THE HOMOLOGICAL ALGEBRA OF ARTIN GROUPS

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Following Magnus [25] and Brieskorn-Saito [8], an Artin group is an abstract group $G$ defined by presentation as follows. $G$ has as generators a set $X$. The defining relations of $G$ may be described as follows: to certain unordered pairs $\{x, y\}$ of distinct elements of $X$, an integer $m = m_{xy} \geq 2$ is assigned; the corresponding defining relation is $(xy)^q = (yx)^q$ if $m = 2q$ is even and $(xy)^q x = (yx)^q y$ if $m = 2q + 1$ is odd. An important example is Artin's braid group $B^{(n)}$ which is presented as follows: $X = \{s_1, \ldots, s_{n-1}\}$ with defining relations all $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ and all $s_is_j = s_js_i$ whenever $|i - j| \geq 2$.

Associated to each Artin group $G$ is a Coxeter group $W$ obtained from $G$ by imposing, for each $x \in X$, the additional relation $x^2 = 1$. For example, the Coxeter group associated with $B^{(n)}$ is the symmetric group on $n$ symbols.

It is well-known that each Coxeter group $W$ whose generating set $X$ (as above) has finite cardinality $n - 1$ admits a faithful linear representation in $GL_n(\mathbb{R})$. See, for example, [6] or [16]. It turns out that the space of regular orbits of the extension of this representation to $GL_n(\mathbb{C})$ is, if $W$ is finite, an Eilenberg-MacLane space for the corresponding Artin group $G$. (For $B^{(n)}$, see [19]; for all but a few special cases, see [7]; in general, see [17].) This fact has been the foundation for the study of the homological algebra of these groups. (For a complete description of the ordinary integral cohomology of $B^{(n)}$, see [30], which is based on [2] and [20]. For certain of the other Artin groups, see [23].) We remark that the homological algebra of $B^{(n)}$ has also been treated by homotopy-theoretic methods; see, for example, [26], [13] and [14].

Our purpose here is to give the foundations for a purely algebraic treatment of the homological algebra of Artin groups. Our approach is patterned after methods used by Garside [21] to solve the conjugacy problem in $B^{(n)}$ and used again by Brieskorn-Saito [8] to extend Garside's results to any Artin group whose associated Coxeter group is finite: instead of the Artin group $G$, we consider the monoid $S$ with the same presentation as $G$ (the Artin monoid). Our

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main result, (7.5) below, is a free resolution of certain left modules over the monoid ring RS of an Artin monoid S (with coefficients in an arbitrary commutative ring R). These RS-modules have as underlying R-module R itself with the action of S defined as follows: each x ∈ X acts as multiplication by a fixed but arbitrary element α of R. In particular, these RS-modules include R as a trivial RS-module (α = 1) so that the ordinary homological algebra of S can be recovered. (We remark that, in somewhat greater generality, the special case α = 0 will play an important role in §3 below. We also remark on a slight difference in notation between here and §7: α here will turn into −α in §7.)

The question arises as to when the RS-resolutions described above extend to RG-resolutions of suitable RG-modules. It turns out that if the Coxeter group W is locally finite, then the extension of scalars functor from the category of left RS-modules to the category of left RG-modules is exact (see §2 and §7). The following is an easy consequence of this remark and the actual descriptions of the resolutions discussed in the previous paragraph:

**Theorem A.** Let G be an Artin group whose associated Coxeter group is finite. Then G is of type FL [9, p. 199].

For the proof of Theorem A, see (7.7) and the proof of (7.8). Theorem A also follows from [17]; the Eilenberg-MacLane space for G described in [17] is clearly homotopy-equivalent to a finite complex. A consequence of Theorem A is the fact that G (as in Theorem A) is torsion-free (7.8). With a little bit more work (see §8), we are able to show:

**Theorem B.** Let G be as in Theorem A. Then G is a duality group [3, p. 138].

See (8.3). In this generality, Theorem B is new. For example, the fact that Artin's braid group B(n) is a duality group follows from Proposition VIII.10.2 of [9], using the fact that B(n) is torsion-free (which follows from [19] cited above) and the fact that B(n) has a subgroup of finite index which is a duality group (the kernel of the natural homomorphism from B(n) to the symmetric group on n symbols is an iterated semidirect product of free groups: for this fact, see Lemma 1.8.2 of [4]; for the fact that semidirect products of duality groups are duality groups, see Theorem 9.10 of [3]; for the fact that free groups are duality groups; see p. 233 of [9].)

An obvious question is: what happens when the associated Coxeter group is not locally finite? Combining (2.4) with (7.6), it follows that if G is an Artin group whose Coxeter group is not locally finite, then the extension of scalars functor from RS-modules to RG-modules is not exact. Nonetheless, the author conjectures that if α is a unit in R, then the extension of the RS-complex (C•, ∂•) in (7.5) from RS to RG is exact in positive dimension.

For convenience, this paper has been divided into three parts. The first part
consists of the author's attempt to extract from the proof of Theorem A those ideas that do not essentially involve the fact that $G$ is an Artin group (or, alternately, the fact that $S$ is an Artin monoid). The author hopes that these ideas will prove useful in other contexts. The second part concerns Artin groups; in particular, the theory developed in part I is applied to Artin groups in order to give proofs of Theorems A and B. The third part treats examples. Here is an outline:

I. Homological machinery
   §1. Diagrams
   §2. Monoids and homological group theory
   §3. Exactness and duality
   §4. Homology approximation

II. Artin groups
   §5. Preliminaries
   §6. Fundamental elements
   §7. Proof of Theorem A
   §8. Proof of Theorem B

III. Examples
   §9. $A_3$, $B_3$ and $H_3$
   §10. $B^{(n)}$

Each of parts I, II and III will include their own introduction.

Part of this paper (essentially Theorem A and its proof) was the main result in the author's Ph.D. dissertation [27] written under John Stallings. The author would like to thank Professor Stallings for his continued patient support and for many helpful suggestions.

Part I. Homological machinery.

Part I develops some homological preliminaries that will be applied to Artin groups in Part II. §1 introduces the notion of a diagram (1.2). (Essentially, a diagram is a functor $K \rightarrow \Lambda$. Here, $K$ is a simplicial complex, with the "empty simplex" adjoined, viewed as a category with objects the simplexes of $K$ and morphisms inclusion. Also, $\Lambda$ is an associative ring with 1 viewed as a category with one object, morphisms $\Lambda$ and composition given by multiplication in $\Lambda$.) Out of a diagram, we build a chain complex $(C_*, \partial_*)$: see (1.3) and (1.4); and a cochain complex $(C^*, \partial^*)$: see (1.5) and (1.6). An important goal below will be to study exactness properties of the complexes we have just described: see §3 and §7. We conclude §1 with a sample exactness theorem (1.7).

§2 consists primarily of standard facts concerning the extension of scalars functor from the category of left $RS$-modules to the category of left $RG$-modules. Here, $G$ is a group, $S$ is a submonoid of $G$ and $R$ is a commutative ring with 1.
In §3, we define the notion of an “exact” subset $X$ of a monoid $S$ (3.1), use this notion to define a diagram in $RS$ and show that the resulting complex $(C_*, \partial_*)$ of $RS$-modules is exact in positive dimension (3.2). If, in addition, $X$ is finite and satisfies the “duality” condition (3.3), then the dual complex $(C^*, \partial^*)$ is exact in positive codimension (3.5).

