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THE HOMOLOGICAL ALGEBRA OF ARTIN GROUPS

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Following Magnus [25] and Brieskorn-Saito [8], an Artin group is an abstract group G defined by presentation as follows. G has as generators a set X. The defining relations of G may be described as follows: to certain unordered pairs $\{x, y\}$ of distinct elements of X, an integer $m = m_{xy} \geqq 2$ is assigned; the corresponding defining relation is $(xy)^q = (yx)^q$ if $m = 2q$ is even and $(xy)^q x = (yx)^q y$ if $m = 2q + 1$ is odd. An important example is Artin's braid group $B^{(n)}$ which is presented as follows: $X = \{s_1, \ldots, s_{n-1}\}$ with defining relations all $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and all $s_i s_j = s_j s_i$ whenever $|i - j| \geqq 2$.

Associated to each Artin group G is a Coxeter group W obtained from G by imposing, for each $x \in X$, the additional relation $x^2 = 1$. For example, the Coxeter group associated with $B^{(n)}$ is the symmetric group on n symbols.

It is well-known that each Coxeter group W whose generating set X (as above) has finite cardinality $n - 1$ admits a faithful linear representation in $\mathrm{GL}_n(\mathsf{R})$. See, for example, [6] or [16]. It turns out that the space of regular orbits of the extension of this representation to $\mathrm{GL}_n(\mathsf{C})$ is, if W is finite, an Eilenberg-Maclane space for the corresponding Artin group G. (For $B^{(n)}$, see [19]; for all but a few special cases, see [7]; in general, see [17].) This fact has been the foundation for the study of the homological algebra of these groups. (For a complete description of the ordinary integral cohomology of $B^{(n)}$, see [30], which is basedon [2] and [20]. For certain of the other Artin groups, see [23].) We remark that the homological algebra of $B^{(n)}$ has also been treated by homotopy-theoretic methods; see, for example, [26], [13] and [14].

Our purpose here is to give the foundations for a purely algebraic treatment of the homological algebra of Artin groups. Our approach is patterned after methods used by Garside [21] to solve the conjugacy problem in $B^{(n)}$ and used again by Brieskorn-Saito [8] to extend Garside's results to any Artin group whose associated Coxeter group is finite: instead of the Artin group G, we consider the monoid S with the same presentation as G (the Artin monoid). Our

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main result, (7.5) below, is a free resolution of certain left modules over the monoid ring RS of an Artin monoid S (with coefficients in an arbitrary commutative ring R). These RS-modules have as underlying R-module R itself with the action of S defined as follows: each $x \in X$ acts as multiplication by a fixed but arbitrary element α of R. In particular, these RS-modules include R as a trivial RS-module ($\alpha = 1$) so that the ordinary homological algebra of S can be recovered. (We remark that, in somewhat greater generality, the special case $\alpha = 0$ will play an important role in §3 below. We also remark on a slight difference in notation between here and §7: α here will turn into $-\alpha$ in §7.)

The question arises as to when the RS-resolutions descrived above extend to RG-resolutions of suitable RG-modules. It turns out that if the Coxeter group W is locally finite, then the extension of scalars functor from the category of left RS-modules to the category of left RG-modules is exact (see §2 and §7). The following is an easy consequence of this remark and the actual descriptions of the resolutions discussed in the previous paragraph:

THEOREM A. *Let G be an Artin group whose associated Coxeter group is finite. Then G is of type* FL [9, p. 199].

For the proof of Theorem A, see (7.7) and the proof of (7.8). Theorem A also follows from [17]; the Eilenberg-Maclane space for G described in [17] is clearly homotopy-equivalent to a finite complex. A consequence of Theorem A is the fact that G (as in Theorem A) is torsion-free (7.8). With a little bit more work (see §8), we are able to show:

THEOREM B. *Let G be as in Theorem* A. *Then G is a duality group* [3, p. 138].

See (8.3). In this generality, Theorem B is new. For example, the fact that Artin's braid group $B^{(n)}$ is a duality group follows from Proposition VIII.10.2 of [9], using the fact that $B^{(n)}$ is torsion-free (which follows from [19] cited above) and the fact that $B^{(n)}$ has a subgroup of finite index which is a duality group (the kernel of the natural homomorphism from $B^{(n)}$ to the symmetric group on n symbols is an iterated semidirect product of free groups: for this fact, see Lemma 1.8.2 of [4]; for the fact that semidirect products of duality groups are duality groups, see Theorem 9.10 of [3]; for the fact that free groups are duality groups; see p. 233 of [9].)

An obvious question is: what happens when the associated Coxeter group is not locally finite? Combining (2.4) with (7.6), it follows that if G is an Artin group whose Coxeter group is not locally finite, then the extension of scalars functor from RS-modules to RG-modules is not exact. Nonetheless, the author conjectures that if α is a unit in R, then the extension of the RS-complex (C_*, ∂_α) in (7.5) from RS to RG is exact in positive dimension.

For convenience, this paper has been divided into three parts. The first part consists of the author's attempt to extract from the proof of Theorem A those ideas that do not essentially involve the fact that G is an Artin group (or, alternately, the fact that S is an Artin monoid). The author hopes that these ideas will prove useful in other contexts. The second part concerns Artin groups; in particular, the theory developed in part I is applied to Artin groups in order to give proofs of Theorems A and B. The third part treats examples. Here is an outline:

Each of parts I, II and III will include their own introduction.

Part of this paper (essentially Theorem A and its proof) was the main result in the author's Ph.D. dissertation [27] written under John Stallings. The author would like to thank Professor Stallings for his continued patient support and for amny helpful suggestions.

Part I. Homological machinery.

Part I develops some homological preliminaries that will be applied to Artin groups in Part II. §1 introduces the notion of a diagram (1.2). (Essentially, a diagram is a functor $K \to \Lambda$. Here, K is a simplicial complex, with the "empty simplex" adjoined, viewed as a category with objects the simplexes of K and morphisms inclusion. Also, Λ is an associative ring with 1 viewed as a category with one object, morphisms Λ and composition given by multiplication in Λ.) Out of a diagram, we build a chain complex (C_*, ∂_*): see (1.3) and (1.4); and a cochain complex (C^*, ∂^*): see (1.5) and (1.6). An important goal below will be to study exactness properties of the complexes we have just described: see §3 and §7. We conclude §1 with a sample exactness theorem (1.7).

§2 consists primarily of standard facts concerning the extension of scalars functor from the category of left RS-modules to the category of left RG-modules. Here, G is a group, S is a submonoid of G and R is a commutative ring with 1.

In §3, we define the notion of an "exact" subset X of a monoid S (3.1), use this notion to define a diagram in RS and show that the resulting complex (C_*, ∂_*) of RS-modules is exact in positive dimension (3.2). If, in addition, X is finite and satisfies the "duality" condition (3.3), then the dual complex (C^*, ∂^*) is exact in positive codimension (3.5).

It turns out that the exactness theorems (3.2) and (3.5) are not very useful in homological group theory; in particular, see (3.6). Nonetheless, the resolutions (C_*, ∂_*) and (C^*, ∂^*) in §3, when they occur in the seting of Artin groups (see §7 and §8), turn out to be "top-degree approximations" of complexes which, by (4.1), are resolutions; these resolutions turn out to be useful in homological group theory. The proof of (4.1) is essentially an adaption of Stallings' notion [28] of a "slow contracting homotopy" to the situation at hand. Our excessive concern in §4 with the fact that certain R-modules are *free* R-modules, as in (4.2), (4.3b) and (4.4b), results from the current state of knowledge about duality groups; for further discussion, see the introduction to Part II and §8.

§1. *Diagrams.*

Let X be a set and let $<$ be a strict local ordering of X ($<$ is transitive, irreflexive and satisfies the law of trichotomy). If A is a finite subset of X and $x \in X$, then $x(A)$ will denote the number of $y \in A$ such that $y < x$. Let $A \subseteq X$ and $x \in X$. If $x \in A$, then $A - x$ will denote the set difference $A - \{x\}$. If $x \notin A$, then $A + x$ will denote the set union $A \cup \{x\}$.

(1.1) LEMMA. *Let A be a finite subset X and let $x, y \in X$.*
(a) *If $x, y \in A$ and $x \neq y$, then* $(-1)^{x(A)+y(A-x)} + (-1)^{y(A)+x(A-y)} = 0$.
(b) *If $x \in A$ and $y \notin A$, then* $(-1)^{x(A)+y(A-x)} + (-1)^{y(A)+x(A+y)} = 0$.
(c) *If $x, y \notin A$ and $x \neq y$, then* $(-1)^{x(A)+y(A+x)} + (-1)^{y(A)+x(A+y)} = 0$.

PROOF. In a), we may assume, by symmetry, that $x < y$. Then $y(A - x) = y(A) - 1$ and $x(A - y) = x(A)$, so the exponents differ by 1, as required. The proofs of b) and c) are similar.

Given X as above, let $\mathscr{P}$ be a collection of finite subsets of X which satisfy:
P0) $\emptyset \in \mathscr{P}$.
P1) for each $x \in X$, $\{x\} \in \mathscr{P}$.
P2) if $A \in \mathscr{P}$ and $B \subseteq A$, then $B \in \mathscr{P}$.

Except for P0), this is the definition of a simplicial complex. Given X and $\mathscr{P}$ as above, define $\mathscr{P}^{(2)} = \{(A, B) \mid A \in \mathscr{P} \text{ and } B \subseteq A\}$. Let Λ be an associative ring (with 1).

(1.2) DEFINITION. A *diagram* of $(X, \mathscr{P})$ in Λ is a function $D\colon \mathscr{P}^{(2)} \to \Lambda$ which satisfies

a) if $A \in \mathscr{P}$, then $D(A, A) = 1$.
b) if $C \subseteq B \subseteq A \in \mathscr{P}$, then $D(A, C) = D(A, B)D(B, C)$.

If $\mathscr{P}$ consists of all finite subsets of X, then D will be called a *full* diagram. Note that if D is a diagram of $(X, \mathscr{P})$ in Λ and $\phi\colon \Lambda \to \Lambda'$ is a ring homomorphism (preserving 1), then the formula $D_\phi(A, B) = \phi(D(A, B))$ defines a diagram of $(X, \mathscr{P})$ in Λ'.

With X, $\mathscr{P}$, Λ and D as above, let C_k denote the free left Λ-module with a generator denoted $[A]$ corresponding to each k-element $A \in \mathscr{P}$.

(1.3) DEFINITION. For $k > 0$, *define* $\partial_k\colon C_k \to C_{k-1}$ on generators by

$$\partial_k([A]) = \sum_{x \in A} (-1)^{x(A)} D(A, A - x)[A - x]$$

and extend to C_k by Λ-linearity.

(1.4) LEMMA. *If* $k \geqq 2$, *then* $\partial_{k-1}\partial_k = 0$.

PROOF. Let $A \in \mathscr{P}$ have cardinality $\geqq 2$, let $x, y \in A$ and suppose that $x \neq y$. In the expansion of $\partial_{k-1}\partial_k([A])$, $[A - x - y]$ appears twice; the coefficients are the same by (1.2b) and the signs are opposite by (1.1a).

In particular, the pair (C_*, ∂_*) is a Λ-complex. We call ∂_* the *differential* associated to D. We shall be interested in sufficient conditions for (C_*, ∂_*) to be exact in positive dimension: if $k > 0$, then $\ker \partial_k = \operatorname{im} \partial_{k+1}$. In this situation, we call D *exact* and call $C_0/\operatorname{im} \partial_1$ the *module* of D.

With the same notation as above, assume further that X is finite. The cardinality of X will be denoted $n - 1$. In this situation, let C^k denote the free right Λ-module with a generator denoted $\langle A \rangle$ corresponding to each k-element $A \in \mathscr{P}$.

(1.5) DEFINITION. For $0 \leqq k < n - 1$, *define* $\partial^k\colon C^k \to C^{k+1}$ on generators by

$$\partial^k(\langle A \rangle) = \sum_{x \notin A} (-1)^{x(A)} \langle A + x \rangle D(A + x, A)$$

and extend to C^k by Λ-linearity. (Here, we will be most interested in the situation when D is a full diagram; otherwise, we adopt the convention that $\langle A + x \rangle = 0$ whenever $A + x \notin \mathscr{P}$.)

(1.6) LEMMA. *If* $0 \leqq k < n - 2$, *then* $\partial^{k+1}\partial^k = 0$.

PROOF. Mimic the proof of (1.4), using (1.1c) in place of (1.1a).

The cochain complex (C^*, ∂^*) may be naturally identified with the Λ-dual of (C_*, ∂_*): $C^k = \operatorname{Hom}_\Lambda(C_k, \Lambda)$ with its natural right Λ-module structure. Under this identification, $\langle A \rangle$ is given by

$$\langle A\rangle([B]) = \begin{cases} 1 & \text{if } B = A \\ 0 & \text{otherwise.} \end{cases}$$

We shall be interested in sufficient conditions for (C^*, ∂^*) to be exact in positive codimension: if $0 < k < n - 1$, then $\ker \partial^k = \operatorname{im} \partial^{k-1}$. In this situation, we call D *co-exact* and call $C^{n-1}/\operatorname{im} \partial^{n-2}$ the *co-module of* D.

We conclude this section by giving a sufficient condition for a diagram D to be exact in all dimensions. (In other words, D is exact in positive dimension, as above, and the module of D is the zero-module.) With X, $\mathscr{P}$, Λ and D as above, $z \in X$ is called a *cone-point* for D provided for each $A \in \mathscr{P}$, either $z \in A$ or $A + z \in \mathscr{P}$ and, in the second case, $D(A + z, A)$ is a unit of Λ.

(1.7) THEOREM. *If the diagram D has a cone-point z, then (C_*, ∂_*) is exact in all dimensions.*

PROOF. For each $k \geqq 0$, define $s_k: C_k \to C_{k+1}$ by

$$s_k([A]) = \begin{cases} 0 & \text{if } z \in A \\ (-1)^{z(A)} D(A + z, A)^{-1}[A + z] & \text{if } z \notin A \end{cases}$$

and extend to C_k by Λ-linearity. Defining $C_{-1} = 0$, $\partial_0 = 0$ and $s_{-1} = 0$, we show that if $k \geqq 0$ and $W \in C_k$, then $(s_{k-1}\partial_k + \partial_{k+1}s_k)(W) = W$ from which (1.7) follows easily. Since each s_k and ∂_k is Λ-linear, it suffices to check that for each k-element $A \in \mathscr{P}$, $(s_{k-1}\partial_k + \partial_{k+1}s_k)([A]) = [A]$. There are two cases: $z \in A$ and $z \notin A$. For the remainder of the proof, we will omit subscripts from ∂ and s.

