* looping, and vice versa. Thus P' will have the same length and end point as P, and both paths contain exactly one loop. See Fig. 2. Complementing is *legal* if at least one of P and P' visits the top point, because then and only then will the new path P' also be alternatingly labeled (by reasoning similar to the lemma above). This will always be the case for the path segments that we consider. (Observe that any alternatingly labeled path of length $m(s,t)$ in the Hecke graph must visit the top point.)

PROPOSITION 3.2. *In the set of alternatingly labeled paths of length $m(s,t)$ with given starting point in the Hecke graph, there is a bijection between stst...-paths and tsts...-paths that preserves the end point and the number of loops of the path (and hence also the number of edges traveled downwards).*

PROOF. Given an alternatingly labeled path P_s from u to v, starting with an edge labeled s, we shall construct a path P_t with the same starting point u and end point v and number of loops, but beginning with t. It will be clear that the construction is an involution, and hence a bijection.

First, in the case without loops, the path P_s, being an alternating path of length $m(s,t)$, ends in the opposite node of the $2m(s,t)$-circuit; we take P_t to be the other half of the circuit.

If there is exactly one loop in P_s, let P_t be the complement.

Otherwise, we shall construct P_t piecewise. By the lemma above, we can cut the path P_s at the top point between any two loops. Complement each of the path segments to obtain P_t.

When the parameter $m(s,t)$ is even, we want to say even more. Here the construction is slightly more delicate, but relying on the simple fact that the loop of the complement of a path segment P has the same label as the loop of P if and only if the length of P is even.

PROPOSITION 3.3. *If $m(s,t)$ is even, then there is a bijection as in Proposition 3.2 that also leaves the number of loops labeled s and t respectively invariant (and hence also the number of edges traveled downwards with respective labels).*

PROOF. We follow the proof of Proposition 3.2 up to the definition of the path segments. Note that, with possible exception for the first and the last segment, every path segment has odd length. If the first or last path segment has even length, we immediately complement it (or both) to get the corresponding piece of P_t. Thus, all remaining segments are of odd length, and since the length of the entire path is even (and remains even after removing segments of even length), there must be an even number of segments. Number the segments in order $1, 2, \ldots, 2k$. When constructing the segment pair of P_t corresponding to segments $2i-1$ and $2i$, we must distinguish between two cases:

a) The two loops of this segment pair have different labels (i.e. one labeled s, the other t). Then complement both segments to get the piece of P_t. Since both segments complemented are of odd length, this complemented piece will also have one loop labeled s and one labeled t.

b) The two loops of this segment pair have the same label, say s. The edge labeled t incident to the intermediate top point leads to a node w. Connect the segments again and cut them at w instead. Complement the two even segments obtained in this way to get the corresponding piece of P_t. But how do we know that it is legal to complement the segment that do not contain the intermediate top point? For legality, it is enough that this segment has at least one edge on the other side of the loop, because then the complement will reach the top point. And indeed, the segment must have an odd number of edges (and hence at least one) on that side, since the total length of the segment is even, and the rest of it has edges labeled s in both ends and is therefore of odd length.

4. The recurrence of the R-polynomials.

For a moment, let $\mathscr{H}$ denote the Hecke algebra over the ring $\mathbf{Z}[q, q^{-1}]$, with all $a_s = q - 1$ and all $b_s = q$. This is the world where Kazhdan-Lusztig theory (see [5] or [6]) lives. Very briefly, one can define an involution ι on $\mathscr{H}$ by $\iota(q) = q^{-1}$ and $\iota(T_w) = (T_{w^{-1}})^{-1}$. One can then find a new basis $\{C_w\}$ of $\mathscr{H}$ that is invariant under ι. Basically, the *Kazhdan-Lusztig polynomials* are polynomials in q that appear as coefficients when the new basis elements C_w are expressed in the old basis $\{T_w\}$. The concept of Kazhdan-Lusztig polynomials seems to be very deep and important in several mathematical disciplines, and combinatorics is one of them; e.g. a combinatorial explanation is wanted to resolve the conjecture of Kazhdan-Lusztig that these polynomials have nonnegative coefficients.