It turns out that the exactness theorems (3.2) and (3.5) are not very useful in homological group theory; in particular, see (3.6). Nonetheless, the resolutions $(C_*, \partial_*)$ and $(C^*, \partial^*)$ in §3, when they occur in the setting of Artin groups (see §7 and §8), turn out to be “top-degree approximations” of complexes which, by (4.1), are resolutions; these resolutions turn out to be useful in homological group theory. The proof of (4.1) is essentially an adaption of Stallings’ notion [28] of a “slow contracting homotopy” to the situation at hand. Our excessive concern in §4 with the fact that certain $R$-modules are free $R$-modules, as in (4.2), (4.3b) and (4.4b), results from the current state of knowledge about duality groups; for further discussion, see the introduction to Part II and §8.

§1. Diagrams.

Let $X$ be a set and let $<$ be a strict local ordering of $X$ ($<$ is transitive, irreflexive and satisfies the law of trichotomy). If $A$ is a finite subset of $X$ and $x \in X$, then $x(A)$ will denote the number of $y \in A$ such that $y < x$. Let $A \subseteq X$ and $x \in X$. If $x \in A$, then $A - x$ will denote the set difference $A - \{x\}$. If $x \notin A$, then $A + x$ will denote the set union $A \cup \{x\}$.

(1.1) Lemma. Let $A$ be a finite subset $X$ and let $x, y \in X$.

(a) If $x, y \in A$ and $x \neq y$, then $(-1)^{x(A) + y(A - x)} + (-1)^{y(A) + x(A - y)} = 0$.

(b) If $x \in A$ and $y \notin A$, then $(-1)^{x(A) + y(A - x)} + (-1)^{y(A) + x(A + y)} = 0$.

(c) If $x, y \notin A$ and $x \neq y$, then $(-1)^{x(A) + y(A + x)} + (-1)^{y(A) + x(A + y)} = 0$.

Proof. In a), we may assume, by symmetry, that $x < y$. Then 

$y(A - x) = y(A) - 1$ and $x(A - y) = x(A)$, so the exponents differ by 1, as required. The proofs of b) and c) are similar.

Given $X$ as above, let $\mathcal{P}$ be a collection of finite subsets of $X$ which satisfy:

P0) $\emptyset \in \mathcal{P}$.

P1) For each $x \in X$, $\{x\} \in \mathcal{P}$.

P2) If $A \in \mathcal{P}$ and $B \subseteq A$, then $B \in \mathcal{P}$.

Except for P0, this is the definition of a simplicial complex. Given $X$ and $\mathcal{P}$ as above, define $\mathcal{P}^{(2)} = \{(A, B) | A \in \mathcal{P}$ and $B \subseteq A\}$. Let $A$ be an associative ring (with 1).

(1.2) Definition. A diagram of $(X, \mathcal{P})$ in $A$ is a function $D: \mathcal{P}^{(2)} \rightarrow A$ which satisfies
a) if \( A \in \mathcal{P} \), then \( D(A, A) = 1 \).

b) if \( C \subseteq B \subseteq A \in \mathcal{P} \), then \( D(A, C) = D(A, B)D(B, C) \).

If \( \mathcal{P} \) consists of all finite subsets of \( X \), then \( D \) will be called a full diagram. Note that if \( D \) is a diagram of \((X, \mathcal{P})\) in \( A \) and \( \phi: A \to A' \) is a ring homomorphism (preserving 1), then the formula \( D_\phi(A, B) = \phi(D(A, B)) \) defines a diagram of \((X, \mathcal{P})\) in \( A' \).

With \( X, \mathcal{P}, A \) and \( D \) as above, let \( C_k \) denote the free left \( A \)-module with a generator denoted \([A] \) corresponding to each \( k \)-element \( A \in \mathcal{P} \).

(1.3) DEFINITION. For \( k > 0 \), define \( \partial_k: C_k \to C_{k-1} \) on generators by
\[
\partial_k([A]) = \sum_{x \in A} (-1)^{x(A)}D(A, A - x)[A - x]
\]
and extend to \( C_k \) by \( A \)-linearity.

(1.4) LEMMA. If \( k \geq 2 \), then \( \partial_{k-1}\partial_k = 0 \).

PROOF. Let \( A \in \mathcal{P} \) have cardinality \( \geq 2 \), let \( x, y \in A \) and suppose that \( x \neq y \). In the expansion of \( \partial_{k-1}\partial_k([A]), [A - x - y] \) appears twice; the coefficients are the same by (1.2b) and the signs are opposite by (1.1a).

In particular, the pair \((C_*, \partial_*)\) is a \( A \)-complex. We call \( \partial_* \) the differential associated to \( D \). We shall be interested in sufficient conditions for \((C_*, \partial_*)\) to be exact in positive dimension: if \( k > 0 \), then \( \ker \partial_k = \text{im} \partial_{k+1} \). In this situation, we call \( D \) exact and call \( C_0/\text{im} \partial_1 \) the module of \( D \).

With the same notation as above, assume further that \( X \) is finite. The cardinality of \( X \) will be denoted \( n - 1 \). In this situation, let \( C_k \) denote the free right \( A \)-module with a generator denoted \( \langle A \rangle \) corresponding to each \( k \)-element \( A \in \mathcal{P} \).

(1.5) DEFINITION. For \( 0 \leq k < n - 1 \), define \( \partial^k: C^k \to C^{k+1} \) on generators by
\[
\partial^k(\langle A \rangle) = \sum_{x \in A} (-1)^{x(A)}\langle A + x \rangle D(A + x, A)
\]
and extend to \( C^k \) by \( A \)-linearity. (Here, we will be most interested in the situation when \( D \) is a full diagram; otherwise, we adopt the convention that \( \langle A + x \rangle = 0 \) whenever \( A + x \notin \mathcal{P} \).)

(1.6) LEMMA. If \( 0 \leq k < n - 2 \), then \( \partial^{k+1}\partial^k = 0 \).

PROOF. Mimic the proof of (1.4), using (1.1c) in place of (1.1a).

The cochain complex \((C^*, \partial^*)\) may be naturally identified with the \( A \)-dual of \((C_*, \partial_*): C^k = \text{Hom}_A(C_k, A) \) with its natural right \( A \)-module structure. Under this identification, \( \langle A \rangle \) is given by
\[ \langle A \rangle([B]) = \begin{cases} 1 & \text{if } B = A \\ 0 & \text{otherwise} \end{cases} \]

We shall be interested in sufficient conditions for \((C^*, \partial^*)\) to be exact in positive codimension: if \(0 < k < n - 1\), then \(\ker \partial^k = \im \partial^{k-1}\). In this situation, we call \(D\) co-exact and call \(C^{n-1}/\im \partial^{n-2}\) the co-module of \(D\).

We conclude this section by giving a sufficient condition for a diagram \(D\) to be exact in all dimensions. (In other words, \(D\) is exact in positive dimension, as above, and the module of \(D\) is the zero-module.) With \(X, \mathcal{P}, \Lambda\) and \(D\) as above, \(z \in X\) is called a cone-point for \(D\) provided for each \(A \in \mathcal{P}\), either \(z \in A\) or \(A + z \in \mathcal{P}\) and, in the second case, \(D(A + z, A)\) is a unit of \(\Lambda\).

(1.7) Theorem. If the diagram \(D\) has a cone-point \(z\), then \((C^*, \partial^*)\) is exact in all dimensions.

Proof. For each \(k \geq 0\), define \(s_k: C_k \to C_{k+1}\) by
\[
s_k([A]) = \begin{cases} 0 & \text{if } z \in A \\ (-1)^{z(A)}D(A + z, A)^{-1}[A + z] & \text{if } z \notin A \end{cases}
\]
and extend to \(C_k\) by \(\Lambda\)-linearity. Defining \(C_{-1} = 0\), \(\partial_0 = 0\) and \(s_{-1} = 0\), we show that if \(k \geq 0\) and \(W \in C_k\), then \((s_{k-1} \partial_k + \partial_{k+1}s_k)(W) = W\) from which (1.7) follows easily. Since each \(s_k\) and \(\partial_k\) is \(\Lambda\)-linear, it suffices to check that for each \(k\)-element \(A \in \mathcal{P}\), \(s_{k-1} \partial_k + \partial_{k+1}s_k)([A]) = [A]\). There are two cases: \(z \in A\) and \(z \notin A\). For the remainder of the proof, we will omit subscripts from \(\partial\) and \(s\).