If $z \in A$, then $s([A]) = 0$ so that

$$\begin{aligned} (s\partial + \partial s)([A]) &= s\partial([A]) \\ &= s\left(\sum_{x \in A} (-1)^{x(A)} D(A, A - x)[A - x]\right) \\ &= s((-1)^{z(A)} D(A, A - z)[A - z]) \\ &= [A] \end{aligned}$$

where the third equality follows from the fact that $s([A - x]) = 0$ if $x \neq z$.

If $z \notin A$ (in particular, if $A = [\emptyset]$), then

$$\begin{aligned} \partial s([A]) &= \partial((-1)^{z(A)} D(A + z, A)^{-1}[A + z]) \\ &= \sum_{x \in A + z} (-1)^{z(A) + x(A+z)} D(A + z, A)^{-1} D(A + z, A + z - x)[A + z - x] \\ &= [A] - \sum_{x \in A} (-1)^{z(A-x) + x(A)} D(A + z, A)^{-1} D(A + z, A + z - x) \\ &\quad \times [A + z - x] \end{aligned}$$

where the third inequality follows from (1.1b). Also,

$$s\partial([A]) = s\left(\sum_{x\in A}(-1)^{x(A)}D(A, A-x)[A-x]\right)$$
$$= \sum_{x\in A}(-1)^{x(A)+z(A-x)}D(A, A-x)D(A-x+z, A-x)^{-1}[A-x+z].$$

Since $D(A+z, A)^{-1}D(A+z, A+z-x) = D(A, A-x)D(A-x+z, A-x)^{-1}$ follows from (1.2b), we have $(\partial s + s\partial)([A]) = [A]$ here as well.

§2. *Monoids and homological group theory.*

Let G be a group and let S be a submonoid of G. In other words, S is a multiplicatively closed subset of G which contains the identity element 1 of G. In particular, S is a two-sided cancellation monoid: if $u, v, w \in S$ and either $uw = vw$ or $wu = wv$, then $u = v$. With G and S as above, we say that S *fills* G (on the left) provided for each $g \in G$, there exist $u, v \in S$ such that $g = u^{-1}v$.

(2.1) LEMMA. *If S fills G, then for each $x, y \in S$, there exist $u, s \in S$ such that $ux = vy$ in S.*

PROOF. Given $x, y \in S$, let $g = xy^{-1} \in G$. By hypothesis, there exist $u, v \in S$ such that $g = u^{-1}v$. It follows easily that $ux = vy$ in S, as required.

If S is a monoid and $x, y \in S$, then $w \in S$ is called a *common left multiple* of x and y provided there exist, $u, v \in S$ such that $w = ux = vy$ in S. A two-sided cancellation monoid in which every pair of elements has a common left multiple is said to satisfy the *left Ore condition.* By (2.1), if a monoid S fills a group G, then S satisfies the left Ore condition. This fact has the following converse:

(2.2) THEOREM. *Suppose that the monoid S satisfies the left Ore condition. Then, up to isomorphism, there exists a unique group G such that S fills G.*

PROOF. For a construction of G, see [12, p. 35]. From [12], it is also clear that any monoid homomorphism from S to a group H factors through a group homomorphism from G to H. To prove uniqueness of G, it suffices to show that if $\phi\colon G \to H$ is a group homomorphism whose restriction to S is injective, then ϕ is injective. But if $g \in G$ satisfies $\phi(g) = 1$, then, writing $g = u^{-1}v$ with $u, v \in S$, it follows that $\phi(u) = \phi(v)$. By hypothesis, $u = v$ in S so $g = 1$, as required.

We turn to homological properties of groups and monoids that fill them. Let R be a commutative ring (with 1), let G be a group and let S be a submonoid of G. Then the monoid ring RS is a subring of the group ring RG. Thus, by "restriction of scalars", left RG-modules can be viewed as left RS-modules. Conversely, viewing RG as a left RG-, right RS-module, there is an "extension of scalars" functor $(-)\mapsto RG \otimes_{RS}(-)$ from the category of left RS-modules to the category of left RG-modules.

(2.3) THEOREM. *If S fills G, then the functor $(-)\mapsto RG \otimes_{RS}(-)$ is exact.*

PROOF. See [10, p. 191]. Briefly, the proof of (2.3) proceeds by showing that, as a right RS-module, RG is the direct limit of the system $\{w^{-1}(RS) \mid w \in S\}$ of right RS-modules. Clearly, each $w^{-1}(RS)$ is a free right RS-module. By (2.1), the system above is directed and has limit RG. It follows that RG is a flat right RS-module so that $(-)\mapsto RG \otimes_{RS}(-)$ is exact.

We conclude this section by noting that (2.3) has the following converse:

(2.4) THEOREM. *Let G be a group and let S be a submonoid of G. If S does not fill the subgroup of G that it generates, then the functor $(-)\mapsto RG \otimes_{RS}(-)$ is not exact.*

PROOF. Clearly, S is a two-sided cancellation monoid. By hypothesis and (2.2), S cannot satisfy the Ore condition. It follows that there exist $x, y \in S$ such that for all $u, v \in S$, $ux \neq vy$. Let M denote the free left RS-module with basis $\{e_1, e_2\}$. Define a left RS-module homomorphism ϕ: $M \to RS$ by $\phi(e_1) = x$ and $\phi(e_2) = y$. Clearly, ϕ is injective. Note that $RG \otimes_{RS} M$ is a free left RG-module with basis $\{1 \otimes e_1, 1 \otimes e_2\}$. Also note that $x^{-1} \otimes e_1 - y^{-1} \otimes e_2 \neq 0$ in $RG \otimes_{RS} M$ and belongs to the kernel of the extension of ϕ to $RG \otimes_{RS} M$. It follows that the extension of ϕ to $RG \otimes_{RS} M$ is not injective, so that $(-)\mapsto RG \otimes_{RS}(-)$ is not exact.

§3. *Exactness and duality.*

Let S be a monoid and let A be a subset of S. We call $w \in S$ a *common left multiple* of A provided there is a function f: $A \to S$ such that for each $a \in A$, $w = f(a)a$. (Note that any element of S is a common left multiple of the empty set $\emptyset$ and that if $B \subseteq A$, then a common left multiple of A is a common left multiple of B.) We call $w \in S$ a *least common left multiple* of A provided w is a common left multiple of A and for each common left multiple u of A, there exists $v \in S$ such that $u = vw$. (It follows that the identity $1 \in S$ is a least common left multiple of $\emptyset$ and that if $x \in X$, then x is a least common left of $\{x\}$.)

(3.1) DEFINITION. Let S be a monoid. A subset X of S is called *exact* provided any finite subset of X which has a common left multiple in S also has a least common left multiple in S.

If X is an exact subset of the monoid S, define $\mathscr{P}$ to consist of all finite subsets of X which have a common left multiple in S. It follows from the remarks preceding

(3.1) that $\mathscr{P}$ satisfies conditions P0), P1) and P2) of §1. Assume that for each $A \in \mathscr{P}$ a least common left multiple Δ_A of A in S has been chosen such that $\Delta_\phi = 1$ and if $x \in X$, then $\Delta_{\{x\}} = x$. Note that if $A \in \mathscr{P}$ and $B \subseteq A$, then Δ_A is a left multiple of Δ_B.

Let S be a right cancellation monoid: if $x, y, z \in S$ satisfy $xz = yz$, then $x = y$. Equivalently, if $x \in S$ is a left multiple of $z \in S$, then there exists a unique $y \in S$ such that $x = yz$; in this situation, we write $y = xz^{-1}$. Clearly, if $x \in S$, then x is a left multiple of x and $xx^{-1} = 1$. Similarly, if $x, y, z \in S$ satisfy: x is a left multiple of y and y is a left multiple of z, then x is a left multiple of z and $xz^{-1} = (xy^{-1})(yz^{-1})$.

Let S be a right cancellation monoid, let X be an exact subset of S and let R be a commutative ring (with 1). Define $D_X\colon \mathscr{P}^{(2)} \to RS$ by $D_X(A, B) = \Delta_A \Delta_B^{-1}$. (As noted above, if $B \subseteq A \in \mathscr{P}$, then Δ_A is a left multiple of Δ_B.) Clearly, (1.2) is satisfied, so that D_X is a diagram of $(X, \mathscr{P})$ in RS.

(3.2) THEOREM. *If S is a right cancellation monoid and X is an exact subset of S, then D_X is an exact diagram.*

PROOF. Totally order X. If $A \in \mathscr{P}$ has $k (> 0)$ elements, then

$$\partial_k([A]) = \sum_{x \in A} (-1)^{x(A)} \Delta_A \Delta_{A-x}^{-1} [A - x].$$

We define for each $k \geqq 0$ an R-linear homomorphism $s_k\colon C_k \to C_{k+1}$ such that if $w \in S$ and $\emptyset \neq A \in \mathscr{P}$, then

$$(\partial_{k+1} s_k + s_{k-1} \partial_k)(w[A]) = w[A]$$

where A has k elements. It will follow that if $k > 0$, then $\ker \partial_k = \operatorname{im} \partial_{k+1}$, as required.

To define s_k, first define $\xi\colon S \to X \cup \{0\}$ as follows: if $w \in S$, then $\xi(w) = 0$ if and only if for each $x \in X$, w is not a left multiple of x. Otherwise, $\xi(w)$ is a chosen element of X of which w is a left multiple (so that $w\xi(w)^{-1}$ is defined).

As a left R-module, C_k is free on all $w[A]$ as w ranges over all elements of S and A ranges over all k-element sets in $\mathscr{P}$. With this in mind, define

$$s_k(w[A]) = \begin{cases} 0 & \text{if } \xi(w\Delta_A) \in A \cup \{0\} \\ (-1)^{z(A)} w \Delta_A \Delta_{A+z}^{-1} [A + z] & \text{if } z = \xi(w\Delta_A) \notin A \cup \{0\} \end{cases}$$

and extend to C_k by R-linearity. (If $z = \xi(w\Delta_A) \notin A \cup \{0\}$, then there exists $w_1 \in S$ such that $w\Delta_A = w_1 z$, so that $w\Delta_A$ is a common left multiple of $A + z$ and therefore a left multiple of Δ_{A+z}. Also, $\xi(w\Delta_A) = 0$ can only arise if $A = \emptyset$: if $A \neq \emptyset$, then $\xi(w\Delta_A) \in X$.)

For the remainder of the proof, we omit subscripts from s and ∂. Let $w \in S$ and $\emptyset \neq A \in \mathscr{P}$. Since $A \neq \emptyset$, there are two cases: $\xi(w\Delta_A) \in A$ and $\xi(w\Delta_A) \notin A$. If $z = \xi(w\Delta_A) \in A$, then $s(w[A]) = 0$ so that

$$\begin{aligned}(s\partial + \partial s)(w[A]) &= s\partial(w[A])\\ &= s\left(\sum_{x\in A}(-1)^{x(A)}w\Delta_A\Delta_{A-x}^{-1}[A-x]\right)\\ &= s((-1)^{z(A)}w\Delta_A\Delta_{A-z}^{-1}[A-z])\\ &= w[A]\end{aligned}$$

where the third equality follows from the fact that $z = \xi((w\Delta_A\Delta_{A-x}^{-1})\Delta_{A-x}) \in A - x$ if $x \neq z$. If $z = \xi(w\Delta_A) \notin A$, then

$$\begin{aligned}\partial s(w[A]) &= \partial((-1)^{z(A)}w\Delta_A\Delta_{A+z}^{-1}[A+z])\\ &= \sum_{x\in A+z}(-1)^{z(A)+x(A+z)}w\Delta_A\Delta_{A+z-x}^{-1}[A+z-x]\\ &= w[A] - \sum_{x\in A}(-1)^{z(A-x)+x(A)}w\Delta_A\Delta_{A+z-x}^{-1}[A+z-x]\end{aligned}$$

where the third equality uses (1.1b). Also

$$\begin{aligned}s\partial(w[A]) &= s\left(\sum_{x\in A}(-1)^{x(A)}w\Delta_A\Delta_{A-x}^{-1}[A-x]\right)\\ &= \sum_{x\in A}(-1)^{x(A)+z(A-x)}w\Delta_A\Delta_{A-x+z}^{-1}[A-x+z]\end{aligned}$$

since for each $x \in A$, $z \notin A - x$. Thus $(s\partial + \partial s)(w[A]) = w[A]$ in this case as well.

The module $C_0/\operatorname{im}\partial_1$ of D_X can be described as follows: $C_0/\operatorname{im}\partial_1$ is isomorphic to RS modulo the left ideal generated by X. It follows that $C_0/\operatorname{im}\partial_1$ is a free R-module with basis corresponding to $\{w \in S \mid \xi(w) = 0\}$. (Note that if S is a group and $X \neq \emptyset$, then the module of D_X is the zero-module. (See Lemma (4.2).)

In the following definition, we use the notion of a (least) common right multiple.

(3.3) Definition. Let S be a right cancellation monoid and let $X \subseteq S$ be exact. Then X is said to satisfy the *duality condition* provided whenever $A \in \mathscr{P}$ and $x, y \in A$ satisfy $x \neq y$, it follows that $\Delta_A\Delta_{A-x-y}^{-1}$ is a least common right multiple of $\{\Delta_A\Delta_{A-x}^{-1}, \Delta_A\Delta_{A-y}^{-1}\}$.

It follows from the right cancellation property in S that the duality condition for X is independent of the choice of Δ_A's.

