MATH. SCAND. 75 (1994), 59–66

# SOME NOTEWORTHY PROPERTIES OF ZERO DIVISORS IN INFINITE RINGS

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In a ring $R$ with no nonzero nilpotent elements, every zero divisor is two-sided, and its left and right annihilators coincide. If $R$ has nonzero nilpotents, the situation is considerably more complicated: $R$ may have one-sided zero divisors which are not two-sided, two-sided zero divisors with trivial two-sided annihilator, or nonnilpotent elements with nontrivial two-sided annihilator. Denote by $D = D(R)$, $T = T(R)$, $S = S(R)$, and $N = N(R)$ respectively the sets of zero divisors, two-sided zero divisors, zero divisors with nonzero two-sided annihilator, and nilpotent elements; and note that

$$D \supseteq T \supseteq S \supseteq N. \tag{1}$$

Assume throughout the paper that $R$ is infinite and $D \neq \{0\}$.

Ganesan [4, 5] proved long ago that $D$ must be infinite, and more recently Hirano [6] showed that $T$ must be infinite. It is clear that $N$ may be finite; but $S$ must be infinite – a fact which is immediate from Theorem 4 of [1]. In Section 1 we provide additional information about the set-theoretic structure of $D$ and $S$.

A recent extension of Ganesan's result [2] asserts that if $D \neq N$, then $D \backslash N$ must be infinite. This suggests investigating whether $S \backslash N$, $T \backslash S$, and $D \backslash T$ must be infinite, and we carry out such an investigation in Section 2. In the final section we construct examples probing the question of what combinations of equality and proper inclusion may in fact occur in (1). Of the eight theoretical possibilities, all except perhaps one really are possible.

For $Y$ an element or subset of $R$, we let $A_l(Y)$, $A_r(Y)$, and $A(Y)$ denote the left, right, and two-sided annihilators of $Y$; and we denote by $D_l = D_l(R)$ and $D_r = D_r(R)$ the sets of left and right zero divisors. We use the word "ideal" to mean two-sided ideal. When we say that a subring of $R$ has finite index, we mean that it has finite index as a subgroup of $(R, +)$.

Received July 30, 1993.

## 1. The Sets $D$, $T$, and $S$.

For arbitrary $y \in R$, considering the map $x \mapsto xy$ from $R$ onto $Ry$ shows that either $Ry$ is infinite, or $A_l(y)$ is of finite index and therefore infinite. In particular, if $y$ is a nonzero left zero divisor, then either $Ry$ or $A_l(y)$ is an infinite subset of $D$; hence $D$ is infinite. This proof is essentially given in [4] and [5], and we have repeated it for the reader's convenience. To say something more precise, we need the following result.

LEMMA 1. (i) *If $I$ is a finite left (right) ideal of $R$, then $A_l(I)$ $(A_r(I))$ is an ideal of $R$ of finite index.*

(ii) *If $I$ is a finite ideal, then $A(I)$ is an ideal of finite index.*

(iii) *If $y \in R$ and $A_l(y)$ has finite index, then $A_l(y)$ contains an ideal of finite index.*

(iv) *If $A(y)$ has finite index, it contains an ideal of finite index.*

PROOF. (i) Clearly $A_l(I)$ is an ideal. Moreover, for each $y \in I$, $Ry$ is finite, hence $A_l(y)$ has finite index. Since the intersection of finitely many subgroups of finite index is again of finite index, we conclude that $A_l(I) = \bigcap_{y \in I} A_l(y)$ has finite index.

(ii) If $I$ is a finite ideal, then $A_l(I)$ and $A_r(I)$ are both ideals of finite index, and so is $A_l(I) \cap A_r(I) = A(I)$.

(iii) Since $Ry$ is finite, $A_l(Ry)$ is an ideal of finite index by (i); and if $y \in Ry$, we are finished. Otherwise, we note that $A_l(Ry) \cap A_l(y)$ is of finite index, and it is an ideal because it is the left annihilator of the left ideal generated by $y$.

(iv) If $A(y)$ has finite index, so do $A_l(y)$ and $A_r(y)$; hence they contain ideals $I_1$ and $I_2$ of finite index by (iii). Then $I_1 \cap I_2$ is an ideal of finite index contained in $A(y)$.