Essential to this theory are the R-polynomials, which arise when basis elements such as T_w are inverted in $\mathscr{H}$. They are polynomials in q indexed by two elements of W: $R_{x,w}(q)$. From the proof where the R-polynomials originate, it follows that they can be computed recursively by choosing any generator s such that $sw < w$ in Bruhat order, and then set

$$R_{x,w} = \begin{cases} R_{sx,sw} & \text{if } sx < x, \\ (q-1)R_{x,sw} + qR_{sx,sw} & \text{if } sx > x, \end{cases}$$

with initial conditions $R_{1,1} = 1$ and $R_{x,1} = 0$ for all $x \neq 1$. (In fact, it is more efficient to use that $R_{w,w} = 1$ and $R_{x,w} = 0$ for all $x > w$, and this is the version stated in Humphreys. However, it is not hard to show that these initial conditions are equivalent to those above, which we prefer here.)

As in the case of the existence of the Hecke algebra, it is not clear by itself that this computation is well-defined, since it seemingly depends on the choice of the generator s. In other words, if one would like to use the recurrence as the definition (as suggested by Brenti [2]), one must have a proof of the fact that the particular choices do not really matter.

Let us now return to the general setting of the generic Hecke algebras of section 2. Using the Hecke graph again, we shall prove the fact that in a generic Hecke algebra over A, the following general recurrence gives a proper definition of "R-polynomials" $R_{x,w}$:

For $s \in S$ such that $sw < w$ we define

$$R_{x,w} = \begin{cases} R_{sx,sw} & \text{if } sx < x, \\ a'_s R_{x,sw} + b'_s R_{sx,sw} & \text{if } sx > x \end{cases} \tag{2}$$

with initial conditions $R_{1,1} = 1$ and $R_{x,1} = 0$ for all $x \neq 1$. Here the a'_s and b'_s are new parameters in A, as usual with the restriction that parameters corresponding to conjugate generators be equal. Clearly, this recurrence specialises to the one of ordinary R-polynomials when we take $A = \mathbf{Z}[q, q^{-}1]$, and set $a_s = a'_s = q - 1$ and $b_s = b'_s = q$ for all $s \in S$.

As a matter of fact, to prove this we will need a slightly different version of the Hecke graph. Begin with the Hasse diagram of the weak order of W as before, but this time for every generator $s \in S$ add a loop labeled s to every node whose s-edge leads *upwards* instead of downwards. It should be immediate that Proposition 3.2 and 3.3 remain valid for this *dual Hecke graph.*

THEOREM 4.1. *In a generic Hecke algebra over A, the $R_{x,w}$ are unambiguously defined by the recurrence (2).*

PROOF. We shall prove the theorem by induction on $l(w)$; we know that $R_{x,w}$ is unambiguously defined when $w = 1$, i.e. when $l(w) = 0$. Suppose the theorem is proved for all $R_{x,v}$ with $l(w)$. Define

$$\sigma_s(R_{x,w}) = \begin{cases} R_{sx,sw} & \text{if } sx < x, \\ a'_s R_{x,sw} + b'_s R_{sx,sw} & \text{if } sx > x \end{cases}$$

It is sufficient to prove that if there are two different choices s and t of generators such that $sw < w$ and $tw < w$ (so w is the maximal coset representative of $W_{s,t}w$), then $\sigma_s(R_{x,w}) = \sigma_t(R_{x,w})$. Let u be the minimal coset representative, so $u = (sts\cdots)w = (tst\cdots)w$, and we have both $w > sw > tsw > \ldots > u$ and $w > tw > stw > tstw > \ldots > u$. Thus, by the induction hypothesis we have

$$\sigma_s(R_{x,w}) = \sigma_t\sigma_s(R_{x,w}) = \sigma_s\sigma_t\sigma_s(R_{x,w}) = \ldots = (\cdots\sigma_s\sigma_t\sigma_s)(R_{x,w})$$

where the last operator is an alternating composition of length $m(s,t)$. Analogously, we have

$$\sigma_t(R_{x,w}) = \ldots = (\cdots\sigma_t\sigma_s\sigma_t)(R_{x,w})$$

Consequently, it suffices to prove that $(\cdots\sigma_s\sigma_t\sigma_s)(R_{x,w}) = (\cdots\sigma_t\sigma_s\sigma_t)(R_{x,w})$. But the terms in this expansion correspond to alternating paths of length $m(s,t)$ from x in the dual Hecke graph essentially as before:

- The group element x' at the end point of the path gives an element $R_{x',u}$.

- Every step *downward* contributes a factor 1.
- Every step *upwards*, labeled s, contributes a factor b'_s.
- Every step in a *loop*, labeled s, contributes a factor a'_s.

We can now use Proposition 3.2 and 3.3 as in section 3, to deduce the desired equality.

REMARK 4.2. Also in this section there is the possibility of an additional parameter $c'_s \in A$, in analogy with the remark of section 3.

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