(3.4) Lemma. *Let S be a two-sided cancellation monoid and suppose that $X \subseteq S$ satisfies the duality condition. If $A \in \mathscr{P}$ and $B \subseteq A$, then $\Delta_A\Delta_B^{-1}$ is a least common right multiple of $\{\Delta_A\Delta_{A-z}^{-1} \mid z \in A - B\}$.*

PROOF. Note that if $z \in A - B$, then $B \subseteq A - z$ so that $\Delta_A \Delta_B^{-1} = (\Delta_A \Delta_{A-z}^{-1})(\Delta_{A-z} \Delta_B^{-1})$. Thus $\Delta_A \Delta_B^{-1}$ is a common right multiple of $\{\Delta_A \Delta_{A-z}^{-1} \mid z \in A - B\}$.

We prove (3.4) by induction on the cardinality of $A - B$. If this cardinality is 0 or 1, (3.4) is easy. In general, choose $x, y \in A - B$ with $x \neq y$. By the inductive hypothesis, $\Delta_A \Delta_{B+x}^{-1}$ is a least common right multiple of $\{\Delta_A \Delta_{A-z}^{-1} \mid z \in A - B - x\}$ and $\Delta_A \Delta_{B+y}^{-1}$ is a least common right multiple of $\{\Delta_A \Delta_{A-z}^{-1} \mid z \in A - B - y\}$.

Assume that $w \in S$ is a common right multiple of $\{\Delta_A \Delta_{A-z}^{-1} \mid z \in A - B\}$. Thus for each $z \in A - B$, there exists $w_z \in S$ such that $w = \Delta_A \Delta_{A-z}^{-1} w_z$. Since $\Delta_A \Delta_{B+x}^{-1}$ is a least common right multiple of $\{\Delta_A \Delta_{A-z}^{-1} \mid z \in A - B - x\}$, there exists $u \in S$ such that $w = \Delta_A \Delta_{B+x}^{-1} u$. Similarly, there exists $v \in S$ such that $w = \Delta_A \Delta_{B+y}^{-1} v$. It follows that

$$\begin{aligned}
(\Delta_A \Delta_{B+x+y}^{-1})(\Delta_{B+x+y} \Delta_{B+x}^{-1})u &= \Delta_A \Delta_{B+x}^{-1} u \\
&= w \\
&= \Delta_A \Delta_{B+y}^{-1} v \\
&= (\Delta_A \Delta_{B+x+y}^{-1})(\Delta_{B+x+y} \Delta_{B+y}^{-1})v
\end{aligned}$$

so that, by left cancellation, $(\Delta_{B+x+y} \Delta_{B+x}^{-1})u = (\Delta_{B+x+y} \Delta_{B+y}^{-1})v$. By (3.3), there exists $w_0 \in S$ such that $(\Delta_{B+x+y} \Delta_{B+x}^{-1})u = (\Delta_{B+x+y} \Delta_B^{-1})w_0$. Clearly, $w = (\Delta_A \Delta_B^{-1})w_0$, as required.

(3.5) THEOREM. *Let S be a two-sided cancellation monoid and let X be a finite subset of S which has a common left multiple and satisfies the duality condition. Then D_X is co-exact.*

PROOF. We use (3.4) to reduce the proof of (3.5) to (3.2). For each $x \in X$, define $\tilde{x} = \Delta_X \Delta_{X-x}^{-1}$ and for each $A \subseteq X$, define $\tilde{A} = \{\tilde{x} \mid x \in A\}$. By (3.5), if $\tilde{A} \subseteq \tilde{X}$, then $\Delta_X \Delta_{X-A}^{-1}$ is a least common right multiple of $\tilde{A}$, so that $\tilde{X}$ is exact (on the right). Thus (3.2) applies: the (right) diagram $D_{\tilde{X}}$ is exact. The associated complex, which we denote $(\tilde{C}_*, \tilde{\partial}_*)$, may be described as follows: each $\tilde{C}_k$ is a free right RS-module on all $[\tilde{A}]$ for A a k-element subset of X and

$$\tilde{\partial}_k([\tilde{A}]) = \sum_{x \in A} (-1)^{x(A)} [A - x]^{\sim} \Delta_{X-A+x} \Delta_{X-A}^{-1}.$$

We show that the complexes $(\tilde{C}_*, \tilde{\partial}_*)$ and (C^*, ∂^*) are chain isomorphic. Let $n - 1$ denote the cardinality of X and for $0 \leqq k \leqq n - 1$, define $\phi_k \colon \tilde{C}_k \to C^{(n-1)-k}$ on generators by

$$\phi_k([\tilde{A}]) = (-1)^{A(X)} \langle X - A \rangle$$

and extend to $\tilde{C}_k$ by RS-linearity. Here, $A(X) = \sum_{x \in A} x(X)$. Each ϕ_k is an isomorphism of right RS-modules. Note that if $x \in A \subseteq X$, then $(-1)^{A(X) + x(X-A)} =$

$(-1)^{x(A)+(A-x)(X)}$. It follows that if $0 \leqq k \leqq n-1$, then $\partial^{n-k-1}\phi_k = \phi_k\tilde{\partial}_k$. Since $(\tilde{C}_*, \tilde{\partial}_*)$ is exact in positive dimension, (C^*, ∂^*) is exact in positive codimension, as required.

We conclude this section by noting a situation in which (3.2) is particularly uninteresting:

(3.6) THEOREM. *In the situation of* (3.2), *if X contains a unit of S, then the module of D_X is the zero module and the contracting homotopy of (C_*, ∂_*) can be chosen to the RS-linear.*

PROOF. The first conclusion follows from the remark following the proof of (3.2). To prove the second conclusion, we appeal to (1.7): note that if $z \in X$ is a unit and $A \in \mathscr{P}$ satisfies $z \notin A$, then Δ_A is a common left multiple of $A + z$. It follows that z is a cone point of D_X so that (1.7) applies; all of (3.6) follows easily.

§4. *Homology approximation.*

We begin with a homology "approximation" lemma. Let R be a commutative ring with 1. For each $k \geqq 0$, let C_k be an R-module graded by the non-negative integers: $C_k = \oplus_{p=0}^{\infty} C_k(p)$. An R-module homomorphism $\partial_k\colon C_k \to C_{k-1}$ is said to be *homogeneous* provided each $\partial_k(C_k(p)) \subseteq C_{k-1}(p)$. A second R-module homomorphism $d_k\colon C_k \to C_{k-1}$ is said to be *dominated* by ∂_k provided whenever $c \in C_k(p)$, it follows that $(d_k - \partial_k)(c) \in \oplus_{q=0}^{p-1} C_{k-1}(q)$. Finally, let (C_*, ∂_*) be a chain complex with each C_k graded and each ∂_k homogeneous as above. Note that if (C_*, ∂_*) is contractible in positive dimension (for each $k \geqq 0$, there exists an R-module homomorphism $s_k\colon C_k \to C_{k+1}$ such that if $k > 0$, then $\partial_{k+1}s_k + s_{k-1}\partial_k$ is the identity on C_k), then the s_k's can be chosen to be homogeneous (each $s_k(C_k(p)) \subseteq C_{k+1}(p)$).

(4.1) THEOREM. *Let (C_*, ∂_*) and (C_*, d_*) be chain complexes. Assume that each C_k is graded, each ∂_k is homogeneous, each d_k is dominated by ∂_k and (C_*, ∂_*) is contractible in positive dimension* (*all as above*). *Then*

a) *(C_*, d_*) is contractible in positive dimension.*

b) *$C_0/\operatorname{im}\partial_1$ and $C_0/\operatorname{im} d_1$ are isomorphic R-modules.*

PROOF. Clearly, we may assume that $C_k = 0$ if $k < 0$. It suffices to show that (C_*, ∂_*) and (C_*, d_*) are isomorphic chain complexes over R. Choose a contracting homotopy $\{s_*\}$ for (C_*, ∂_*) so that each s_k is homogeneous. Note that if $k > 0$, then $\partial_k s_{k-1}\partial_k = \partial_k$. For $k \geqq 0$, define $f_k\colon C_k \to C_k$ by the following formula: $f_k = d_{k+1}s_k + (1 - \partial_{k+1}s_k)$. Clearly, if $k > 0$, then

$$\begin{aligned} d_k f_k - f_{k-1}\partial_k &= d_k(1 - \partial_{k+1}s_k) - d_k s_{k-1}\partial_k \\ &= d_k(1 - \partial_{k+1}s_k - s_{k-1}\partial_k) \\ &= 0. \end{aligned}$$

(The first equality uses $d_k d_{k+1} = 0$ and $\partial_k s_{k-1}\partial_k = \partial_k$.) Thus $f_*: (C_*, \partial_*) \to (C_*, d_*)$ is a morphism of chain complexes.

To see that each f_k is an isomorphism of R-modules, note that $f_k = 1 - (\partial_{k+1} - d_{k+1})s_k$. Since ∂_{k+1} dominates d_{k+1} and s_k is homogeneous, $(\partial_{k+1} - d_{k+1})s_k$ is locally nilpotent: if $c \in C_k$, then there exists $N \geqq 0$ such that $((\partial_{k+1} - d_{k+1})s_k)^N(c) = 0$. (In fact, if $c \in C_k(p)$, then $((\partial_{k+1} - d_{k+1})s_k)^{p+1}(c) = 0$.) It follows that each f_k is invertible. (In fact, $f_k^{-1} = \sum_{n=0}^{\infty} ((\partial_{k+1} - d_{k+1})s_k)^n$.)

We view ∂_k as a "top-degree approximation" of d_k.

In order to apply (4.1) to (3.2) and (3.5), we assume that the monoid S is equipped with a homomorphism $w \mapsto |w|: S \to \mathsf{N}$. (Here N denotes the set of non-negative integers, viewed as a monoid under addition. By definition, if $u, v \in S$, then $|uv| = |u| + |v|$. Also, $|1| = 0$.) In this situation, we will say that S admits an N-valued length homomorphism.

If S admits an N-valued length homomorphism and $X \subseteq S$ is exact, then defining C_* as in §3, we extend length notation to each C_k as follows. First, if $w \in S$ and $A \in \mathscr{P}$ has cardinality k, then $|w[A]| = |w\Delta_A|$. Also, if $W \in C_k$ is an R-linear combination of $w[A]$'s as above, then $|W|$ is the largest $|w[A]|$ among the $w[A]$'s that occur with non-zero coefficient. (Recall that C_k is a free R-module on the $w[A]$'s.) We let $C_k(p) = \{W \in C_k \mid |W| = p\}$. Clearly, each ∂_k and s_k in (3.2) is homogeneous.

(4.2) Lemma. *In the situation of* (3.2), $C_0/\text{im } \partial_1$ *is a free R-module.*

Proof. In the notation of (3.2)

$$s_0(w[\emptyset]) = \begin{cases} 0 & \text{if } \xi(w) = 0 \\ wz^{-1}[z] & \text{if } z = \xi(w) \in X. \end{cases}$$

Note that if $\xi(w) \in X$, then $\partial_1 s_0(w[\emptyset]) = w[\emptyset]$. It follows easily that $s_0\partial_1 s_0 = s_0$ and $\partial_1 s_0 \partial_1 = \partial_1$. We claim that C_0 is an (internal) direct sum $\text{im } \partial_1 \oplus \ker s_0$ as an R-module. First, if $W \in C_0$, then $W = \partial_1 s_0 W + (1 - \partial_1 s_0)W$. Since $s_0\partial_1 s_0 = s_0$, $(1 - \partial_1 s_0)W \in \ker s_0$ so that $C_0 = \text{im } \partial_1 + \ker s_0$. To show $\text{im } \partial_1 \cap \ker s_0 = 0$, assume $W \in C_1$ satisfies $\partial_1 W \in \ker s_0$. Then $\partial_1 W = \partial_1 s_0 \partial_1 W = 0$, as required.

It follows that $C_0/\text{im } \partial_1$ is isomorphic to $\ker s_0$ as an R-module. Clearly, $\ker s_0$ is a free module on $\{w[\emptyset] \mid \xi(w) = 0\}$, as required.

(4.3) Corollary. *In this situation (and notation) of* (3.2), *suppose that S admits* *an* $\mathbf{N}$*-valued length homomorphism and that* (C_*, d_*) *is a chain complex such that each* d_k *isdominated by* ∂_k*. Then*

a) (C_*, d_*) *is exact in positive dimension.*

b) $C_0/\mathrm{im}\, d_1$ *is a free R-module.*

PROOF. By (3.2) and its proof, the hypotheses of (4.1) are satisfied. Thus (C_*, d_*) is contractible in positive dimension by (4.1a) from which (4.3a) follows. Also, (4.3b) follows from (4.1b) and (4.2).

In (4.3), it is only necessary to assume that each d_k is R-linear. If, in fact, each d_* is RS-linear, let (C^*, d^*) denote the RS-dual complex of (C_*, d_*): each $C^k = \mathrm{Hom}_{RS}(C_k, RS)$ and if $f \in C^k$, $w \in C_{k+1}$, then $(d^k f)(w) = f(d_{k+1} w)$.

(4.4) COROLLARY. *In the situation (and notation) of* (3.5), *suppose that S admits an* $\mathbf{N}$*-valued length homomorphism and that* (C_*, d_*) *is a chain complex such that each* d_k *is RS-linear and dominated by* ∂_k*. Then*

a) (C^*, d^*) *is exact in positive codimension.*

b) $C^{n-1}/\mathrm{im}\, d^{n-2}$ *is a free R-module.*

PROOF. The proof of (3.5) reduces (4.4) to (4.3).

We remark that in (4.3) or (4.4), if the length homomorphism is trivial (for each $w \in S$, $|w| = 0$), then each $d_k = \partial_k$ so that (4.3) or (4.4) give no new information.

Part II. Artin groups.

In Part II, we apply the homological machinery developed in Part I to Artin groups. In §5, we recall the definitions of Artin groups, Artin monoids and Coxeter groups and we recall the single most important fact (5.1) about Artin monoids: the Kurzungslemma of Brieskorn-Saito [8] (proved by Garside [21] for the braid groups and certain other groups). The easy consequence (5.2) of (5.1) implies that any subset X of an Artin monoid is exact (3.1), so that (3.2) applies as well; in the Artin group setting, we will only apply (3.2) when X is the generating set X_M of the Artin group (or monoid) as in §5.