COROLLARY 1. *The set $D$ is a union of nonzero one-sided ideals; and if one of these is finite, then $D$ contains an ideal of $R$ of finite index.*

PROOF. Obviously, $D$ is the union of the nonzero one-sided ideals $A_l(x)$, $x \in D_r \backslash \{0\}$ and $A_r(y)$, $y \in D_l \backslash \{0\}$. If one of these, say $A_l(u)$, is finite, then by Lemma 1 (i) $A_l(A_l(u))$ is an ideal of finite index which is contained in $D$.

REMARK. It is natural to ask what happens if all the above $A_l(x)$ and $A_r(y)$ are finite. In fact, by Theorem 4 of [1], this cannot occur.

With respect to the set $T$, note that it is the union of the subrings $A_l(y) \cap A_r(z)$ with $y, z \in R \backslash \{0\}$. If $R$ is prime, then for $x \in T$ we have $\{0\} \neq A_r(x)RA_l(x) \subseteq A_l(x) \cap A_r(x) = A(x)$; and therefore $T = S$. There are, however, prime rings for which $T \neq D$ – for instance, the ring of linear transformations of an infinite-dimensional vector space over a division ring, which has left invertible elements which are not invertible.

In a ring with no nonzero nilpotent elements, we have $D = T = S$; and for each $x \in R$, $A_l(x) = A_r(x) = A(x)$ is an ideal. Thus, in view of Lemma 1, we get

COROLLARY 2. *If $N = \{0\}$, then $D = S$ is a union of nonzero ideals; and if one of these ideals is finite, then $D$ contains an ideal of finite index.*

We proceed now to the case where $N$ is finite and nonzero.

THEOREM 1. *If $N$ is finite and nonzero, then for each $y \in N$, $A(y)$ contains an ideal of $R$ of finite index; and so does $A(N)$.*

PROOF. Clearly $R$ has no infinite zero subrings, hence by a result of [8], $A(y)$ is of finite index for each $y \in N$. The theorem now follows from Lemma 1 (iv) and the finiteness of $N$.

An immediate consequence is a result which we have recently established in another context.

COROLLARY 3 ([3, Theorem 3]). *If $R$ is an infinite prime ring with only finitely many nilpotent elements, then $R$ is a domain.*

We are now ready to state our structure theorem for $S$, which obviously contains the previously-mentioned result that $S$ must be infinite.

THEOREM 2. *The set $S$ is either a union of infinite ideals of $R$, or contains an ideal of $R$ of finite index, or contains an infinite zero ring. Moreover, if $A(x)$ has finite index for all $x \in S\backslash\{0\}$, then $S$ is an ideal of $R$ of finite index.*

PROOF. If $S$ is neither a union of infinite ideals nor contains an ideal of finite index, then by Corollary 2 and Theorem 1. $S$ must contain infinitely many nilpotent elements. But by a recent result of ours [7, Theorem 6], any ring with $N$ infinite contains an infinite zero ring, necessarily in $S$. The final assertion of our theorem is easily established, using Lemma 1 (iv).

REMARK. In [8] it is shown that for any ring $R$, the set $H(R) = \{x \in R \mid A(x)$ has finite index$\}$ is an ideal of $R$. In the final sentence of Theorem 2, $S$ is of course equal to $H(R)$.

## 2. The Sets $S\backslash N$, $T\backslash S$, and $D\backslash T$.

A subset of a ring is said to be *power closed* if it contains all positive powers of its elements; and it is said to be *root closed* if whenever it contains a positive power of an element, it also contains the element itself. It is clear that if $U \supseteq V$ are two power closed and root closed subsets of a ring, then $U\backslash V$ is also power closed and root closed.

LEMMA 2. *The sets $S\backslash N$, $T\backslash S$, and $D\backslash T$ are power closed and root closed.*

PROOF. It suffices to prove that each of $N$, $S$, $T$, and $D$ is power closed and root closed. All the parts of the proof are completely straightforward except showing that $S$ is root closed. Suppose, then, that $x^n \in S$ and $0 \neq y \in A(x^n)$. We aim to show that $A(x^{n-1}) \neq \{0\}$, so that root closure follows by backward induction. We may assume that $y \notin A(x)$ and, without loss, that $xy \neq 0$. If $xyx \neq 0$, then $xyx \in A(x^{n-1})\backslash\{0\}$; otherwise $x^k y \in A(x^{n-1})\backslash\{0\}$, where $k$ is the largest integer for which $x^k y \neq 0$.