If \(z \in A\), then \(s([A]) = 0\) so that
\[
(s\partial + \partial s)([A]) = s\partial([A])
\]
\[
= s \left( \sum_{x \in A} (-1)^{z(A)}D(A, A - x)[A - x] \right)
\]
\[
= s((-1)^{z(A)}D(A, A - z)[A - z])
\]
\[
= [A]
\]
where the third equality follows from the fact that \(s([A - x]) = 0\) if \(x \neq z\).

If \(z \notin A\) (in particular, if \(A = [\emptyset]\)), then
\[
\partial s([A]) = \partial((-1)^{z(A)}D(A + z, A)^{-1}[A + z])
\]
\[
= \sum_{x \in A + z} (-1)^{z(A + x(A + z))}D(A + z, A)^{-1}D(A + z, A + z - x)[A + z - x]
\]
\[
= [A] - \sum_{x \in A} (-1)^{z(A - x) + x(A)}D(A + z, A)^{-1}D(A + z, A + z - x)
\]
\[
\times [A + z - x]
\]
where the third inequality follows from (1.1b). Also,
\[ s\hat{\partial}([A]) = s \left( \sum_{x \in A} (-1)^{x(A)} D(A, A - x)[A - x] \right) \]
\[ = \sum_{x \in A} (-1)^{x(A) + x(A - x)} D(A, A - x) D(A - x + z, A - x)^{-1}[A - x + z]. \]
Since \( D(A + z, A)^{-1}D(A + z, A + z - x) = D(A, A - x)D(A - x + z, A - x)^{-1} \)
follows from (1.2b), we have \((\hat{s}\partial + s\hat{\partial})([A]) = [A]\) here as well.

§2. Monoids and homological group theory.

Let \( G \) be a group and let \( S \) be a submonoid of \( G \). In other words, \( S \) is a multiplicatively closed subset of \( G \) which contains the identity element 1 of \( G \). In particular, \( S \) is a two-sided cancellation monoid: if \( u, v, w \in S \) and either \( uw = vw \) or \( uw = vw \), then \( u = v \). With \( G \) and \( S \) as above, we say that \( S \) fills \( G \) (on the left) provided for each \( g \in G \), there exist \( u, v \in S \) such that \( g = u^{-1}v \).

(2.1) Lemma. If \( S \) fills \( G \), then for each \( x, y \in S \), there exist \( u, s \in S \) such that \( ux = vy \) in \( S \).

Proof. Given \( x, y \in S \), let \( g = xy^{-1} \in G \). By hypothesis, there exist \( u, v \in S \) such that \( g = u^{-1}v \). It follows easily that \( ux = vy \) in \( S \), as required.

If \( S \) is a monoid and \( x, y \in S \), then \( w \in S \) is called a common left multiple of \( x \) and \( y \) provided there exist, \( u, v \in S \) such that \( w = ux = vy \) in \( S \). A two-sided cancellation monoid in which every pair of elements has a common left multiple is said to satisfy the left Ore condition. By (2.1), if a monoid \( S \) fills a group \( G \), then \( S \) satisfies the left Ore condition. This fact has the following converse:

(2.2) Theorem. Suppose that the monoid \( S \) satisfies the left Ore condition. Then, up to isomorphism, there exists a unique group \( G \) such that \( S \) fills \( G \).

Proof. For a construction of \( G \), see [12, p. 35]. From [12], it is also clear that any monoid homomorphism from \( S \) to a group \( H \) factors through a group homomorphism from \( G \) to \( H \). To prove uniqueness of \( G \), it suffices to show that if \( \phi: G \to H \) is a group homomorphism whose restriction to \( S \) is injective, then \( \phi \) is injective. But if \( g \in G \) satisfies \( \phi(g) = 1 \), then, writing \( g = u^{-1}v \) with \( u, v \in S \), it follows that \( \phi(u) = \phi(v) \). By hypothesis, \( u = v \) in \( S \) so \( g = 1 \), as required.

We turn to homological properties of groups and monoids that fill them. Let \( R \) be a commutative ring (with 1), let \( G \) be a group and let \( S \) be a submonoid of \( G \). Then the monoid ring \( RS \) is a subring of the group ring \( RG \). Thus, by "restriction of scalars", left \( RG \)-modules can be viewed as left \( RS \)-modules. Conversely, viewing \( RG \) as a left \( RG \)-, right \( RS \)-module, there is an "extension of scalars"
functor \((-\)) \mapsto RG \otimes_{RS} (-)\) from the category of left RS-modules to the category of
left RG-modules.

(2.3) **Theorem.** If \(S\) fills \(G\), then the functor \((-\)) \mapsto RG \otimes_{RS} (-)\) is exact.

**Proof.** See [10, p. 191]. Briefly, the proof of (2.3) proceeds by showing that, as
a right RS-module, \(RG\) is the direct limit of the system \(\{w^{-1}(RS) | w \in S\}\) of right
RS-modules. Clearly, each \(w^{-1}(RS)\) is a free right RS-module. By (2.1), the system
above is directed and has limit \(RG\). It follows that \(RG\) is a flat right RS-module so
that \((-\)) \mapsto RG \otimes_{RS} (-)\) is exact.

We conclude this section by noting that (2.3) has the following converse:

(2.4) **Theorem.** Let \(G\) be a group and let \(S\) be a submonoid of \(G\). If \(S\) does not fill
the subgroup of \(G\) that it generates, then the functor \((-\)) \mapsto RG \otimes_{RS} (-)\) is not
exact.

**Proof.** Clearly, \(S\) is a two-sided cancellation monoid. By hypothesis and (2.2),
\(S\) cannot satisfy the Ore condition. It follows that there exist \(x, y \in S\) such that for
all \(u, v \in S, ux \neq vy\). Let \(M\) denote the free left RS-module with basis \(\{e_1, e_2\}\).
Define a left RS-module homomorphism \(\phi: M \to RS\) by \(\phi(e_1) = x\) and \(\phi(e_2) = y\).
Clearly, \(\phi\) is injective. Note that \(RG \otimes_{RS} M\) is a free left RG-module with basis
\(\{1 \otimes e_1, 1 \otimes e_2\}\). Also note that \(x^{-1} \otimes e_1 - y^{-1} \otimes e_2 \neq 0\) in \(RG \otimes_{RS} M\)
and belongs to the kernel of the extension of \(\phi\) to \(RG \otimes_{RS} M\). It follows that the
extension of \(\phi\) to \(RG \otimes_{RS} M\) is not injective, so that \((-\)) \mapsto RG \otimes_{RS} (-)\) is not
exact.

§3. Exactness and duality.

Let \(S\) be a monoid and let \(A\) be a subset of \(S\). We call \(w \in S\) a common left multiple
of \(A\) provided there is a function \(f: A \to S\) such that for each \(a \in A, w = f(a)a\).
(Note that any element of \(S\) is a common left multiple of the empty set \(\emptyset\) and that if
\(B \subseteq A\), then a common left multiple of \(A\) is a common left multiple of \(B\).) We call
\(w \in S\) a least common left multiple of \(A\) provided \(w\) is a common left multiple of
\(A\) and for each common left multiple \(u\) of \(A\), there exists \(v \in S\) such that \(u = vw\). (It
follows that the identity \(1 \in S\) is a least common left multiple of \(\emptyset\) and that if \(x \in X,\)
then \(x\) is a least common left of \(\{x\}\).