In §6, we study fundamental elements of an Artin monoid S_M: these are the least common left multiples Δ_A of the subsets A of X_M which have a common left multiple. The early results in §6 are due to Brieskorn-Saito [8] (and to Garside [21] for the braid groups). Where convenient, we have indicated the proofs. The main result of §6 is (6.12) which is used in §7 to obtain a group-theoretically interesting "top-degree" approximation to the RS_M-complex associated to X_M (3.2).

In §7, we prove Theorem A. We begin by defining the "diagram polynomial" $D_\alpha(w) \in RS_M$ of an arbitrary $w \in S_M$ (7.1): $D_\alpha(w)$ is an "α-signed" sum of the right factors of w. In (7.4), we show that the functions $D_\alpha(A, B) = D_\alpha(\Delta_A \Delta_B^{-1})$ define a diagram in RS_M. Combining (3.2) with (4.1), it follows that the corresponding RS_M-complex is exact in positive dimension (7.5). The remainder of §7 is concerned with extending (7.5) from RS_M to RG_M; it turns out that (2.3) applies if and only if the associated Coxeter group W_M is locally finite. In particular, we obtain (7.7) from which Theorem A follows easily.

In §8, we prove Theorem B. To prove that if W_M is finite, then G_M is a duality group, we use Theorem 9.2 of [3] and adopt the following notation: $n - 1$ denotes the cardinality of X_M. Condition 9.2i of [3] (G_M is of type FP) follows from Theorem A. To prove condition 9.2ii of [3] ($H^k(G_M, RG_M) = 0$, if $k \neq n - 1$), we show (8.1) that X_M satisfies the duality condition (3.3), from which condition 9.2ii of [3] follows easily. The remainder (and bulk) of §8 is devoted to a proof that G_M satisfies condition 9.2iii of [3] ($H^{n-1}(G, RG)$ is a flat R-module). In fact, we show that $H^{n-1}(G, RG)$ is a free R-module. Our proof of this fact involves a "Möbius function" μ for S_M, defined just before (8.6). The Möbius function of a monoid with the "finite factorization property" was studied by Cartier and Foata in [11]. In particular, their results about "partially-commutative" monoids are an easy consequence of (8.6) below. We also remark that, by a theorem of Tits [29], certain results about the weak Bruhat ordering of Coexter groups (see, for example, [5]) are an easy consequence of our (8.6) below.

§5. *Preliminaries on Artin groups.*

Following [6], a *Coxeter matrix* is a symmetric matrix M each of whose entries $m(i,j)$ is a positive integer or ∞ such that $m(i,j) = 1$ if and only if $i = j$. Given a Coxeter matrix M, we define an *Artin monoid* S_M, an *Artin group* G_M and a *Coxeter group* W_M.

To define S_M, we introduce the following notation: if x and y are elements of a semigroup and $m \in \mathbb{N}$, then

$$\langle xy \rangle^m = \begin{cases} (xy)^k & \text{if } m = 2k \\ (yx)^k y & \text{if } m = 2k + 1. \end{cases}$$

Given M, let X be a set in fixed one-to-one correspondence with the rows (or columns) of M; a typical element of X_M is denoted a_i. S_M is defined by presentation to have generators X_M and relations all

$$\langle a_i a_j \rangle^{m(i,j)} = \langle a_j a_i \rangle^{m(j,i)}$$

with the convention that $m(i,j) = \infty$ stands for "no relation". Also, G_M is defined to be the group with the same presentation as S_M and W_M is defined to be G_M (or, equivalently, S_M) modulo the additional relations that each $a_i^2 = 1$. Note the natural homomorphisms from S_M to G_M and from G_M to W_M extending the identity on X_M.

We will use the tools developed in §1, §3 and §4 to study the homological algebra of the Artin monoids S_M. When possible, we will use §2 to extend to G_M.

For the moment, we focus on S_M. The presentation of S_M has several special properties. First, the defining relations of S_M are equalities between words of equal length, so that S_M admits an $\mathbb{N}$-valued length homomorphism $w \mapsto |w|$ where each $|a_i| = 1$ (see §4). Second, both sides of each defining relation of S_M involve the same generators. It follows that if $A \subseteq X_M$ then the submonoid of S_M generaed by A has the "obvious" presentation: generators A and relations those defining relations of S_M that only involve elements of A. Finally, each defining relation is invariant under reflection: if each side of a defining relation is replaced with its mirror image, then the same relation arises (interchanging sides, if $m(i,j)$ is odd). As a consequence, certain properties of S_M "on the left" have as immediate corollaries the analogous properties "'on the right".

The following "Kurzungslemma" of Brieskorn and Saito will be crucial below:

(5.1) THEOREM. *Let S_M be an Artin monoid. Suppose that $a_i, a_j \in X_M$ and $u, v \in S_M$ satisfy $ua_i = va_j$. Then $m(i,j) \neq \infty$ and there exists $z \in S_M$ such that $u = z\langle a_i a_j\rangle^{m(i,j)-1}$ and $v = z\langle a_j a_i\rangle^{m(i,j)-1}$.*

PROOF. This is Lemma 2.1 of [8].

In particular, if $m(i,j) \neq \infty$, then $\langle a_i a_j\rangle^{m(i,j)}$ is a least common left multiple of a_i and a_j. (If $m(i,j) = \infty$, then a_i and a_j have no common left multiple.) If $i = j$, (5.1) states that a_i cancels on the right ($m(i,i) = 1$) so that S_M is a right cancellation monoid. The mirror image of (5.1) also holds. In particular, S_M is also a left cancellation monoid.

(5.2) COROLLARY. *Let S_M be an Artin monoid. Then any subset of S_M which has a common left multiple is finite and has a unique laest common left multiple.*

PROOF. This is Proposition 4.1 of [8].

It follows from (5.2) that any subset X of S_M is exact (3.1), so that (3.2) applies. We will only apply (3.2) when $X = X_M$.

For the following consequence of (5.1), we adopt the following notation: if S_M is an Artin monoid and $A \subseteq X_M$, then S_A will denote the submonoid of S_M generated by A.

(5.3) COROLLARY. *Let S_M be an Artin monoid and let $A \subseteq X_M$. If $B \subseteq S_A$ has a common left multiple in S_M, then the least common left multiple of B belongs to S_A.*

PROOF. This follows from (5.1). See [8].

§6. *Fundamental elements.*

Let S_M be an Artin monoid with generators X_M. In this section, we write S for S_M and X for X_M. As in §5, if $A \subseteq X$, we write S_A for the submonoid of S generated by A. As noted in §5, each S_A has the "obvious" presentation (and therefore is an Artin monoid).

We let $\mathscr{P}$ denote the collection of all subsets of X which have a common left multiple. By (5.2), each $A \in \mathscr{P}$ is finite, so that the notation here agrees with that in §3. By (5.2), each $A \in \mathscr{P}$ has a least common left multiple which, as in §3, we denote Δ_A. (By (5.2), each Δ_A is uniquely determined by A.) Since, by (5.1), S is a right cancellation monoid (and, from above, X is exact), (3.2) applies; we will save this observation until we can apply (4.3) as well. Meanwhile, we study the Δ_A's.

(6.1) Lemma. *If $A \in \mathscr{P}$, then $\Delta_A \in S_A$.*

Proof. This follows from (5.2).

(6.2) Lemma. *If $A \in \mathscr{P}$ and $a \in A$, then there is a unique $s \in A$ such that $\Delta_A a = s\Delta_A$.*

Proof. See Lemma 5.2 of [8]. (Existence of s follows by noting that $\Delta_A a$ is a common left multiple of A; uniqueness follows from right cancellation; a length argument shows that $s \in X$; since $\Delta_A a \in S_A$, we have $s \in S_A$; thus, $s \in X \cap S_A = A$.)

In (6.2), we write $s = a^A$. Left cancellation and the finiteness of A imply that the function $a \mapsto a^A$ is a permutation of A.

(6.3) Lemma. *The function $a \mapsto a^A$ extends to an automorphism of S_A.*

Proof. Again see Lemma 5.2 of [8]. (Here, it suffices to show that $a \mapsto a^A$ respects the defining relations of A. For this, note that if $a_i, a_j \in X$ and $m \in \mathsf{N}$ satisfy $\langle a_i a_j \rangle^m = \langle a_j a_i \rangle^m$, then m is a multiple of $m(i,j)$.)

The image of $w \in S_A$ under the atomorphism of S_A determined by $a \mapsto a^A$ will be denoted w^A. It follows that $\Delta_A w = w^A \Delta_A$ and if $u, v \in S_A$, then $(uv)^A = u^A v^A$.

(6.4) Lemma. *If $u, v \in S_A$ satisfy $uv = \Delta_A$, then $v^A u = \Delta_A$.*

Proof. Note that $v^A uv = v^A \Delta_A = \Delta_A v$ and cancel v.

(6.5) Lemma. *Δ_A is the least common right multiple of A and equals its own mirro-image.*

Proof. See Lemma 5.1 of [8]. (Here, the left-right symmetry of the defining relations of S is crucial.)

(6.6) Corollary. *If $a \in A$, then $(a^A)^A = a$. In particular, Δ_A^2 is central in S_A.*

Proof. See Lemma 5.2 of [8]. (By (6.5), the mirror-image of $a^A \Delta_A = \Delta_A a$ is $a\Delta_A = \Delta_A a^A$.)

For the following proposition, we define $w \in S$ to be *square-free* provided w cannot be written as $w = uaav$ with $u, v \in S$ and $a \in X$.

(6.7) LEMMA. *If $A \in \mathscr{P}$, then Δ_A is square-free.*

PROOF. See Lemma 5.4 of [8]. (If $\Delta_A =$ uaav, then $\Delta_A = v^A uaa$; as in the proof of (6.2), show that $v^A ua$ is a common left multiple of A, a contradiction.)

(6.8) LEMMA. *Let $w \in S$ and $a \in X$. If w is square-free and cannot be written as $w_1 a$ in S, then wa is square-free.*

PROOF. This is Lemma 3.4 of [8].

The next few propositions all involve factorizations of the fundamental elements Δ_A. Only the first of these appears in [8].

(6.9) THEOREM. *Let $A \in \mathscr{P}$. Suppose that $\Delta_A = uv$ with $u, v \in S_A$. Then for each $a \in A$, either there exists $u_1 \in S_A$ such that $u = u_1 a$ or there exists $v_1 \in S_A$ such that $v = av_1$.*

PROOF. See Lemma 5.3 of [8].

By (6.7), the phrase "but not both" can be added to the statement of (6.9). Note that (6.8) and (6.9) have the following consequence: if A is a finite subset of X, then Δ_A exists if and only if S_A contains finitely many square-free elements.

We shall need some extensions of (6.9). These require the following observation: any non-empty subset of S has a greatest common right factor. ($Z \subseteq S$ has a common right factor w provided there is a function $f\colon Z \to S$ such that for each $z \in Z$, $Z = f(z)w$; w is a greatest common right factor of Z provided it is a common right factor of Z and, in addition, is a left multiple of any common right factor.) To prove this, note that the set of common right factors of Z has a common left multiple (any element of Z) and therefore by (5.2) a least common left multiple w; clearly, w is a greatest common right factor of Z. (See [8].)

(6.10) THEOREM. *Let $A \in \mathscr{P}$. Suppose that $\Delta_A = uv$ with $u, v \in S_A$. Then for each $B \subseteq A$, u and v can be factored uniquely as $u = u_1 u_2$ and $v = v_1 v_2$ in S_A so that $u_2 v_1 = \Delta_B$.*

PROOF. Let u_2 be the greatest common right factor of u and Δ_B. Then u_2 is a right factor of u, say $u = u_1 u_2$. Also, u_2 is a right factor, and therefore by (6.4), a left factor of Δ_B, say $\Delta_B = u_2 v_1$.

We claim that no $b \in B$ is a right factor of u_1. To prove this, note that if $b \in B$ is a right factor of u_1, then bu_2 is a right factor of u. Now b is not a left factor of u_2, since Δ_A is square-free (6.7). Since u_2 is a right factor of Δ_B and b is not a left factor of u_2, (6.9) applied to Δ_B shows that bu_2 is a right factor of Δ_B. Thus bu_2 is a common right factor u and Δ_B, contradicting the definition of u_2.

Since no $b \in B$ is a right factor of u_1, (6.9) applied to the factorization $\Delta_A = (u_1)(u_2 v)$ of Δ_A shows that every $b \in B$ is a left factor of $u_2 v$. In other words, $u_2 v$ is a common right multiple of B; by (6.5), there exists $v_2 \in S_A$ such that $u_2 v = \Delta_B v_2$. Since $\Delta_B = u_2 v_1$, we get $v = v_1 v_2$, as required.

To prove uniqueness, let u_1, u_2, v_1 and v_2 be as above and assume also that $u = u_1' u_2'$, $v = v_1' v_2'$ and $\Delta_B = u_2' v_1'$. Then u_2' is a right factor of u and, by (6.4), a right factor of Δ_B. By definition of u_2, there exists $z \in S_A$ such $u_2 = z u_2'$. Since $u_2 \in S_B$, we have $z \in S_B$. Then

$$\begin{aligned} \Delta_A &= uv \\ &= u_1 u_2 v_1' v_2' \\ &= u_1 z u_2' v_1' v_2' \\ &= u_1 z \Delta_B v_2' \end{aligned}$$

so that the assumption $z \neq 1$ would contradict (6.7). Thus $z = 1$, so $u_2' = u_2$. Cancellation easily gives $u_1' = u_1$, $v_1' = v_1$ and $v_2' = v_2$, as required.

Theorem (6.10) has two corollaries. The first is a generalization to an ascending union of subsets of A.

(6.11) COROLLARY. *Let $A \in \mathscr{P}$. Suppose that $\Delta_A = uv$ with $u, v \in S_A$. If $B_k \subseteq B_{k-1} \subseteq \ldots \subseteq B_1 \subseteq A$, then u and v can be factored uniquely as $u = u_0 u_1 \ldots u_k$ and $v = v_k v_{k-1} \ldots v_0$ so that $u_k v_k = \Delta_k$ and if $1 \leqq i < k$, then $u_i \Delta_{B_{i+1}} v_i = \Delta_{B_i}$.*

PROOF. This follows from (6.10) by induction on k.

For the second corollary, we use the following notation: if $w \in S$, then $R(w)$ denotes the set of right factors of w.