LEMMA 3 ([9] or [3, Theorem 1]). *If $R$ is an infinite ring with $R \neq N$, then $R\backslash N$ is infinite.*

THEOREM 3. *If $S \neq N$, the $S\backslash N$ is infinite.*

PROOF. If $S\backslash N$ contains an element $a$ with infinitely many distinct powers, then Lemma 2 gives the result at once. Thus, we assume that for $a \in S\backslash N$ we have distinct positive integers $m$, $n$ for which $a^n = a^m$; and it follows by a standard argument that $S\backslash N$ contains a nonzero idempotent $e$. Consider the Pierce decomposition

$$R = eRe + eR(1-e) + (1-e)Re + (1-e)R(1-e),$$

the 1 being purely formal; and observe that each summand is contained in $S$. Since $R$ is infinite, at least one summand must be infinite.

Suppose first that $eRe$ is infinite. Since $e \in eRe$, $eRe \neq N(eRe)$; thus $eRe\backslash N(eRe)$ is infinite by Lemma 3, and $S\backslash N$ is infinite.

If $(1-e)R(1-e)$ is infinite and different from $N((1-e)R(1-e))$, the same argument works; so suppose $(1-e)R(1-e)$ is infinite and is equal to $N((1-e)R(1-e))$. For $y \in (1-e)R(1-e)$, $e(e+y) = e$, so $e + y \notin N$; and if $y^n = 0 \neq y^{n-1}$, then $y^{n-1} \in A(e+y)\backslash\{0\}$. Therefore $\{e + y \mid y \in (1-e)R(1-e)\}$ is an infinite subset of $S\backslash N$.

We complete the proof by discussing the case where $eR(1-e)$ is infinite, the fourth case being similar to it. Choose $v \neq 0$ in $A(e) = (1-e)R(1-e)$. For each $y \in eR(1-e)$, we have $(e+y)^2 = e + y \neq 0$ and $yv - v \in A(e+y)$; moreover, $yv \neq v$, since otherwise $0 = y^2 v = yv = v$. Therefore, $e + eR(1-e)$ is an infinite subset of $S\backslash N$.

We now proceed to construct two rings with $N$ finite, one showing that $T\backslash S$ may be finite and non-empty, the other showing that $D\backslash T$ may be finite and non-empty.

EXAMPLE 1. Let $V$ be an infinite Boolean ring without unit, so that all its elements are zero divisors; adjoin to $V$ a unit $e$ of additive order 2, and denote the extended ring by $W$. Let $U = \{0, 1\}$ be the 2-element group, and make it into a $(W, W)$-bimodule by defining $1e = e1 = 1$ and $V1 = 0 = 1V$. Let

$$R = \begin{bmatrix} W & U \\ U & 0 \end{bmatrix},$$

with the usual matrix multiplication and $UU = \{0\}$.

It is readily seen that $N = \begin{bmatrix} 0 & U \\ U & 0 \end{bmatrix}$, hence $|N| = 4$. All other elements with $(1, 1)$-entry different from $e$ are in $S$, for if $v \in V\backslash\{0\}$ and $v' \in V\backslash\{0\}$ such that $vv' = 0$, then $\begin{bmatrix} v' & 0 \\ 0 & 0 \end{bmatrix} \in A\left(\begin{bmatrix} v & u_1 \\ u_2 & 0 \end{bmatrix}\right)\backslash\{0\}$ and $\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \in A\left(\begin{bmatrix} e+v & u_1 \\ u_2 & 0 \end{bmatrix}\right)\backslash\{0\}$.

Each of the remaining four elements $\begin{bmatrix} e & U \\ U & 0 \end{bmatrix}$ has left annihilator $\begin{bmatrix} 0 & U \\ 0 & 0 \end{bmatrix}$ and right annihilator $\begin{bmatrix} 0 & 0 \\ U & 0 \end{bmatrix}$, so these four are the elements of $T\backslash S$. Note that in this example, $D = T$.