(3.1) **Definition.** Let \(S\) be a monoid. A subset \(X\) of \(S\) is called exact provided
any finite subset of \(X\) which has a common left multiple in \(S\) also has a least
common left multiple in \(S\).

If \(X\) is an exact subset of the monoid \(S\), define \(\mathcal{P}\) to consist of all finite subsets of
\(X\) which have a common left multiple in \(S\). It follows from the remarks preceding
(3.1) that \( P \) satisfies conditions P0), P1) and P2) of §1. Assume that for each \( A \in P \)
a least common left multiple \( \Delta_A \) of \( A \) in \( S \) has been chosen such that \( \Delta_{A_\phi} = 1 \) and if \( x \in X \), then \( \Delta_{A}(x) = x \). Note that if \( A \in P \) and \( B \subseteq A \), then \( \Delta_A \) is a left multiple of \( \Delta_B \).

Let \( S \) be a right cancellation monoid: if \( x, y, z \in S \) satisfy \( xz = yz \), then \( x = y \).
Equivalently, if \( x \in S \) is a left multiple of \( z \in S \), then there exists a unique \( y \in S \) such
that \( x = yz \); in this situation, we write \( y = xz^{-1} \). Clearly, if \( x \in S \), then \( x \) is a left
multiple of \( x \) and \( xx^{-1} = 1 \). Similarly, if \( x, y, z \in S \) satisfy: \( x \) is a left multiple
of \( y \) and \( y \) is a left multiple of \( z \), then \( x \) is a left multiple of \( z \) and \( xz^{-1} = (xy^{-1})(yz^{-1}) \).

Let \( S \) be a right cancellation monoid, let \( X \) be an exact subset of \( S \) and let \( R \) be
a commutative ring (with 1). Define \( D_X: P(\mathbb{Z}) \to RS \) by \( D_X(A, B) = \Delta_A \Delta_B^{-1} \). (As
noted above, if \( B \subseteq A \in P \), then \( \Delta_A \) is a left multiple of \( \Delta_B \).) Clearly, (1.2) is
satisfied, so that \( D_X \) is a diagram of \((X, P) \) in \( RS \).

(3.2) THEOREM. If \( S \) is a right cancellation monoid and \( X \) is an exact subset of \( S \),
then \( D_X \) is an exact diagram.

PROOF. Totally order \( X \). If \( A \in P \) has \( k(> 0) \) elements, then
\[
\partial_k([A]) = \sum_{x \in A} (-1)^{\pi(A)} \Delta_A \Delta_A^{-1} x [A - x].
\]
We define for each \( k \geq 0 \) an \( R \)-linear homomorphism \( s_k: C_k \to C_{k+1} \) such that if \( w \in S \) and \( \emptyset \neq A \in P \), then
\[
(\partial_{k+1}s_k + s_{k-1}\partial_k)(w[A]) = w[A]
\]
where \( A \) has \( k \) elements. It will follow that if \( k > 0 \), then \( \ker \partial_k = \operatorname{im} \partial_{k+1} \), as
required.

To define \( s_k \), first define \( \zeta: S \to X \cup \{0\} \) as follows: if \( w \in S \), then \( \zeta(w) = 0 \) if and
only if for each \( x \in X \), \( w \) is not a left multiple of \( x \). Otherwise, \( \zeta(w) \) is a chosen
element of \( X \) of which \( w \) is a left multiple (so that \( w\zeta(w) \) is defined).

As a left \( R \)-module, \( C_k \) is free on all \( w[A] \) as \( w \) ranges over all elements of \( S \) and
\( A \) ranges over all \( k \)-element sets in \( P \). With this in mind, define
\[
s_k(w[A]) = \begin{cases} 0 & \text{if } \zeta(wA) \in A \cup \{0\} \\((-1)^{\pi(A)}wA \Delta_A^{-1}[A + z] & \text{if } z = \zeta(wA) \notin A \cup \{0\} \end{cases}
\]
and extend to \( C_k \) by \( R \)-linearity. (If \( z = \zeta(wA) \notin A \cup \{0\} \), then there exists \( w_1 \in S \)
such that \( wA = w_1z \), so that \( wA \) is a common left multiple of \( A + z \) and
therefore a left multiple of \( A_{A+z} \). Also, \( \zeta(wA) = 0 \) can only arise if \( A = \emptyset \): if \( A \neq \emptyset \), then \( \zeta(wA) \in X \).

For the remainder of the proof, we omit subscripts from \( s \) and \( \partial \). Let \( w \in S \) and
\( \emptyset \neq A \in P \). Since \( A \neq \emptyset \), there are two cases: \( \zeta(wA) \in A \) and
\( \zeta(wA) \notin A \). If \( z = \zeta(wA) \in A \), then \( s(w[A]) = 0 \) so that
\[(s \partial + \partial s)(w[A]) = s \partial(w[A])\]
\[= s \left( \sum_{x \in A} (-1)^{x(A)} w \Delta_A A^{-1}_{A - x} [A - x] \right)\]
\[= s((-1)^{x(A)} w \Delta_A A^{-1}_{A - z} [A - z])\]
\[= w[A]\]

where the third equality follows from the fact that \(z = \xi((w \Delta_A A^{-1}_{A - x}) A_{A - z}) \in A - x\) if \(x \neq z\). If \(z = \xi(w \Delta_A) \notin A\), then
\[\partial s(w[A]) = \partial((-1)^{x(A)} w \Delta_A A^{-1}_{A + z} [A + z])\]
\[= \sum_{x \in A + z} (-1)^{x(A) + x(A + z)} w \Delta_A A^{1}_{A + z - x} [A + z - x]\]
\[= w[A] - \sum_{x \in A} (-1)^{x(A) + x(A - x)} w \Delta_A A^{-1}_{A + z - x} [A + z - x]\]

where the third equality uses (1.1b). Also
\[s \partial(w[A]) = s \left( \sum_{x \in A} (-1)^{x(A)} w \Delta_A A^{-1}_{A - x} [A - x] \right)\]
\[= \sum_{x \in A} (-1)^{x(A) + z(A - x)} w \Delta_A A^{-1}_{A - x + z} [A - x + z]\]

since for each \(x \in A, z \notin A - x\). Thus \((s \partial + \partial s)(w[A]) = w[A]\) in this case as well.

The module \(C_0/im \partial_1\) of \(D_X\) can be described as follows: \(C_0/im \partial_1\) is isomorphic to \(RS\) modulo the left ideal generated by \(X\). It follows that \(C_0/im \partial_1\) is a free \(R\)-module with basis corresponding to \(\{w \in S \mid \xi(w) = 0\}\). (Note that if \(S\) is a group and \(X \neq \emptyset\), then the module of \(D_X\) is the zero-module. (See Lemma (4.2).)

In the following definition, we use the notion of a (least) common right multiple.

(3.3) DEFINITION. Let \(S\) be a right cancellation monoid and let \(X \subseteq S\) be exact. Then \(X\) is said to satisfy the duality condition provided whenever \(A \in \mathcal{P}\) and \(x, y \in A\) satisfy \(x \neq y\), it follows that \(A \Delta A^{-1}_{A - x - y}\) is a least common right multiple of \(\{A \Delta A^{-1}_{A - x}, A \Delta A^{-1}_{A - y}\}\).

It follows from the right cancellation property in \(S\) that the duality condition for \(X\) is independent of the choice of \(A\)'s.

(3.4) LEMMA. Let \(S\) be a two-sided cancellation monoid and suppose that \(X \subseteq S\) satisfies the duality condition. If \(A \in \mathcal{P}\) and \(B \subseteq A\), then \(A \Delta B^{-1}\) is a least common right multiple of \(\{A \Delta A^{-1}_{A - x} \mid z \in A - B\}\).
PROOF. Note that if \( z \in A - B \), then \( B \subseteq A - z \) so that \( A_A A_B^{-1} = (A_A A_{A- z})^{-1} \). Thus \( A_A A_B^{-1} \) is a common right multiple of \( \{ A_A A_{A- z}^{-1} | z \in A - B \} \).