(6.12) COROLLARY. *Let $A \in \mathscr{P}$ and $B \subseteq A$. Then the function $R(\Delta_A \Delta_B^{-1}) \times R(\Delta_B) \to R(\Delta_A)$ defined by $(v_1, v_2) \mapsto v_1 v_2$ is a bijection.*

PROOF. We first check that if $v_1 \in R(\Delta_A \Delta_B^{-1})$ and $v_2 \in R(\Delta_B)$, then $v_1 v_2 \in R(\Delta_A)$. If $u_1 v_1 = \Delta_A \Delta_B^{-1}$ and $u_2 v_2 = \Delta_B$, then by (6.4) and (6.6)

$$\begin{aligned} \Delta_A &= u_1 v_1 u_2 v_2 \\ &= u_1 v_1 v_2 u_2^B \\ &= (u_2^B)^A u_1 v_1 v_2 \end{aligned}$$

so that $v_1 v_2 \in R(\Delta_A)$.

To check surjectivity, suppose that $v \in R(\Delta_A)$, say $uv = \Delta_A$. By (6.6), $\Delta_A = vu^A$. By (6.10), we can write $v = v_1 v_2$ and $u^A = u_1 u_2$ with $v_2 u_1 = \Delta_B$. By (6.4),

$u_1^B v_2 = \Delta_B$ so $v_2 \in R(\Delta_B)$. Since $\Delta_A = v_1 \Delta_B u_2 = u_2^A v_1 \Delta_B$, we get $u_2^A v_1 = \Delta_A \Delta_B^{-1}$ so $v_1 \in R(\Delta_A \Delta_B^{-1})$.

Finally, to check injectivity, note that if $uv = \Delta_A$ and $v = v_1 v_2$ with $v_1 \in R(\Delta_A \Delta_B^{-1})$ and $v_2 \in R(\Delta_B)$, then the proof of (6.10) applied to $\Delta_A = vu^A$ shows that v_2 is the greatest common right factor of Δ_B and v. Thus v_2 is uniquely determined by v. By cancellation in S, v_1 is therefore also uniquely determined by v.

§7. *Proof of Theorem* A

Let M be a fixed Coxeter matrix. In this section, we write X, S, G and W for X_M, S_M, G_M and W_M. Let R be a commutative ring with 1.

(7.1) DEFINITION. If $w \in S$ and $\alpha \in R$, then $D_\alpha(w) \in RS_M$ *is defined by*

$$D_\alpha(w) = \sum_{uv=w} \alpha^{|u|} v$$

where the summation ranges over all ordered pairs (u, v) of elements of S whose product is w.

As usual, $|u|$ denotes the length of u. The sum is finite, since each element of S has only finitely many right factors and, by right cancellation, u is uniquely determined by v and uv. For all $\alpha \in R$, we set $\alpha^0 = 1$, so that, for example, each $D_0(w) = w$.

We use the notation $\mathscr{P}$, Δ_A and $\Delta_A \Delta_B^{-1}$ as in §6. For $A \in \mathscr{P}$ and $B \subseteq A$, define $D_\alpha(A, B) \in RS$ by $D_\alpha(A, B) = D_\alpha(\Delta_A \Delta_B^{-1})$. Our first goal is to show that (1.2) is satisfied.

(7.2) THEOREM. *Let $A \in \mathscr{P}$ and $B \subseteq A$. Then $D_\alpha(\Delta_A) = D_\alpha(\Delta_A \Delta_B^{-1}) D_\alpha(\Delta_B)$.*

PROOF.

$$\begin{aligned} D_\alpha(\Delta_A) &= \sum_{uv=\Delta_A} \alpha^{|u|} v \\ &= \sum_{u_1 v_1 = \Delta_A \Delta_B^{-1},\, u_2 v_2 = \Delta_B} \alpha^{|u_1| + |u_2|} v_1 v_2 \\ &= \Bigg(\sum_{u_1 v_1 = \Delta_A \Delta_B^{-1}} \alpha^{|u_1|} v_1 \Bigg) \Bigg(\sum_{u_2 v_2 = \Delta_B} \alpha^{|u_2|} v_2 \Bigg) \\ &= D_\alpha(\Delta_A \Delta_B^{-1}) D_\alpha(\Delta_B) \end{aligned}$$

where the second equality uses (6.12).

To show that (1.2b) follows from (7.2), we shall need some cancellation in RS. Recall from §4 that if $W \in RS$, then $|W|$ denotes the maximum $|w|$ for w appearing with non-zero coefficient in w. Call $W \in RS$ *monic* provided $W = w + W_1$ with $w \in S$, $|w| = |W|$ and $|W_1| < |w|$. Since S is a right cancellation monoid, if $U, V, W \in RG$ satisfy $UW = VW$ and W is monic, then $U = V$. Note that each $D_\alpha(w)$ is monic.

(7.3) COROLLARY. *If* $C \subseteq B \subseteq A \in \mathscr{P}$, *then* $D_\alpha(\Delta_A \Delta_C^{-1}) = D_\alpha(\Delta_A \Delta_B^{-1}) D_\alpha(\Delta_B \Delta_C^{-1})$.

PROOF. By several applications of (7.2),

$$\begin{aligned} D_\alpha(\Delta_A \Delta_C^{-1}) D_\alpha(\Delta_C) &= D_\alpha(\Delta_A) \\ &= D_\alpha(\Delta_A \Delta_B^{-1}) D_\alpha(\Delta_B) \\ &= D_\alpha(\Delta_A \Delta_B^{-1}) D_\alpha(\Delta_B \Delta_C^{-1}) D_\alpha(\Delta_C). \end{aligned}$$

Since $D_\alpha(\Delta_C)$ is monic, (7.3) follows.

(7.4) COROLLARY. *The functions* $D_\alpha(A, B) = D_\alpha(\Delta_A \Delta_B^{-1})$, *for* $B \subseteq A \in \mathscr{P}$, *define a diagram of* $(X, \mathscr{P})$ *in* RS.

PROOF. Clearly, $\Delta_A \Delta_A^{-1} = 1$ and $D_\alpha(1) = 1$ which gives (1.2a). (7.3) gives (1.2b).

We let ∂_α denote the differential associated to the diagram $D_\alpha(A, B) = D_\alpha(\Delta_A \Delta_B^{-1})$. (In this section, we omit the dimension subscript from ∂_α.) Interpreting §1 in the current situation, C_k is the free left RS-module with basis consistng of all $[A]$ with A a k-element subset of X such that Δ_A exists; ∂_α is given by

$$\partial_\alpha([A]) = \sum_{x \in A} (-1)^{x(A)} D_\alpha(\Delta_A \Delta_{A-x}^{-1})[A - x]$$

(7.5) THEOREM. *For each* $\alpha \in R$, *the* RS-*complex* (C_*, ∂_α) *is exact in positive dimension.*

PROOF. By (5.1), S is a right-cancellation monoid. By (4.2), any subset of S which has a common left multiple has a least common left multiple; it follows that X is exact (3.1). Thus (3.2) applies: (C_*, ∂_0) is exact in positive dimension. Clearly, each ∂_α is dominated by ∂_0 and the other hypotheses of (4.3) are satisfied. From (4.3), we conclude that each (C_*, ∂_α) is exact in positive dimension, as required.

Note that the module $C_0/\partial_\alpha(C_1)$ of D_α is R with the following S-action: each $w \in S$ acts as multiplication by $(-\alpha)^{|w|}$. (To verify this, note that if $x \in X$, then $D_\alpha(x) = x + \alpha$.) In particular, (C_*, ∂_{-1}) is a resolution of R as a trivial RS-module.

We would like to show that for each Artin group G, the extension of the complex (C_*, ∂_α) of (7.5) from RS to RG is exact in positive dimension. Unfortunately, we only know how to do this in the situation in which (2.3) applies.

(7.6) THEOREM. *The following are equivalent:*

a) *Any two elements of S have a common left multiple.*

b) *For each finite subset A of X, Δ_A exists.*

c) *For each finite subset A of X, W_A is finite.*

PROOF. (If $A \subseteq X$, then W_A denotes the subgroup of W generated by A.) If X is finite, then (7.6) follows from [8]. If X is infinite, then (7.6) follows from the finite case and the fact that if $A \subseteq X$, then S_A and W_A have the obvious presentation (for S_A, this was noted in §5; for W_A, see [6].)

Note that (7.6c) holds if and only if every finitely-generated subgroup of W is finite. If G satisfies (7.6), we say that G is *locally of finite type.* If, in addition, X is finite (so W is finite), we say that G is *of finite type.*

(7.7) COROLLARY. *If G is locally of finite type, then the extension of (C_*, ∂_α) from RS to RG is exact in positive dimension.*

PROOF. By (5.1) and its mirror-image, S is a two-sided cancellation monoid. Thus, by hypothesis and (7.6a), S satisfies the Ore condition. Therefore, (7.7) follows from (7.5) and (2.3).

(7.8) COROLLARY. *If G is locally of finite type, then G is torsion-free.*

PROOF. First note that if $A \subseteq X$, then G_A ($=$ the subgroup of G generated by A) has the "obvious" presentation. (As already noted, S_A has the obvious presentation and therefore is an Artin monoid. By hypothesis, S satisfies (7.6b) so that S_A satisfies (7.6b). Thus S_A satisfies (7.6a) and therefore satisfies the Ore condition, so that (2.2) applies. Since the natural homomorphism from S_A to G is injective, it follows easily from uniqueness in (2.2) that G_A is the Artin group corresponding to the Artin monoid S_A.) Also, since S_A inherits condition (7.6b), G_A is locally of finite type.

Clearly, each cyclic subgroup of G is contained in G_A for some finite $A \subseteq X$. Thus it suffices to prove (7.8) under the stronger assumption that G is of finite type. Under this assumption, we apply (7.7) with $R = \mathbf{Z}$ and $\alpha = -1$; the extension of (C_*, ∂_{-1}) from $\mathbf{Z}S$ ro $\mathbf{Z}G$ is then a finite free resolution of $\mathbf{Z}$ as a trivial $\mathbf{Z}G$-module, so that G is torsion-free.

§8. *Proof of Theorem* B.

Let M be a fixed Coxeter matrix we use notation ($X, S, G, W, D_\alpha, \mathscr{P}, \Delta_A$, etc.) as in §7. We show that if G is of finite type, then G is a duality group. We first show that (3.5) applies to (C_*, ∂_0).

(8.1) LEMMA. *X satisfies the duality condition* (3.3).

PROOF. By (5.1), S is a right cancellation monoid. By (5.2), X is exact. Thus we need to show: if $A \in \mathscr{P}$ and $x, y \in A$ satisfy $x \neq y$, then $\Delta_A \Delta_{A-x-y}^{-1}$ is the least common right multiple of $\Delta_A \Delta_{A-x}^{-1}$ and $\Delta_A \Delta_{A-y}^{-1}$ in S. From the fact that

$$\begin{aligned}\Delta_A \Delta_{A-x-y}^{-1} &= (\Delta_A \Delta_{A-x}^{-1})(\Delta_{A-x} \Delta_{A-x-y}^{-1}) \\ &= (\Delta_A \Delta_{A-y}^{-1})(\Delta_{A-y} \Delta_{A-x-y}^{-1})\end{aligned}$$

we conclude that $\Delta_A \Delta_{A-x}^{-1}$ and $\Delta_A \Delta_{A-y}^{-1}$ have a common right multiple and therefore, by the mirror image of (5.2), they have a (unique) least common right multiple. Let $(\Delta_A \Delta_{A-x}^{-1})u = (\Delta_A \Delta_{A-y}^{-1})v$ be the least common right multiple of $\Delta_A \Delta_{A-x}^{-1}$ and $\Delta_A \Delta_{A-y}^{-1}$. Using the expressions above for $\Delta_A \Delta_{A-x-y}^{-1}$ as a right multiple of $\Delta_A \Delta_{A-x}^{-1}$ and $\Delta_A \Delta_{A-y}^{-1}$ and using left cancellation in S, we conclude that there exists $z \in S$ such that $\Delta_{A-x} \Delta_{A-x-y}^{-1} = uz$ and $\Delta_{A-y} \Delta_{A-x-y}^{-1} = vz$. To prove (8.1), it suffices to show that $z = 1$.

If $z \neq 1$, then some $a \in A$ is a right factor of both $\Delta_{A-x} \Delta_{A-x-y}^{-1}$ and $\Delta_{A-y} \Delta_{A-x-y}^{-1}$. We show that this contradicts (6.7). Since $\Delta_{A-x} = (\Delta_{A-x} \Delta_{A-x-y}^{-1})$ Δ_{A-x-y} and every element of $A - x - y$ is a left factor of Δ_{A-x-y}, it follows from (6.7) that the only element of A which could be a right factor of $\Delta_{A-x} \Delta_{A-x-y}^{-1}$ is y itself. (In fact, by (6.8), y is a right factor of $\Delta_{A-x} \Delta_{A-x-y}^{-1}$.) Similarly, the only element of A which could be a right factor of $\Delta_{A-y} \Delta_{A-x-y}^{-1}$ is x itself. Since $x \neq y$, this contradicts $z \neq 1$. Thus $z = 1$, as required.

From (8.1), we easily obtain the dual of (7.5) when G has finite type. As in §4, we let (C^*, ∂^α) denote the RS-dual complex of (C_*, ∂_α). (As in §7, we suppress the dimension superscript on ∂^α.)

(8.2) THEOREM. *If Δ_X exists, then for each $\alpha \in R$, the complex (C^*, ∂^α) is exact in positive codimension.*

PROOF. By (5.1) and its mirror-image, S is a two-sided cancellation monoid. By hypothesis, X has a common left multiple. Thus, by (5.2), X is finite. By (8.1), X satisfies the duality condition. Thus (3.5) applies: (C^*, ∂^0) is exact in positive codimension. As in the proof of (7.5), each ∂_α is dominated by ∂_0. Clearly, the other hypotheses of (4.4) are satisfied. From (4.4a), we conclude that each (C^*, ∂^α) is exact in positive codimension.

Note that "Δ_X exists" is equivalent to "G is of finite type."