Example 2. *With $W$ and $U$ as in the previous example, take*

$$R = \begin{bmatrix} W & U \\ 0 & 0 \end{bmatrix}.$$

Then $|N| = |U| = 2$; and as above, all elements with $(1, 1)$-entry different from $e$ are in $S$. Only two elements remain, $\begin{bmatrix} e & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$; and these are left units with left annihilator $\begin{bmatrix} 0 & U \\ 0 & 0 \end{bmatrix}$, hence are in $D\backslash T$. Of course, $T = S$ in this example.

Theorem 4. *If $T \neq S$ and $N$ is infinite, then $T\backslash S$ is infinite.*

Proof. As in the proof of Theorem 3, we use Lemma 2 to reduce to the case where there exists a nonzero idempotent $e \in T\backslash S$. In this case,

$$A(e) = (1 - e)R(1 - e) = \{0\},$$

but $A_l(e) = R(1 - e) \neq \{0\}$ and $A_r(e) = (1 - e)R \neq \{0\}$. It follows that

$$(A_l(e))^2 = (A_r(e))^2 = A_r(e)A_l(e) = \{0\}, \tag{2}$$

and that

$$ey = y \quad \text{for all} \quad y \in A_l(e). \tag{3}$$

We show first that $e + A_l(e) \subseteq T\backslash S$.

It is easy to verify that for $b \in A_l(e)$, $A_l(e + b) = A_l(e)$; and for $x \in A_l(e + b) \cap A_r(e + b)$, it now follows from (2) and (3) that $(e + b)x = x = 0$, so that

$A(e+b) = \{0\}$. Now for $0 \neq c \in (1-e)R$, (2) and (3) yield $(e+b)(bc-c) = 0$; and the fact that $b^2 = 0$ makes it clear that $bc - c \neq 0$. We have now shown that $A_r(e+b) \neq \{0\}$, so that $e + b \in T \backslash S$ as claimed.

Of course a similar argument shows that $e + A_r(e) \subseteq T \backslash S$, so we may proceed under the assumption that $A_l(e)$ and $A_r(e)$ are both finite. Since $R = eRe + A_l(e) + A_r(e)$, it follows easily from (2) that $I = A_l(e) + A_r(e) + A_l(e)A_r(e)$ is a finite ideal of $R$ with $I^3 = \{0\}$. Since $N$ is infinite, we must have $N\left(\frac{R}{I}\right)$ infinite; and since $A_l(e) + A_r(e) \subseteq I$, every element of $\frac{R}{I}$ is of form $x + I$ with $x \in eRe$. It follows that $N(eRe)$ is infinite. We complete the proof by showing that $e + N(eRe) \subseteq T \backslash S$.

Clearly, $e + N(eRe) \subseteq T$, since $e \in T$. Moreover, for $y \in N(eRe)$, $e + y$ is invertible in $eRe$; consequently, $A_l(e+y) = A_l(e)$ and $A_r(e+y) = A_r(e)$, so that $A(e+y) = 0$ and $e + y \in T \backslash S$.

Recalling that $S$ is always infinite, we now get from Theorems 3 and 4 the following result.

COROLLARY 4. *If $T \neq N$, then $T \backslash N$ is infinite.*

THEOREM 5. *Let $D \backslash T$ be nonempty.*
(i) *If $N$ is infinite, then $D \backslash T$ is infinite.*
(ii) *If $N$ is finite, then $|D \backslash T| \geqq |N|$.*

PROOF. As before, we proceed at once to the case where $D \backslash T$ contains an idempotent $e$; and assume without loss that $A_l(e) = \{0\}$, so that $e$ is a right unit in $R$ and $\{0\} \neq A_r(e) = A_r(R)$. We need only show that $e + N \subseteq D \backslash T$.

If $b \in N$ with $b^{2k} = 0$, then $A_r(e+b) \supseteq A_r(R) \neq \{0\}$; moreover, if $x \in A_l(e+b)$, then $xe = -xb = -xeb = xb^2 = \ldots = xb^{2k} = 0$, so $A_l(e+b) = \{0\}$ and $e + b \in D \backslash T$ as required.

COROLLARY 5 [2, Theorem 1]. *If $D \neq N$, then $D \backslash N$ is infinite.*

PROOF. If $N$ is finite, we are finished, by Ganesan's original result. If $N$ is infinite, we apply Corollary 4 if $T \neq N$ and Theorem 5 if $T = N$.