We prove (3.4) by induction on the cardinality of \( A - B \). If this cardinality is 0 or 1, (3.4) is easy. In general, choose \( x, y \in A - B \) with \( x \neq y \). By the inductive hypothesis, \( A_A A_{A- x}^{-1} \) is a least common right multiple of \( \{ A_A A_{A- z}^{-1} | z \in A - B - x \} \) and \( A_A A_{A- y}^{-1} \) is a least common right multiple of \( \{ A_A A_{A- z}^{-1} | z \in A - B - y \} \).

Assume that \( w \in S \) is a common right multiple of \( \{ A_A A_{A- z}^{-1} | z \in A - B \} \). Thus for each \( z \in A - B \), there exists \( w_z \in S \) such that \( w = A_A A_{A- z}^{-1} w_z \). Since \( A_A A_B^{-1} \) is a least common right multiple of \( \{ A_A A_{A- z}^{-1} | z \in A - B - x \} \), there exists \( u \in S \) such that \( w = A_A A_{A- x}^{-1} u \). Similarly, there exists \( v \in S \) such that \( w = A_A A_{A- y}^{-1} v \). It follows that

\[
(\begin{array}{c}
(A_A A_{A- x}) (A_B A_{A- x+y})^{-1} u = A_A A_{A- x}^{-1} u \\
= w \\
= A_A A_{A- y}^{-1} v \\
= (A_A A_{A- x+y}) (A_B A_{A- x+y})^{-1} v
\end{array})
\]

so that, by left cancellation, \( (A_B A_{A- x+y})^{-1} u = (A_B A_{A- x+y})^{-1} v \). By (3.3), there exists \( w_0 \in S \) such that \( (A_B A_{A- x+y})^{-1} u = (A_B A_{A- y})^{-1} w_0 \). Clearly, \( w = (A_A A_{A- y})^{-1} w_0 \), as required.

(3.5) Theorem. Let \( S \) be a two-sided cancellation monoid and let \( X \) be a finite subset of \( S \) which has a common left multiple and satisfies the duality condition. Then \( D_X \) is co-exact.

PROOF. We use (3.4) to reduce the proof of (3.5) to (3.2). For each \( x \in X \), define \( \tilde{x} = A_X A_{X- x}^{-1} \) and for each \( A \subseteq X \), define \( \tilde{A} = \{ x | x \in A \} \). By (3.5), if \( \tilde{A} \subseteq \tilde{X} \), then \( A = A_{X- A}^{-1} \) is a least common right multiple of \( \tilde{A} \), so that \( \tilde{X} \) is exact (on the right). Thus (3.2) applies: the (right) diagram \( D_X \) is exact. The associated complex, which we denote \( (\tilde{C}_*, \tilde{\partial}_*) \), may be described as follows: each \( \tilde{C}_k \) is a free right RS-module on all \( [\tilde{A}] \) for \( A \) a \( k \)-element subset of \( X \) and

\[
\tilde{\partial}_k([\tilde{A}]) = \sum_{x \in \tilde{A}} (-1)^{x(A)} [A - x]^{-1} A_{X- x + A} A_{X- A}^{-1}.
\]

We show that the complexes \( (\tilde{C}_*, \tilde{\partial}_*) \) and \( (C^*, \partial^*) \) are chain isomorphic. Let \( n - 1 \) denote the cardinality of \( X \) and for \( 0 \leq k \leq n - 1 \), define \( \phi_k: \tilde{C}_k \rightarrow C^{(n-1)-k} \) on generators by

\[
\phi_k([\tilde{A}]) = (-1)^{d(x)} [X - A]
\]

and extend to \( \tilde{C}_k \) by RS-linearity. Here, \( A(X) = \sum_{x \in A} x(X) \). Each \( \phi_k \) is an isomorphism of right RS-modules. Note that if \( x \in A \subseteq X \), then \((-1)^{d(x) + x(X - A)} = \)
\((-1)^{\chi(A)} + (A - x)(X)\). It follows that if \(0 \leq k \leq n - 1\), then \(\bar{\partial}^{n-k-1}\phi_k = \phi_k \tilde{\phi}_k\). Since \((\bar{C}_*, \bar{\partial}_*)\) is exact in positive dimension, \((C^*, \partial^*)\) is exact in positive codimension, as required.

We conclude this section by noting a situation in which (3.2) is particularly uninteresting:

(3.6) Theorem. In the situation of (3.2), if \(X\) contains a unit of \(S\), then the module of \(D_X\) is the zero module and the contracting homotopy of \((C_*, \partial_*)\) can be chosen to be the RS-linear.

Proof. The first conclusion follows from the remark following the proof of (3.2). To prove the second conclusion, we appeal to (1.7): note that if \(z \in X\) is a unit and \(A \in \mathcal{A}\) satisfies \(z \notin A\), then \(\Delta_A\) is a common left multiple of \(A + z\). It follows that \(z\) is a cone point of \(D_X\) so that (1.7) applies; all of (3.6) follows easily.


We begin with a homology “approximation” lemma. Let \(R\) be a commutative ring with 1. For each \(k \geq 0\), let \(C_k\) be an \(R\)-module graded by the non-negative integers: \(C_k = \bigoplus_{p=0}^{\infty} C_k(p)\). An \(R\)-module homomorphism \(\bar{\partial}_k: C_k \to C_{k-1}\) is said to be homogeneous provided each \(\bar{\partial}_k(C_k(p)) \subseteq C_{k-1}(p)\). A second \(R\)-module homomorphism \(d_k: C_k \to C_{k-1}\) is said to be dominated by \(\bar{\partial}_k\) provided whenever \(c \in C_k(p)\), it follows that \((d_k - \bar{\partial}_k)(c) \in \bigoplus_{q=0}^{p-1} C_{k-1}(q)\). Finally, let \((C_*, \partial_*)\) be a chain complex with each \(C_k\) graded and each \(\partial_k\) homogeneous as above. Note that if \((C_*, \partial_*)\) is contractible in positive dimension (for each \(k \geq 0\), there exists an \(R\)-module homomorphism \(s_k: C_k \to C_{k+1}\) such that if \(k > 0\), then \(\partial_{k+1}s_k + s_{k-1}\partial_k\) is the identity on \(C_k\), then the \(s_k\)'s can be chosen to be homogeneous (each \(s_k(C_k(p)) \subseteq C_{k+1}(p)\)).

(4.1) Theorem. Let \((C_*, \partial_*)\) and \((C_*, d_*)\) be chain complexes. Assume that each \(C_k\) is graded, each \(\partial_k\) is homogeneous, each \(d_k\) is dominated by \(\bar{\partial}_k\) and \((C_*, \partial_*)\) is contractible in positive dimension (all as above). Then

a) \((C_*, d_*)\) is contractible in positive dimension.

b) \(C_0/\text{im} \bar{\partial}_1\) and \(C_0/\text{im} d_1\) are isomorphic \(R\)-modules.

Proof. Clearly, we may assume that \(C_k = 0\) if \(k < 0\). It suffices to show that \((C_*, \partial_*)\) and \((C_*, d_*)\) are isomorphic chain complexes over \(R\). Choose a contracting homotopy \(\{s_\bullet\}\) for \((C_*, \partial_*)\) so that each \(s_k\) is homogeneous. Note that if \(k > 0\), then \(\partial_k s_{k-1} \partial = \partial_k\). For \(k \geq 0\), define \(f_k: C_k \to C_k\) by the following formula: 

\[ f_k = d_{k+1}s_k + (1 - \partial_{k+1}s_k). \]

Clearly, if \(k > 0\), then
\[ d_k f_k - f_{k-1} \partial_k = d_k (1 - \partial_{k+1} s_k) - d_k s_{k-1} \partial_k \]
\[ = d_k (1 - \partial_{k+1} s_k - s_{k-1} \partial_k) \]
\[ = 0. \]

(The first equality uses \( d_k d_{k+1} = 0 \) and \( \partial_k s_{k-1} \partial_k = \partial_k \).) Thus \( f_* : (C_*, \partial_*) \to (C_*, d_*) \) is a morphism of chain complexes.