(8.3) THEOREM. *Let G be an Artin group of finite type. Then G is a duality group.*

PROOF. We use the characterization of duality groups given in Theorem 9.2 of [3]. Applying (7.7) with $\alpha = -1$, it follows that G is of type FP (in fact, of type FL) so that G satisfies condition 9.2i of [3].

Toprove that G satisfies condition 9.2ii of [3], we let $n - 1$ denote the cardinality of X. By (5.1) and its mirror-image, S is a two-sided cancellation monoid. By the mirror-image of (7.6), S satisfies the (right) Ore condition. Thus the mirror-image of (2.3) holds as well. Combining this observation with (8.2), we conclude that the extension of (C^*, ∂^α) to RG (as a right RG-complex) is exact in positive codimension. Noting that each C_k is a finitely-generated free left RS-module, it follows easily that the RG-dual of the extension of (C_*, ∂_α) to RG is isomorphic to the extension of (C^*, ∂^α) to RG. From all this, it follows easily that $H^k(G, RG) = 0$, if $k \neq n - 1$, which is condition 9.2ii of [3].

Finally, we verify that G satisfies condition 9.2iii of [3]: $H^{n-1}(G, RG)$ is a flat R-module. In fact, we will establish a slightly more general result. To state this result, note that, via the inclusion $RS \subseteq RG$, the diagram $D_\alpha(A, B) = D_\alpha(\Delta_A \Delta_B^{-1})$ in RS defines a diagram in RG. Clearly, the extension to RG of the RS-complex associated to D_α in RS is the RG-complex associated to D_α in RG.

(8.4) LEMMA. *If G is an Artin group of finite type and α is a unit of R, then, as R-modulus, the co-module of D_α in RS and the co-module of D_α in RG are isomorphic.*

Recall from (4.4b) that the co-module of D_α in RS is a free R-module. Note that the co-module of D_{-1} in RG is $H^{n-1}(G, RG)$. From (8.4), we conclude that condition 9.2iii of [3] is satisfied. From Theorem 9.2 of [3], we conclude that G is a duality group, which completes the proof of (8.3).

The remainder of this section will be devoted to a proof of (8.4). The proof will involve some new ideas. First, a monoid S is said to have the *finite factorization property* provided for each $w \in S$ there are only finitely many ordered pairs $(u, v) \in S \times S$ such that $uv = w$. Clearly, an Artin monoid has the finite factorization property.

Let S be a monoid with the finite factorization property and let Λ be an associative ring (with 1). If $f, g \colon S \to \Lambda$ are arbitrary functions, then the *convolution* $f * g$ of f and g is the function from S to Λ defined as follows:

$$f * g(w) = \sum_{uv = w} f(u)g(v)$$

where, as in (7.1), the summation ranges over all pairs of elements of S whose product is w. We leave the proof of the following to the reader.

(8.5) LEMMA. *Convolution is associative. The function $\delta \colon S \to \Lambda$ defined by $\delta(1) = 1$ and $\delta(w) = 0$ if $w \neq 1$ is a two-sided identity for convolution.*

Let S be a monoid with the finite factorization property and let R be a commutative ring (with 1). We study convolution with $\Lambda = RS$. Define $I \colon S \to RS$ by

$I(w) = w$. If S admits an $\mathbb{N}$-valued length homomorphism $w \mapsto |w|$ and $\alpha \in R$, define $e_\alpha: S \to RS$ by $e_\alpha(w) = \alpha^{|w|}$. (The definition (7.1) of D_α can now be written as $D_\alpha = e_\alpha * I$.)

Finally, we assume that S is an Artin monoid. In this situation, we define $\mu: S \to RS$ by

$$\mu(w) = \begin{cases} (-1)^{|A|}\Delta_A & \text{if } w = \Delta_A \text{ for some } A \subseteq X \\ 0 & \text{otherwise} \end{cases}$$

where $|A|$ denotes the cardinality of A. In particular, $\mu(1) = 1$.

(8.6) THEOREM. *Let S be an Artin monoid. Then $\mu * I = I * \mu = \delta$.*

PROOF. Clearly $I * \mu(1) = I(1)\mu(1) = 1 = \delta(1)$, since if $uv = 1$ in S, then $u = v = 1$.

If $w \neq 1$, let $A = \{a \in X \mid \text{for some } w' \in S, w = w'a\}$. Since $w \neq 1$, $A \neq 0$. By (5.2), A is finite, since w is a common left multiple of A. In particular, Δ_A exists and there exists $w_1 \in S$ such that $w = w_1\Delta_A$. Also, if $B \subseteq X$ is finite and Δ_B exists, then ther exists $w' \in S$ such that $w = w'\Delta_B$ if and only if $B \subseteq A$, in which case $w' = w_1(\Delta_A\Delta_B^{-1})$. Thus

$$\begin{aligned} I * \mu(w) &= \sum_{uv = w} u\mu(v) \\ &= \sum_{B \subseteq X : u\Delta_B = w} u(-1)^{|B|}\Delta_B \\ &= \sum_{B \subseteq A} (-1)^{|B|} w \\ &= (1 - 1)^{|A|} w \\ &= 0. \end{aligned}$$

Thus we have shown that $I * \mu = \delta$. (The proof that $\mu * I = \delta$ follows from the left-right symmetry of the defining relations of S.)

(8.7) COROLLARY. *Let S be an Artin monoid and suppose that Δ_X exists. Then, writing $N = |\Delta_X|$, $\Delta_X - (-1)^{|X|}\alpha^N$ belongs to the right ideal in RS generated by $\{D_\alpha(\Delta_X\Delta_{X-x}^{-1}) \mid x \in X\}$.*

PROOF. Since $D_\alpha = e_\alpha * I$, it follows from (8.5) and (8.6) that $e_\alpha = D_\alpha * \mu$. Applying this equality to Δ_X (and recalling that $N = |\Delta_X|$), we conclude that

$$\begin{aligned}\alpha^N &= e_\alpha(\Delta_X)\\ &= D_\alpha * \mu(\Delta_X)\\ &= \sum_{uv=\Delta_X} D_\alpha(u)\mu(v)\\ &= \sum_{A\subseteq X}(-1)^{|A|}D_\alpha(\Delta_X\Delta_A^{-1})\Delta_A\\ &= (-1)^{|X|}\Delta_X + \sum_{A\subsetneq X}(-1)^{|A|}D_\alpha(\Delta_X\Delta_A^{-1})\Delta_A.\end{aligned}$$

Note that if A is a proper subset of X, then $D_\alpha(\Delta_X\Delta_A^{-1})$ belongs to the right ideal in RS generated by $\{D_\alpha(\Delta_X\Delta_{X-x}^{-1}) \mid x \in X\}$: choosing $x \notin A$, $D_\alpha(\Delta_X\Delta_A^{-1}) = D_\alpha(\Delta_X\Delta_{X-x}^{-1})D_\alpha(\Delta_{X-x}\Delta_A^{-1})$ by (7.2); (8.7) follows easily.

At last, we complete the proof of (8.3) by providing the

PROOF OF (8.4). Write $H^{n-1}(S, RS)$ and $H^{n-1}(G, RG)$ for the co-modules of D_α in RS and RG, respectively. Since S is a two-sided cancellation monoid and, by the mirror-image of (7.6), satisfies the right Ore condition, it follows from the mirror-image of (2.3) that the fuctor $(-) \mapsto (-) \otimes_{RS} RG$ is exact. It follows that $H^{n-1}(G, RG)$ and $H^{n-1}(S, RS) \otimes_{RS} RG$ are isomorphic RG-modules.

Now suppose that α is a unit of R. It follows from (8.7) that Δ_X acts invertibly on $H^{n-1}(S, RS)$. Since each $x \in X$ is both a left factor and a right factor of Δ_X, each $x \in X$ (and therefore all of S) acts invertibly on $H^{n-1}(S, RS)$. It follows that $H^{n-1}(S, RS)$ and $H^{n-1}(S, RS) \otimes_{RS} RG$ are isomorphic R-modules; (8.4) follows easily.

Part III. Examples.

In part III, we apply the results of Part II to certain specific Artin groups of finite type. In §9, we study the three irreducible Artin groups (of finite type) whose generating set X_M has cardinality 3; in accordance with [6, p. 193], these three groups are denoted A_3, B_3 and H_3. In §10, we briefly treat the braid groups $B^{(n)}$. (Already, a notational anomaly has arisen: the Artin group denoted A_3 in §9 is denoted $B^{(4)}$ in §10.)

In §9, we explicitly compute the ordinary integral homology $H_*(G)$ where G is one of A_3, B_3, H_3 or their commutator subgroups A'_3, B'_3, H'_3, respectively. Some of our results are not new. $H_*(A_3)$ was described in [30] for example. A description of $H_*(A'_3)$ follows easily from the fact that A'_3 is a semidirect product of two free groups of rank 2 (see [18]). $H_*(B_3)$ was described in [23]. Our description of $H_*(B'_3)$ contradicts [24], where it is claimed that B'_3 is a free group of rank 4. Apparently, our descriptions of $H_2(H_3)$ and $H_2(H'_3)$ are new.

In § 10, we briefly describe the homological algebra of Artin's braid group $B^{(n)}$. In particular, we explicitly describe each $D_\alpha(\Delta_A \Delta_{A-x}^{-1})$ as in §7. We also describe a finite complex of finitely-generated free abelian groups whose homology is the ordinary integral homology of $B^{(n)}$ (the ordinary integral homology of $B^{(n)}$ is well-known; see [30] or [13]). In addition, we describe a finite complex of finitely-generated free modules over the Laurent-polynomial ring $\mathbf{Z}4[t, t^{-1}]$ whose homology is the ordinary integral homology of the commutator subgroup of $B^{(n)}$.

§9. A_3, B_3 *and* H_3

In this section, we will carry out some explicit computations for Artin groups whose Coxeter groups are of type A_3, B_3 and H_3 in the notation of [6, p. 193]. It follows from [6] that these Coxeter groups are finite, so that (7.7) applies to the corresponding Artin groups. Before turning to these examples, we establish some notation.

Throughout this section, G will be one of the three Artin groups indicated above, α will be -1, ∂ will denote ∂_{-1} and R will be $\mathbf{Z}$. In each case, X will be $\{a, b, c\}$ with the total ordering $a < b < c$. If $A \subseteq X$, then $[A]$ and Δ_A will be written by listing the elements of A; for example, if $A = \{a, c\}$, then $[A]$ and Δ_A will be written $[ac]$ and Δ_{ac}, respectively.

If G is a group, then $H_*(G)$ will denote the ordinary integral homology of G. For each of the three Artin groups G that we consider, $H_1(G)$ will turn out to be a free abelian group of rank 1 or 2. In each case, we write the group-ring $\mathbf{Z}(G/G')$ as a Laurent polynomial ring. Recall Shapiro's lemma (see, for example, [9, p. 73]): if G is a group and N is a normal subgroup of G, then $H_*(N) = H_*(G, \mathbf{Z}(G/N))$. We will use Shapiro's lemma to compute $H_*(G')$ where G' is the commutator subgroup of G.

Finally, given a monoid S and a generating set X of S, we associate to each $w \in S$ a graph $\Gamma(w)$ defined as follows. A vertex of $\Gamma(w)$ is an ordered pair $(u, v) \in S \times S$ which satisfies $uv = w$. (Note the relationship between the vertices of $\Gamma(w)$ and the definition (7.1) of $D_\alpha(w)$.) An edge of $\Gamma(w)$ is an ordered triple $(u, a, v) \in S \times X \times S$ which satisfies $uav = w$; the edge (u, a, v) will be directed from the vertex (u, av) to the vertex (ua, v) and will be labelled a. We will use the graphs $\Gamma(w)$ to avoid explicitly describing the complex (C_*, ∂_*) in two of the three examples below.

In each example, we will give the Coxeter matrix M, give the corresponding presentation of the Artin group G, draw, for each $x \in X$, $\Gamma(\Delta_X \Delta_{X-x}^{-1})$ (as an aid in computing $\partial[abc]$), compute $H_*(G)$ and, at the very least, compute $H_*(G')$.

(9.1) EXAMPLE. $M = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 3 \\ 2 & 3 & 1. \end{pmatrix}$

The corresponding Artin group, denoted A_3, and Artin monoid have the following presentation: generators a, b, c and relations $aba = bab$, $ac = ca$, $bcb = cbc$. The Artin group A_3 is the braid group on 4 strands and, somewhat inconveniently, is conventionally denoted B_4. Here is (C_*, ∂_*):

$$\partial([\emptyset]) = 0$$

$$\partial([a]) = (a - 1)[\emptyset]$$

$$\partial([b]) = (b - 1)[\emptyset]$$

$$\partial([c]) = (c - 1)[\emptyset]$$

$$\partial([ab]) = (ba - a + 1)[b] - (ab - b + 1)[a]$$

$$\partial([ac]) = (a - 1)[c] - (c - 1)[a]$$

$$\partial([bc]) = (cb - b + 1)[c] - (bc - c + 1)[b]$$

$$\partial([abc]) = (cba - ba + a - 1)[bc] - (bacb - acb + ab + cb - b + 1)[ac] + (abc - bc + c - 1)[ab]$$

The formula for $\partial([bc])$, for example, follows from the fact that $\Delta_{bc}\Delta_c^{-1} = cb$ and $\Delta_{bc}\Delta_b^{-1} = bc$. The formula for $\partial([abc])$ follows from the fact that $\Delta_{abc} = abcaba$, $\Delta_{abc}\Delta_{bc}^{-1} = cba$, $\Delta_{abc}\Delta_{ac}^{-1} = bacb$ and $\Delta_{abc}\Delta_{ab}^{-1} = abc$. In Figure 1, we list the corresponding graphs (edges are directed from left to right).

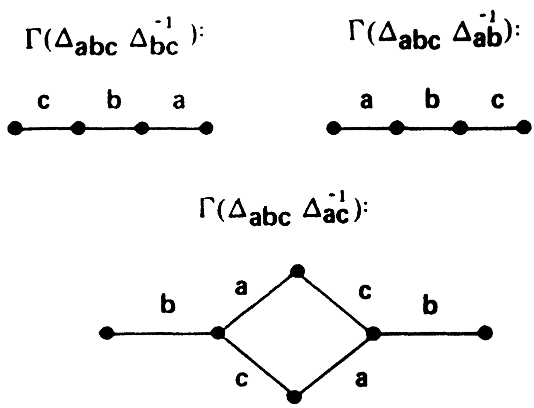


Figure 1.