COROLLARY 6. *Let $R$ have 1. If $T \neq S$, then $T \backslash S$ is infinite; and if $D \neq T$, then $D \backslash T$ is infinite.*

PROOF. If $R$ has 1, then for any idempotent $e \in D$, $e(1-e) = (1-e)e = 0$; hence $e \in S$ and $e$ is in neither of $D \backslash T$ and $T \backslash S$. Therefore, the first step of the proofs of Theorems 4 and 5 applies, without the hypothesis that $N$ is infinite.

COROLLARY 7. *If $R$ is prime (and infinite with $D \neq \{0\}$) and $D \neq T$, then $D \backslash T$ is infinite.*

PROOF. This follows immediately from Corollary 3 and Theorem 5.

## 3. Examples.

There are eight formal conditions obtainable by choosing sequences of equalities and proper inclusions in (1). In this section we give examples showing that each of them except

$$(\alpha) \qquad D \neq T \neq S = N$$

can in fact be satisfied. Whether there exists a ring satisfying ($\alpha$) we have been unable to determine.

All our examples have the property that whenever two adjacent sets in (1) are unequal, the corresponding difference set is infinite. That need not be the case, as Examples 1 and 2 show.

EXAMPLE 3. $D = T = S = N$. Any infinite nil ring will do.

EXAMPLE 4. $D = T = S \neq N$. The ring of $k \times k$ matrices over an infinite division ring, $k \geqq 2$, is in this class. In fact, so are all infinite semisimple artinian rings except division rings.

EXAMPLE 5. $D = T \neq S = N$. Let $V$ be the zero ring on the infinite cyclic group, regarded in the natural way as a $(\mathbf{Z}, \mathbf{Z})$-bimodule, where $\mathbf{Z}$ denotes the ring of integers; and let

$$R = \begin{bmatrix} \mathbf{Z} & V \\ V & 0 \end{bmatrix},$$

with the obvious multiplication. Clearly $N = \begin{bmatrix} 0 & V \\ V & 0 \end{bmatrix}$; and for $x = \begin{bmatrix} m & v \\ v' & 0 \end{bmatrix}$ with $m \in \mathbf{Z}\backslash\{0\}$, $A_l(x) = \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}$ and $A_r(x) = \begin{bmatrix} 0 & 0 \\ V & 0 \end{bmatrix}$, hence $x \in T\backslash S$.

EXAMPLE 6. $D \neq T = S = N$. Take $R = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$, where $F$ is any infinite field.

EXAMPLE 7. $D = T \neq S \neq N$. Let $F$ be an infinite field, and let

$$R = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}.$$

If $x \in R$ has both components nilpotent, then $x \in N$; and if $x$ has exactly one component nilpotent, then $x \in S\backslash N$. Finally, let $x = \left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix}\right)$ with

$a \neq 0 \neq d$. Then $A_r(x) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}\right)$ and $A_l(x) = \left(\begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)$, hence $x \in T \backslash S$.

EXAMPLE 8. $D \neq T = S \neq N$. As previously noted, the ring of linear transformations on an infinite-dimensional vector space is in this class.

EXAMPLE 9. $D \neq T \neq S \neq N$. Let $R_1$ and $R_2$ be infinite rings such that $D_l(R_1) \neq T(R_1)$, $D_r(R_2) \neq T(R_2)$ and $R_2$ has 1. (See Examples 6 and 8.) Take $R = R_1 \oplus R_2$. Clearly $S(R) \neq N(R)$. If $u_1 \in D_l(R_1) \backslash D_r(R_1)$ and $u_2 \in D_r(R_2) \backslash D_l(R_2)$, then $(u_1, u_2) \in T(R) \backslash S(R)$, for $A_r((u_1, u_2)) = A_r(u_1) \oplus 0$ and $A_l((u_1, u_2)) = 0 \oplus A_l(u_2)$. Finally, if $u_1 \in D_l(R_1) \backslash D_r(R_1)$, then $(u_1, 1) \in D_l(R) \backslash D_r(R)$.

ACKNOWLEDGEMENT. The second author acknowledges support from the Natural Sciences and Engineering Research Council of Canada, Grant No. A3961.

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