To see that each \( f_k \) is an isomorphism of R-modules, note that \( f_k = 1 - (\partial_{k+1} - d_{k+1}) s_k \). Since \( \partial_{k+1} \) dominates \( d_{k+1} \) and \( s_k \) is homogeneous, \( (\partial_{k+1} - d_{k+1}) s_k \) is locally nilpotent: if \( c \in C_k \), then there exists \( N \geq 0 \) such that \( ((\partial_{k+1} - d_{k+1}) s_k)^N(c) = 0. \) (In fact, if \( c \in C_k(p) \), then \( ((\partial_{k+1} - d_{k+1}) s_k)^p c = 0. \)) It follows that each \( f_k \) is invertible. (In fact, \( f_k^{-1} = \sum_{n=0}^{\infty} ((\partial_{k+1} - d_{k+1}) s_k)^n. \))

We view \( \partial_k \) as a “top-degree approximation” of \( d_k \).

In order to apply (4.1) to (3.2) and (3.5), we assume that the monoid \( S \) is equipped with a homomorphism \( w \mapsto |w| : S \to \mathbb{N} \). (Here \( \mathbb{N} \) denotes the set of non-negative integers, viewed as a monoid under addition. By definition, if \( u, v \in S \), then \( |uv| = |u| + |v| \). Also, \( |1| = 0 \).) In this situation, we will say that \( S \) admits an \( \mathbb{N} \)-valued length homomorphism.

If \( S \) admits an \( \mathbb{N} \)-valued length homomorphism and \( X \subseteq S \) is exact, then defining \( C_* \) as in §3, we extend length notation to each \( C_k \) as follows. First, if \( w \in S \) and \( A \in \mathcal{P} \) has cardinality \( k \), then \( |w[A]| = |w A_A| \). Also, if \( W \in C_k \) is an \( R \)-linear combination of \( w[A] \)'s as above, then \( |W| \) is the largest \( |w[A]| \) among the \( w[A] \)'s that occur with non-zero coefficient. (Recall that \( C_k \) is a free \( R \)-module on the \( w[A] \)'s.) We let \( C_k(p) = \{ W \in C_k \mid |W| = p \} \). Clearly, each \( \partial_k \) and \( s_k \) in (3.2) is homogeneous.

(4.2) **Lemma.** In the situation of (3.2), \( C_0/\text{im} \partial_1 \) is a free \( R \)-module.

**Proof.** In the notation of (3.2)
\[ s_0(w[\emptyset]) = \begin{cases} 0 & \text{if } \xi(w) = 0 \\ wz^{-1}[z] & \text{if } z = \xi(w) \in X. \end{cases} \]

Note that if \( \xi(w) \in X \), then \( \partial_1 s_0(w[\emptyset]) = w[\emptyset] \). It follows easily that \( s_0 \partial_1 s_0 = s_0 \) and \( \partial_1 s_0 \partial_1 = \partial_1 \). We claim that \( C_0 \) is an (internal) direct sum \( \text{im} \partial_1 \oplus \ker s_0 \) as an \( R \)-module. First, if \( W \in C_0 \), then \( W = \partial_1 s_0 W + (1 - \partial_1 s_0) W \). Since \( s_0 \partial_1 s_0 = s_0, \)
\( (1 - \partial_1 s_0) W \in \ker s_0 \) so that \( C_0 = \text{im} \partial_1 + \ker s_0 \). To show \( \text{im} \partial_1 \cap \ker s_0 = 0 \), assume \( W \in C_1 \) satisfies \( \partial_1 W \in \ker s_0 \). Then \( \partial_1 W = \partial_1 s_0 \partial_1 W = 0 \), as required.

It follows that \( C_0/\text{im} \partial_1 \) is isomorphic to \( \ker s_0 \) as an \( R \)-module. Clearly, \( \ker s_0 \) is a free module on \( \{ w[\emptyset] \mid \xi(w) = 0 \} \), as required.

(4.3) **Corollary.** In this situation (and notation) of (3.2), suppose that \( S \) admits
an N-valued length homomorphism and that \((C_*, d_*)\) is a chain complex such that each \(d_k\) is dominated by \(\partial_k\). Then

\begin{itemize}
  \item[(a)] \((C_*, d_*)\) is exact in positive dimension.
  \item[(b)] \(C_0/\text{im } d_1\) is a free \(R\)-module.
\end{itemize}

**Proof.** By (3.2) and its proof, the hypotheses of (4.1) are satisfied. Thus \((C_*, d_*)\) is contractible in positive dimension by (4.1a) from which (4.3a) follows. Also, (4.3b) follows from (4.1b) and (4.2).

In (4.3), it is only necessary to assume that each \(d_k\) is \(R\)-linear. If, in fact, each \(d_*\) is \(RS\)-linear, let \((C^*, d^*)\) denote the \(RS\)-dual complex of \((C_*, d_*)\): each \(C^k = \text{Hom}_{RS}(C_k, RS)\) and if \(f \in C^k, w \in C_{k+1}\), then \((d^k f)(w) = f(d_{k+1} w)\).

(4.4) **Corollary.** In the situation (and notation) of (3.5), suppose that \(S\) admits an N-valued length homomorphism and that \((C_*, d_*)\) is a chain complex such that each \(d_k\) is \(RS\)-linear and dominated by \(\partial_k\). Then

\begin{itemize}
  \item[(a)] \((C^*, d^*)\) is exact in positive codimension.
  \item[(b)] \(C^{n-1}/\text{im } d^{n-2}\) is a free \(R\)-module.
\end{itemize}

**Proof.** The proof of (3.5) reduces (4.4) to (4.3).

We remark that in (4.3) or (4.4), if the length homomorphism is trivial (for each \(w \in S, |w| = 0\)), then each \(d_k = \partial_k\) so that (4.3) or (4.4) give no new information.

**Part II. Artin groups.**

In Part II, we apply the homological machinery developed in Part I to Artin groups. In §5, we recall the definitions of Artin groups, Artin monoids and Coxeter groups and we recall the single most important fact (5.1) about Artin monoids: the Kurzungslemma of Brieskorn-Saito [8] (proved by Garside [21] for the braid groups and certain other groups). The easy consequence (5.2) of (5.1) implies that any subset \(X\) of an Artin monoid is exact (3.1), so that (3.2) applies as well; in the Artin group setting, we will only apply (3.2) when \(X\) is the generating set \(X_M\) of the Artin group (or monoid) as in §5.

In §6, we study fundamental elements of an Artin monoid \(S_M\): these are the least common left multiples \(\Delta_A\) of the subsets \(A\) of \(X_M\) which have a common left multiple. The early results in §6 are due to Brieskorn-Saito [8] (and to Garside [21] for the braid groups). Where convenient, we have indicated the proofs. The main result of §6 is (6.12) which is used in §7 to obtain a group-theoretically interesting "top-degree" approximation to the \(RS_M\)-complex associated to \(X_M\) (3.2).