The integral homology of the group A_3 may be computed by substituting $a = b = c = 1$ into the complex (C_*, ∂_*) and computing the homology of the resulting complex over Z. This yields

$$\partial([\emptyset]) = 0$$
$$\partial([a]) = \partial([b]) = \partial([c]) = 0$$
$$\partial([ab]) = [b] - [a]$$
$$\partial([ac]) = 0$$
$$\partial([bc]) = [c] - [b]$$
$$\partial([abc]) = -2[ac]$$

It follows easily that $H_0(A_3) = H_1(A_3) = \mathbf{Z}$, $H_2(A_3) = \mathbf{Z}_2$ and $H_k(A_3) = 0$ if $k \neq 0, 1, 2$.

Finally, we compute the integral homology of the commutator subgroup A'_3 of A_3. From above, $A_3/A'_3 = H_1(A_3)$ is an infinite cyclic group. We identify the group ring $\mathbf{Z}(A_3/A'_3)$ with the Laurent-polynomial ring $\mathbf{Z}[t, t^{-1}]$ where t denotes the common image of a, b, c in A_3/A'_3. Using Shapiro's lemma (described above), $H_*(A'_3)$ may be computed by substituting t for a, b, c in (C_*, ∂_*) and computing the homology of the resulting complex over $\mathbf{Z}[t, t^{-1}]$. In describing the resulting complex, we will use the following abbreviation: ϕ_n $(n > 0)$ will stand for the nth cyclotomic polynomial, with the convention that $\phi_1 = t - 1$. (Aside from ϕ_1, the only ϕ_n's that appear here are $\phi_4 = t^2 + 1$ and $\phi_6 = t^2 - t + 1$.) This yields

$$\partial([\emptyset]) = 0$$
$$\partial([a]) = \partial([b]) = \partial([c]) = \phi_1[\emptyset]$$
$$\partial([ab]) = \phi_6([b] - [a])$$
$$\partial([ac]) = \phi_1([c] - [a])$$
$$\partial([bc]) = \phi_6([c] - [b])$$
$$\partial([abc]) = \phi_4(\phi_1[bc] - \phi_6[ac] + \phi_1[ab])$$

It follows easily that $H_0(A'_3) = \mathbf{Z}$, $H_1(A'_3) = H_2(A'_3) = \mathbf{Z} \oplus \mathbf{Z}$ and $H_k(A'_3) = 0$ if $k \neq 0, 1, 2$. We remark that the $\mathbf{Z}[t, t^{-1}]$-module structure of $H_*(A'_3)$, induced by conjugation in A_3, can be deduced from the $\mathbf{Z}[t, t^{-1}]$-complex above.

In examples (9.2) and (9.3) below, we will not explicitly describe the complex (C_*, ∂_*). Except for the formulas for $\partial([bc])$ and $\partial([abc])$, (C_*, ∂_*) in (9.2) and (9.3) will agree with (9.1). The formula for $\partial([bc])$ will be easy enough to describe. The formula for $\partial([abc])$ would take up more space than our apology for not printing it. We will print the graphs $\Gamma(\Delta_{abc}\Delta_{bc}^{-1})$, $\Gamma(\Delta_{abc}\Delta_{ac}^{-1})$ and $\Gamma(\Delta_{abc}\Delta_{ab}^{-1})$; from these graphs, it is easy enough to describe $\partial([abc])$. For this reason, we suggest the following exercise: in (9.1), understand $\partial([abc])$ in terms of the graphs $\Gamma(\Delta_{abc}\Delta_{bc}^{-1})$, $\Gamma(\Delta_{abc}\Delta_{ac}^{-1})$ and $\Gamma(\Delta_{abc}\Delta_{ab}^{-1})$.

(9.2) EXAMPLE. $M = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 4 \\ 2 & 4 & 1 \end{pmatrix}$

The corresponding Artin group, denoted B_3, and Artin monoid have the following presentation: generators a, b, c and relations $aba = bab$, $ac = ca$, $bcbc = cbcb$. As noted above, we will omit the explicit description of (C_*, ∂_*). It is easy enough to check that

$$\partial([bc]) = (bcb - cb + b - 1)[c] - (cbc - bc + c - 1)[b]$$

and that $\partial([abc])$ can be read off from the graphs in Figure 2.

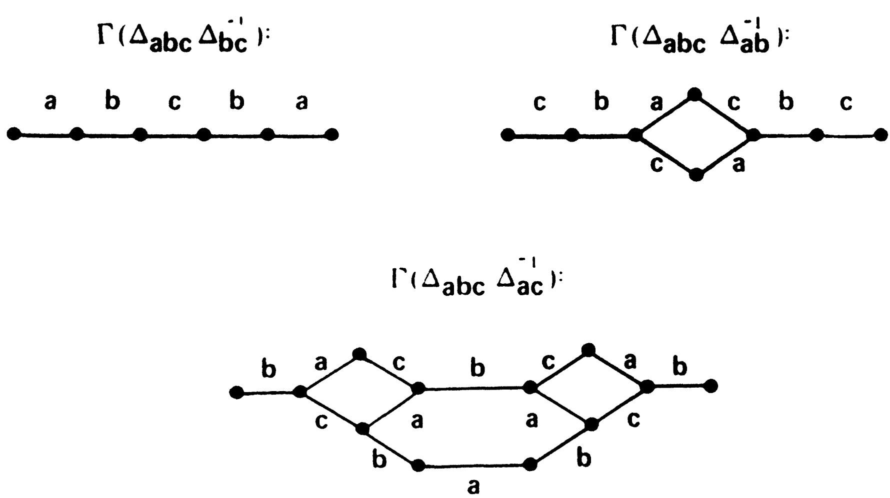


Figure 2

The integral homology of the group B_3 may be computed by substituting $a = b = c = 1$ into the complex (C_*, ∂_*) that results from above. This may be easily seen to yield the following complex over **Z**

$$\partial([\emptyset]) = 0$$

$$\partial([a]) = \partial([b]) = \partial([c]) = 0$$

$$\partial([ab]) = [b] - [a]$$

$$\partial([ac]) = \partial([bc]) = 0$$

$$\partial([abc]) = 0$$

It follows easily that $H_0(B_3) = \mathsf{Z}$, $H_1(B_3) = \mathsf{Z} \oplus \mathsf{Z}$, $H_2(B_3) = \mathsf{Z} \oplus \mathsf{Z}$, $H_3(B_3) = \mathsf{Z}$ and all other $H_k(B_3) = 0$.

Finally, we compute the integral homology of B_3'. Since $B_3/B_3' = \mathsf{Z} \oplus \mathsf{Z}$, we identify $\mathsf{Z}(B_3/B_3')$ with the Laurent-polynomial ring $\mathsf{Z}[s, s^{-1}, t, t^{-1}]$, where, in B_3/B_3', t denotes the image of a and b and s denotes the image of c. Proceeding as in (9.1), we substitute t for a and b and s for c in (C_*, ∂_*). It is not difficult to see that this yields

$$\begin{aligned}
\partial([\emptyset]) &= 0 \\
\partial([a]) = \partial([b]) &= (t - 1)[\emptyset]) \\
\partial([c]) &= (s - 1)[\emptyset]) \\
\partial([ab]) &= (t^2 - t + 1)([b] - [a]) \\
\partial([ac]) &= (t - 1)[c] - (s - 1)[a] \\
\partial([bc]) &= (st + 1)\{(t - 1)[c] - (s - 1)[b]\} \\
\partial([abc]) &= (st^2 - 1)\{(t^2 - t + 1)[bc] - (st + 1)(t^2 - t + 1)[ac] \\
&\qquad + (st + 1)(s - 1)[ab]\}
\end{aligned}$$

To describe $H_*(B_3')$, we let Λ denote $\mathsf{Z}[s, s^{-1}, t, t^{-1}]$ and if $p_1, p_2, \ldots \in \Lambda$, we let $(p_1, p_2, \ldots)$ denote the ideal in Λ generated by $p_1, p_2, \ldots$. Then

$$\begin{aligned}
H_0(B_3') &= \Lambda/(s - 1, t - 1) \\
H_1(B_3') &= \Lambda/(t^2 - t + 1, (st + 1)(s - 1)) \\
H_2(B_3') &= \Lambda/(st^2 - 1)
\end{aligned}$$

and all other $H_k(B_3') = 0$. Note that, as abelian groups, $H_1(B_3')$ and $H_2(B_3')$ are free of rank 4 and countably infinite rank, respectively. (It follows that B_3' is not finitely-related. In fact, B_3' is finitely-generated; it is not hard to show that $a^{-1}b$, ba^{-1}, $ca^{-1}bc^{-1}$ and $cba^{-1}c^{-1}$ generated B_3'.)

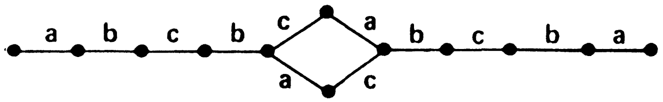
(9.3) Example. $M = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 5 \\ 2 & 5 & 1 \end{pmatrix}$

The corresponding Artin group, denoted H_3, and Artin monoid have the following presentation: generators a, b, c and relations $aba = bab$, $ac = ca$, $bcbcb = cbcbc$. We will omit explicit description of (C_*, ∂_*). Clearly,

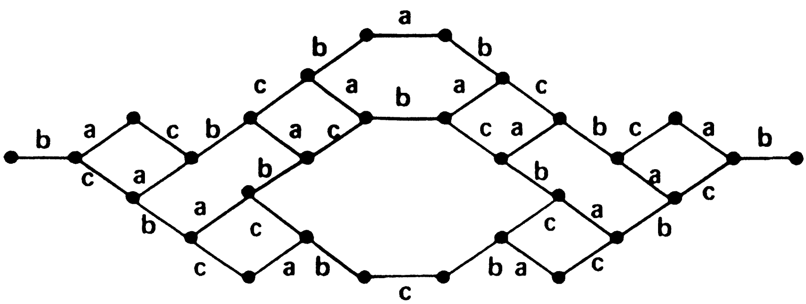
$$\partial([bc]) = (cbcb - bcb + cb - b + 1)[c] - (bcbc - cbc + bc - c + 1)[b]$$

It is also easy to describe $\partial([abc])$ using the graphs in Figure 3.

$\Gamma(\Delta_{abc}\Delta_{bc}^{-1})$:



$\Gamma(\Delta_{abc}\Delta_{ac}^{-1})$:



$\Gamma(\Delta_{abc}\Delta_{ab}^{-1})$:

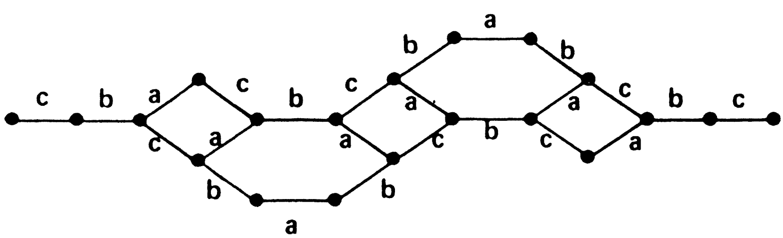


Figure 3.

To compute $H_*(H_3)$, we substitute $a = b = c = 1$ in (C_*, ∂_*). This yields

$$\partial([\emptyset]) = 0$$
$$\partial([a]) = \partial([b]) = \partial([c]) = 0$$
$$\partial([ab]) = [b] - [a]$$
$$\partial([ac]) = 0$$
$$\partial([bc]) = [c] - [b]$$
$$\partial([abc]) = 0$$

It follows $H_k(H_3) = \mathbf{Z}$ if $0 \leqq k \leqq 3$ and $H_k(H_3) = 0$, otherwise. To compute $H_*(H'_3)$, since $H_1(H_3) = \mathbf{Z}$, we identify $\mathbf{Z}(H_3/H'_3)$ with $\mathbf{Z}[t, t^{-1}]$ and proceed as in (9.1). (In addition to ϕ_1 and ϕ_6, which appeared in (9.1), we will also see $\phi_3 = t^2 + t + 1$, $\phi_5 = t^4 + t^3 + t^2 + t + 1$ and $\phi_{10} = t^4 - t^3 + t^2 - t + 1$.) This yields

$$\partial([\emptyset]) = 0$$
$$\partial([a]) = \partial([b]) = \partial([c]) = \phi_1[\emptyset]$$
$$\partial([ab]) = \phi_6([b] - [a])$$
$$\partial([ac]) = \phi_1([c] - [a])$$
$$\partial([bc]) = \phi_{10}([c] - [b])$$
$$\partial([abc]) = \phi_1\phi_3\phi_5(\phi_1\phi_6[bc] - \phi_6\phi_{10}[ac] + \phi_1\phi_{10}[ab])$$

It follows that $H_0(H'_3) = \mathbf{Z}$, $H_2(H'_3)$ is free abelian of rank 7 and each other $H_k(H'_3) = 0$. In particular, H'_3 is a perfect group and any presentation of H'_3 requires at least 7 more relations than generators. We remark that H'_3 is finitely-presented. This can be seen as follows: Δ_{abc} is central in H_3 and the subgroup $\langle \Delta_{abc} \rangle$ generated by Δ_{abc} satisfies $\langle \Delta_{abc} \rangle \cap H'_3 = \{1\}$. It follows that H'_3 is isomorphic to a subgroup of finite index in the finitely-presented group $H_3/\langle \Delta_{abc} \rangle$ and is therefore finitely-presented.

§10. $B^{(n)}$

In this section, we apply the results of §7 to Artin's braid group $B^{(n)}$, which for $n > 0$, is presented as follows: $B^{(n)}$ has generators $X^{(n)} = \{s_1, \dots, s_{n-1}\}$ and relations as given in the introduction. For notational convenience, we let $S^{(n)}$ denote the Artin monoid associated to $B^{(n)}$. Note that $B^{(1)}$ is the trivial group, $B^{(2)}$ is an infinite cyclic group, $B^{(3)}$ is the fundamental group of the complement of the trefoil knot and that $B^{(4)}$ is the group denoted A_3 in §9. The Coxeter group associated to $B^{(n)}$ is the symmetric group on n letters so that $B^{(n)}$ is an Artin group of finite type. In particular, (7.7) applies to $B^{(n)}$. Our goal in this section is to give an explicit description of ∂_α in (7.7) for $B^{(n)}$.

We begin with some terminology and notation. First, two subsets A and B of $X^{(n)}$ are said to be *separated* provided if $s_i \in A$ and $s_j \in B$, then $|i-j| \geqq 2$.

(10.1) LEMMA. *Let $A, B \subseteq X^{(n)}$ be separated.*

a) *If $u \in S_A$ and $v \in S_B$, then $uv = vu$.*

b) $\Delta_{A \cup B} = \Delta_A \Delta_B$.

PROOF. (10.1a) follows from the fact that every element of A commutes with every element of B. For (10.1b), note first that since, by (10.1a), $\Delta_A \Delta_B = \Delta_B \Delta_A$, $\Delta_A \Delta_B$ is a common left multiple of $A \cup B$, so there exists $z \in S^{(n)}$ such that $\Delta_A \Delta_B = z\Delta_{A \cup B}$. Since $B \subseteq A \cup B$, $\Delta_{A \cup B}$ is a common left multiple of B, so there exists $z_1 \in S^{(n)}$ such that $\Delta_{A \cup B} = z_1 \Delta_B$. Since $A \cap B = \emptyset$, by (6.9), every $a \in A$ is a right factor of z_1 so that there exists $z_2 \in S^{(n)}$ such that $z_1 = z_2 \Delta_A$. It follows that $\Delta_A \Delta_B = zz_2 \Delta_A \Delta_B$ in $S^{(n)}$ so that $z = z_2 = 1$. In particular $\Delta_A \Delta_B = \Delta_{A \cup B}$, as required.

Next, we introduce "open interval" notation on $X^{(n)}$: if $0 \leqq i < j \leqq n$, then $X(i,j) = \{s_k \mid i < k < j\}$. Note that each $X(i, i+1)$ is empty and that $X^{(n)} = X(0,n)$. In particular, we let $S(i,j)$ denote the submonoid of $S^{(n)}$ generated by $X(i,j)$ and let $\Delta(i,j)$ denote the least common left multiple of $X(i,j)$. If $A \subseteq X^{(n)}$, then $X(i,j)$ will be called a *subinterval* of A provided: if $i < k < j$, then $s_k \in A$. If, in addition, $s_i, s_j \notin A$, then $X(i,j)$ is called a *full* subinterval of A.

(10.2) COROLLARY. *Let $A \subseteq X^{(n)}$, let $s_k \in A$ and let $X(i,j)$ be the full subinterval of A which contains s_k. Then $\Delta_A \Delta_{A-s_k}^{-1} = \Delta(i,j)(\Delta(i,k)\Delta(k,j))^{-1}$.*

PTOOF. Write $A = B \cup X(i,j)$ with B and $X(i,j)$ disjoint. Clearly, B and $X(i,j)$ are separated. By (10.1), $\Delta_A = \Delta_B \Delta(i,j)$ and $\Delta_{A-s_k} = \Delta_B \Delta(i,k)\Delta(k,j)$. (10.2) follows easily.

(10.1) and (10.2) reduce the computation of Δ_A and $\Delta_A \Delta_{A-s_k}^{-1}$ to certain special cases. We begin by studying $\Delta(0,n)$ and $\Delta(0,n)(\Delta(0,i)\Delta(i,n))^{-1}$.

First, if $j > i$, we define the "descending product" $\Pi(j,i)$ inductively as follows: $\Pi(i+1,i) = 1$ and if $j > i$, then $\Pi(j+1,i) = s_j \Pi(j,i)$. We record the following:

(10.3) LEMMA. *Let $j > i$.*

a) *If $k < i$ or $k > j$, then $s_k \Pi(j,i) = \Pi(j,i)s_k$.*

b) *If $i < k < j-1$, then $s_k \Pi(j,i) = \Pi(j,i)s_{k+1}$.*

c *If $i < k < j$, then $\Pi(j,i) = \Pi(j,k-1)\Pi(k,i)$.*

PROOF. Left to the reader.

Finally, record the well-known.

10.4) THEOREM. *$\Delta(0,1) = 1$ and if $n > 1$, then $\Delta(0,n) = \Delta(0,n-1)\Pi(n,0)$.*

PROOF. See [21] or [4].

Note that $\Pi(n,0) = \Delta(0,n-1)^{-1}\Delta(0,n)$. It is also easy to check that $\Pi(n,0) = \Delta(0,n)\Delta(1,n)^{-1}$. It also follows from (10.4) that if $i < j-1$, then $\Delta(i,j) = \Delta(i,j-1)\Pi(j,i)$: the function $s_k \mapsto s_{k+i}$ from $X(0,j-i)$ to $X(i,j)$ extends to an isomorphism from $S(0,j-i)$ onto $S(i,j)$ and (10.4) applies directly to $\Delta(0,j-i)$.

Next, we study $D_\alpha(\Delta_A)$ and $D_\alpha(\Delta_A\Delta_{A-s_k}^{-1})$ with D_α as defined in (7.1). For simplicity, we write D for D_1 so that if $w \in S^{(n)}$, then

$$D(w) = \sum_{uv=w} v$$

For $w \in S^{(n)}$, we also define

$$D'(w) = \sum_{uv=w} u$$

(10.5) LEMMA. *Let $A \subseteq X^{(n)}$.*

a) $D(\Delta_A) = D'(\Delta_A)$.

b) If $B \subseteq A$, then $D'(\Delta_A) = D'(\Delta_B)D'(\Delta_B^{-1}\Delta_A)$.

PROOF. (6.4) identifies the left factors of Δ_A with the right factors of Δ_A, proving (10.5a). (10.5b) is the mirror-image of (7.2).

(10.6) LEMMA. *$D'(\Pi(n,0)) = \sum_{j=1}^{n} \Pi(n,n-j)$. In particular, if $0 < i < n$, then $D'(\Pi(n,0)) = D'(\Pi(n,i)) + \Pi(n,i-1)D'(\Pi(i,0))$.*

PROOF. The formula for $D'(\Pi(n,0))$ follows from the fact that no defining relation of $S^{(n)}$ applies to $\Pi(n,0)$. The second part of (10.6) follows from the formula for $D'(\Pi(n,0))$.

For convenience, if $0 < i < n$, we let $\delta_i^{(n)}$ denote $\Delta(0,n)(\Delta(0,i)\Delta(i,n))^{-1}$. The formulae for $D(\delta_1^{(n)})$ and $D(\delta_{n-1}^{(n)})$ can be deduced from (10.6). The general case follows from:

(10.7) THEOREM. *Let $1 < i < n-1$. Then $D(\delta_i^{(n)}) = D(\delta_i^{(n-1)}) + D(\delta_{i-1}^{(n-1)})\Pi(n,i-1)$.*

PROOF. First note that

$$D(\Delta(0,n)) = D(\delta_i^{(n)})D(\Delta(0,i))D(\Delta(i,n))$$

by (7.2) and (10.1). Next note that

$$\begin{aligned} D(\Delta(0,n)) &= D'(\Delta(0,n)) \\ &= D'(\Delta(0,n-1))D'(\Pi(n,0)) \\ &= D(\Delta(0,n-1))(D'(\Pi(n,i)) + \Pi(n,i-1)D'(\Pi(i,0))) \end{aligned}$$

by (10.5a), the mirror-image of (7.2), (10.5a) and (10.6) in succession. Analyzing the pieces individually, we have

$$\begin{aligned} &D(\Delta(0,n-1))D'(\Pi(n,i)) \\ &\qquad = D(\delta_i^{(n-1)})D(\Delta(0,i))D(\Delta(i,n-1))D'(\Pi(n,i)) \\ &\qquad = D(\delta_i^{(n-1)})D(\Delta(0,i))D(\Delta(i,n)) \end{aligned}$$

first using (7.2) and (10.1) as above, and then using (10.5a), the mirror-image of (7.2) and then (10.5a) again. We also have

$$\begin{aligned} &D(\Delta(0,n-1))\Pi(n,i-1)D'(\Pi(i,0)) \\ &\qquad = D(\delta_{i-1}^{(n-1)})D(\Delta(0,i-1))D(\Delta(i-1,n-1))\Pi(n,i-1)D'(\Pi(i,0)) \\ &\qquad = D(\delta_{i-1}^{(n-1)})D(\Delta(0,i-1))\Pi(n,i-1)D(\Delta(i,n))D'(\Pi(i,0)) \\ &\qquad = D(\delta_{i-1}^{(n-1)})\Pi(n,i-1)D(\Delta(0,i-1))D'(\Pi(i,0))D(\Delta(i,n)) \\ &\qquad = D(\delta_{i-1}^{(n-1)})\Pi(n,i-1)D(\Delta(0,i))D(\Delta(i,n)) \end{aligned}$$

first using (7.2) and (10.1) as above, second using (10.3b) repeatedly, then using (10.3a) repeatedly and finally using (10.5a), the mirror-image of (7.2) and then (10.5a) again, as above. Comparing the two expressions for $D(\Delta(0,n))$ and then, as in the proof of (7.3), cancelling $D(\Delta(0,i))D(\Delta(i,n))$, (10.7) follows easily.

We remark that (10.7) includes a proof of the formula $\delta_i^{(n)} = \delta_{i-1}^{(n-1)}\Pi(n,i-1)$. Define $\delta_0^{(n)} = \delta_n^{(n)} = 1$, so that $D(\delta_0^{(n)}) = D(\delta_n^{(n)}) = 1$. It is easy to check that the recursion in (10.7) noe holds for $0 < i < n$.

(10.8) COROLLARY. *Let* $0 < i < n$. *Then* $D_\alpha(\delta_i^{(n)}) = \alpha^i D_\alpha(\delta_i^{(n-1)}) + D_\alpha(\delta_{i-1}^{(n-1)})$ $\Pi(n,i-1)$.

PROOF. The case $\alpha = 1$ follows from (10.7) and the remark above. For the general case, we assume that α is an indeterminate over $\mathbf{Z}$ and work over the Laurent-polynomial ring $\mathbf{Z}[\alpha,\alpha^{-1}]$ in order to prove (10.7) over the polynomial ring $\mathbf{Z}[\alpha]$; the general case follows by taking a suitable homomorphism from $\mathbf{Z}[\alpha]$ to the given ring R.

For the remainder of the proof, we let R denote $\mathbf{Z}[\alpha,\alpha^{-1}]$. For $w \in S^{(n)}$, define $\phi_\alpha(w) = \alpha^{|w|}w$. Extend D, D_α and ϕ_α to R-linear functions from $RS^{(n)}$ to itself.

Since α is a unit, ϕ_α is an R-linear (ring) automorphism of $RS^{(n)}$. It is easy to check that $D = \phi_\alpha D_\alpha \phi_\alpha^{-1}$. In particular, if $w \in S^{(n)}$, then $D(w) = \alpha^{-|w|}\phi_\alpha D_\alpha(w)$.

It follows easily from (10.4) that if $n > 0$, then $|\Delta(0,n)| = \frac{1}{2}n(n-1)$. In turn, it follows that if $0 < i < n$, then $|\delta_1^{(n)}| = i(n-i)$. Noting that $|\Pi(n, i-1)| = n - i$, (10.7) follows from (10.6) by a simple computation.

By (7.7), we can use (10.2) and (10.8) to explicitly describe a resolution of R as a trivial left $RB^{(n)}$-module ($\alpha = -1$).

For example, the ordinary integral homology of $B^{(n)}$ may be computed from the complex of free abelian groups that arises from (10.2) and (10.8) by leting $R = \mathbf{Z}$, $\alpha = -1$ and substituting 1 for each s_i. The image of $D_{-1}(\delta_i^{(n)})$ after substituting each $s_i = 1$ will be denoted $\binom{n}{i}_{-1}$. Clearly, each $\binom{n}{0}_{-1} = \binom{n}{n}_{-1} = 1$ and if $0 < i < n$, then $\binom{n}{i}_{-1} = (-1)^i\binom{n-1}{i}_{-1} + \binom{n-1}{i-1}_{-1}$. It is easy to check that if $0 \leqq i \leqq n$, then

$$\binom{n}{i}_{-1} = \begin{cases} 0 & \text{if } n \text{ is even, } i \text{ is odd,} \\ \dbinom{\left[\frac{n}{2}\right]}{\left[\frac{i}{2}\right]} & \text{otherwise} \end{cases}$$

where $[x]$ denotes the greatest integer $\leqq x$ and $\binom{n}{i}$ denotes an ordinary binomial coefficient. In particular, this complex agrees with that described by in [30]; see [30] (or [13]) for a complete description of the ordinary integral (co)homology of $B^{(n)}$.

For a second example, the ordinary integral homology of the commutator subgroup of $B^{(n)}$ may be computed, using Shapiro's lemma as in §9, from the complex of $\mathbf{Z}[t, t^{-1}]$-modules that arises from (10.2) and (10.8) by letting $R = \mathbf{Z}$, $\alpha = -1$ and substituting t for each s_i. The image of $D_{-1}(\delta_i^{(n)})$ after substituting each $s_i = t$ will be denoted $\binom{n}{i}_t$. Clearly, each $\binom{n}{0}_t = \binom{n}{n}_t = 1$ and if $0 < i < n$, then $\binom{n}{i}_t = (-1)^i\binom{n-1}{i}_t + t^{n-i}\binom{n-1}{i-1}_t$. It is easy to check that if $0 \leqq i \leqq n$, then $\binom{n}{i}_t = p_n/p_i p_{n-i}$, where, if $n \geqq 0$, then

$$p_n = \prod_{i=1}^{n} \frac{t^i - (-1)^i}{t+1}$$

so that, in particular, $p_0 = 1$. It is not difficult to use the complex just described to show that if $n \geqq 5$, then the second commutator subgroup of $B^{(n)}$ coincides with the first commutator subgroup (see [22]). Apparently (see [15]), a complete description of the ordinary integral (co)homology of the commutator subgroup of $B^{(n)}$ is not known.

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