In §7, we prove Theorem A. We begin by defining the "diagram polynomial" \(D_\alpha(w) \in RS_M\) of an arbitrary \(w \in S_M\) (7.1): \(D_\alpha(w)\) is an "\(\alpha\)-signed" sum of the right
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factors of \( w \). In (7.4), we show that the functions \( D_x(A, B) = D_x(\Delta A \Delta B^{-1}) \) define a diagram in \( RS_M \). Combining (3.2) with (4.1), it follows that the corresponding \( RS_{M'} \)-complex is exact in positive dimension (7.5). The remainder of §7 is concerned with extending (7.5) from \( RS_M \) to \( RG_M \); it turns out that (2.3) applies if and only if the associated Coxeter group \( W_M \) is locally finite. In particular, we obtain (7.7) from which Theorem A follows easily.

In §8, we prove Theorem B. To prove that if \( W_M \) is finite, then \( G_M \) is a duality group, we use Theorem 9.2 of [3] and adopt the following notation: \( n - 1 \) denotes the cardinality of \( X_M \). Condition 9.2i of [3] \( (G_M \) is of type FP) follows from Theorem A. To prove condition 9.2ii of [3] \( (H^k(G_M, RG_M) = 0, \) if \( k \neq n - 1 \), we show (8.1) that \( X_M \) satisfies the duality condition (3.3), from which condition 9.2ii of [3] follows easily. The remainder (and bulk) of §8 is devoted to a proof that \( G_M \) satisfies condition 9.2iii of [3] \( (H^{n-1}(G, RG) \) is a flat \( R \)-module). In fact, we show that \( H^{n-1}(G, RG) \) is a free \( R \)-module. Our proof of this fact involves a “Möbius function” \( \mu \) for \( S_M \), defined just before (8.6). The Möbius function of a monoid with the “finite factorization property” was studied by Cartier and Foata in [11]. In particular, their results about “partially-commutative” monoids are an easy consequence of (8.6) below. We also remark that, by a theorem of Tits [29], certain results about the weak Bruhat ordering of Coxeter groups (see, for example, [5]) are an easy consequence of our (8.6) below.

§5. Preliminaries on Artin groups.

Following [6], a Coxeter matrix is a symmetric matrix \( M \) each of whose entries \( m(i, j) \) is a positive integer or \( \infty \) such that \( m(i, j) = 1 \) if and only if \( i = j \). Given a Coxeter matrix \( M \), we define an Artin monoid \( S_M \), an Artin group \( G_M \) and a Coxeter group \( W_M \).

To define \( S_M \), we introduce the following notation: if \( x \) and \( y \) are elements of a semigroup and \( m \in \mathbb{N} \), then
\[
\langle xy \rangle^m = \begin{cases} (xy)^k & \text{if } m = 2k \\ (yx)^k & \text{if } m = 2k + 1. \end{cases}
\]

Given \( M \), let \( X \) be a set in fixed one-to-one correspondence with the rows (or columns) of \( M \); a typical element of \( X_M \) is denoted \( a_i \). \( S_M \) is defined by presentation to have generators \( X_M \) and relations all
\[
\langle a_i a_j \rangle^{m(i, j)} = \langle a_j a_i \rangle^{m(j, i)}
\]
with the convention that \( m(i, j) = \infty \) stands for “no relation”. Also, \( G_M \) is defined to be the group with the same presentation as \( S_M \) and \( W_M \) is defined to be \( G_M \) (or, equivalently, \( S_M \)) modulo the additional relations that each \( a_i^2 = 1 \). Note the natural homomorphisms from \( S_M \) to \( G_M \) and from \( G_M \) to \( W_M \) extending the identity on \( X_M \).
We will use the tools developed in §1, §3 and §4 to study the homological algebra of the Artin monoids $S_M$. When possible, we will use §2 to extend to $G_M$.

For the moment, we focus on $S_M$. The presentation of $S_M$ has several special properties. First, the defining relations of $S_M$ are equalities between words of equal length, so that $S_M$ admits an $\mathbb{N}$-valued length homomorphism $w \mapsto |w|$ where each $|a_i| = 1$ (see §4). Second, both sides of each defining relation of $S_M$ involve the same generators. It follows that if $A \subseteq X_M$ then the submonoid of $S_M$ generated by $A$ has the “obvious” presentation: generators $A$ and relations those defining relations of $S_M$ that only involve elements of $A$. Finally, each defining relation is invariant under reflection: if each side of a defining relation is replaced with its mirror image, then the same relation arises (interchanging sides, if $m(i,j)$ is odd). As a consequence, certain properties of $S_M$ “on the left” have as immediate corollaries the analogous properties “on the right”.

The following “Kurzungslemma” of Brieskorn and Saito will be crucial below:

(5.1) Theorem. Let $S_M$ be an Artin monoid. Suppose that $a_i, a_j \in X_M$ and $u, v \in S_M$ satisfy $ua_i = va_j$. Then $m(i,j) \neq \infty$ and there exists $z \in S_M$ such that $u = z \langle a_i a_j \rangle^{m(i,j)-1}$ and $v = z \langle a_j a_i \rangle^{m(i,j)-1}$.

Proof. This is Lemma 2.1 of [8].

In particular, if $m(i,j) \neq \infty$, then $\langle a_i a_j \rangle^{m(i,j)}$ is a least common left multiple of $a_i$ and $a_j$. (If $m(i,j) = \infty$, then $a_i$ and $a_j$ have no common left multiple.) If $i = j$, (5.1) states that $a_i$ cancels on the right ($m(i,i) = 1$) so that $S_M$ is a right cancellation monoid. The mirror image of (5.1) also holds. In particular, $S_M$ is also a left cancellation monoid.

(5.2) Corollary. Let $S_M$ be an Artin monoid. Then any subset of $S_M$ which has a common left multiple is finite and has a unique least common left multiple.

Proof. This is Proposition 4.1 of [8].

It follows from (5.2) that any subset $X$ of $S_M$ is exact (3.1), so that (3.2) applies. We will only apply (3.2) when $X = X_M$.

For the following consequence of (5.1), we adopt the following notation: if $S_M$ is an Artin monoid and $A \subseteq X_M$, then $S_A$ will denote the submonoid of $S_M$ generated by $A$.

(5.3) Corollary. Let $S_M$ be an Artin monoid and let $A \subseteq X_M$. If $B \subseteq S_A$ has a common left multiple in $S_M$, then the least common left multiple of $B$ belongs to $S_A$.

Proof. This follows from (5.1). See [8].

§6. Fundamental elements.

Let $S_M$ be an Artin monoid with generators $X_M$. In this section, we write $S$ for $S_M$ and $X$ for $X_M$. As in §5, if $A \subseteq X$, we write $S_A$ for the submonoid of $S$ generated
by $A$. As noted in §5, each $S_A$ has the “obvious” presentation (and therefore is an Artin monoid).

We let $\mathcal{P}$ denote the collection of all subsets of $X$ which have a common left multiple. By (5.2), each $A \in \mathcal{P}$ is finite, so that the notation here agrees with that in §3. By (5.2), each $A \in \mathcal{P}$ has a least common left multiple which, as in §3, we denote $\Delta_A$. (By (5.2), each $\Delta_A$ is uniquely determined by $A$.) Since, by (5.1), $S$ is a right cancellation monoid (and, from above, $X$ is exact), (3.2) applies; we will save this observation until we can apply (4.3) as well. Meanwhile, we study the $\Delta_A$'s.

(6.1) Lemma. If $A \in \mathcal{P}$, then $\Delta_A \in S_A$.

Proof. This follows from (5.2).

(6.2) Lemma. If $A \in \mathcal{P}$ and $a \in A$, then there is a unique $s \in A$ such that $\Delta_A a = s \Delta_A$.

Proof. See Lemma 5.2 of [8]. (Existence of $s$ follows by noting that $\Delta_A a$ is a common left multiple of $A$; uniqueness follows from right cancellation; a length argument shows that $s \in X$; since $\Delta_A a \in S_A$, we have $s \in S_A$; thus, $s \in X \cap S_A = A$.)

In (6.2), we write $s = a^d$. Left cancellation and the finiteness of $A$ imply that the function $a \mapsto a^d$ is a permutation of $A$.

(6.3) Lemma. The function $a \mapsto a^d$ extends to an automorphism of $S_A$.

Proof. Again see Lemma 5.2 of [8]. (Here, it suffices to show that $a \mapsto a^d$ respects the defining relations of $A$. For this, note that if $a_i, a_j \in X$ and $m \in \mathbb{N}$ satisfy $\langle a_i a_j \rangle^m = \langle a_j a_i \rangle^m$, then $m$ is a multiple of $m(i, j)$.)

The image of $w \in S_A$ under the automorphism of $S_A$ determined by $a \mapsto a^d$ will be denoted $w^d$. It follows that $\Delta_A w = w^d \Delta_A$ and if $u, v \in S_A$, then $(uv)^d = u^d v^d$.

(6.4) Lemma. If $u, v \in S_A$ satisfy $uv = \Delta_A$, then $v^d u = \Delta_A$.

Proof. Note that $v^d u v = v^d \Delta_A = \Delta_A v$ and cancel $v$.

(6.5) Lemma. $A^d$ is the least common right multiple of $A$ and equals its own mirro-image.

Proof. See Lemma 5.1 of [8]. (Here, the left-right symmetry of the defining relations of $S$ is crucial.)

(6.6) Corollary. If $a \in A$, then $(a^d)^d = a$. In particular, $\Delta_A^2$ is central in $S_A$.

Proof. See Lemma 5.2 of [8]. (By (6.5), the mirror-image of $a^d \Delta_A = \Delta_A a$ is $a \Delta_A = \Delta_A a^d$.)


For the following proposition, we define \( w \in S \) to be square-free provided \( w \) cannot be written as \( w = uav \) with \( u, v \in S \) and \( a \in X \).

(6.7) **Lemma.** If \( A \in \mathcal{P} \), then \( \Delta_A \) is square-free.

**Proof.** See Lemma 5.4 of [8]. (If \( \Delta_A = uav \), then \( \Delta_A = v^4uav \); as in the proof of (6.2), show that \( v^4ua \) is a common left multiple of \( A \), a contradiction.)

(6.8) **Lemma.** Let \( w \in S \) and \( a \in X \). If \( w \) is square-free and cannot be written as \( w_1a \) in \( S \), then \( wa \) is square-free.

**Proof.** This is Lemma 3.4 of [8].

The next few propositions all involve factorizations of the fundamental elements \( \Delta_A \). Only the first of these appears in [8].

(6.9) **Theorem.** Let \( A \in \mathcal{P} \). Suppose that \( \Delta_A = uv \) with \( u, v \in S_A \). Then for each \( a \in A \), either there exists \( u_1 \in S_A \) such that \( u = u_1a \) or there exists \( v_1 \in S_A \) such that \( v = av_1 \).

**Proof.** See Lemma 5.3 of [8].

By (6.7), the phrase "but not both" can be added to the statement of (6.9). Note that (6.8) and (6.9) have the following consequence: if \( A \) is a finite subset of \( X \), then \( \Delta_A \) exists if and only if \( S_A \) contains finitely many square-free elements.

We shall need some extensions of (6.9). These require the following observation: any non-empty subset of \( S \) has a greatest common right factor. (\( Z \subseteq S \) has a common right factor \( w \) provided there is a function \( f : Z \rightarrow S \) such that for each \( z \in Z \), \( Z = f(z)w \); \( w \) is a greatest common right factor of \( Z \) provided it is a common right factor of \( Z \) and, in addition, is a left multiple of any common right factor.) To prove this, note that the set of common right factors of \( Z \) has a common left multiple (any element of \( Z \)) and therefore by (5.2) a least common left multiple \( w \); clearly, \( w \) is a greatest common right factor of \( Z \). (See [8].)

(6.10) **Theorem.** Let \( A \in \mathcal{P} \). Suppose that \( \Delta_A = uv \) with \( u, v \in S_A \). Then for each \( B \subseteq A \), \( u \) and \( v \) can be factored uniquely as \( u = u_1u_2 \) and \( v = v_1v_2 \) in \( S_A \) so that \( u_2v_1 = \Delta_B \).

**Proof.** Let \( u_2 \) be the greatest common right factor of \( u \) and \( \Delta_B \). Then \( u_2 \) is a right factor of \( u \), say \( u = u_1u_2 \). Also, \( u_2 \) is a right factor, and therefore by (6.4), a left factor of \( \Delta_B \), say \( \Delta_B = u_2v_1 \).

We claim that no \( b \in B \) is a right factor of \( u_1 \). To prove this, note that if \( b \in B \) is a right factor of \( u_1 \), then \( bu_2 \) is a right factor of \( u \). Now \( b \) is not a left factor of \( u_2 \), since \( \Delta_A \) is square-free (6.7). Since \( u_2 \) is a right factor of \( \Delta_B \) and \( b \) is not a left factor of \( u_2 \), (6.9) applied to \( \Delta_B \) shows that \( bu_2 \) is a right factor of \( \Delta_B \). Thus \( bu_2 \) is a common right factor \( u \) and \( \Delta_B \), contradicting the definition of \( u_2 \).
Since no $b \in B$ is a right factor of $u_1$, (6.9) applied to the factorization $\Delta_A = (u_1)(u_2v)$ of $\Delta_A$ shows that every $b \in B$ is a left factor of $u_2v$. In other words, $u_2v$ is a common right multiple of $B$; by (6.5), there exists $v_2 \in S_A$ such that $u_2v = \Delta_B v_2$. Since $\Delta_B = u_2v_1$, we get $v = v_1v_2$, as required.

To prove uniqueness, let $u_1, u_2, v_1$ and $v_2$ be as above and assume also that $u = u'_1u'_2, v = v'_1v'_2$ and $\Delta_B = u'_2v'_1$. Then $u'_2$ is a right factor of $u$ and, by (6.4), a right factor of $\Delta_B$. By definition of $u_2$, there exists $z \in S_A$ such $u_2 = zu'_2$. Since $u_2 \in S_B$, we have $z \in S_B$. Then

$$\Delta_A = uv$$
$$= u_1u_2v'_1v'_2$$
$$= u_1zu'_1v'_1v'_2$$
$$= u_1zu'_2v'_2$$

so that the assumption $z \neq 1$ would contradict (6.7). Thus $z = 1$, so $u'_2 = u_2$. Cancellation easily gives $u'_1 = u_1, v'_1 = v_1$ and $v'_2 = v_2$, as required.

Theorem (6.10) has two corollaries. The first is a generalization to an ascending union of subsets of $A$.

(6.11) COROLLARY. Let $A \in \mathcal{P}$. Suppose that $\Delta_A = uv$ with $u, v \in S_A$. If $B_k \subseteq B_{k-1} \subseteq \ldots \subseteq B_1 \subseteq A$, then $u$ and $v$ can be factored uniquely as $u = u_0u_1 \ldots u_k$ and $v = v_0v_{k-1} \ldots v_0$ so that $u_kv_k = \Delta_k$ and if $1 \leq i < k$, then $u_iA_{B_{i+1}}v_i = \Delta_{B_i}$.

PROOF. This follows from (6.10) by induction on $k$.

For the second corollary, we use the following notation: if $w \in S$, then $R(w)$ denotes the set of right factors of $w$.

(6.12) COROLLARY. Let $A \in \mathcal{P}$ and $B \subseteq A$. Then the function $R(\Delta_A) \times R(\Delta_B) \to R(\Delta_A)$ defined by $(v_1, v_2) \mapsto v_1v_2$ is a bijection.

PROOF. We first check that if $v_1 \in R(\Delta_A) \Delta_B^{-1}$ and $v_2 \in R(\Delta_B)$, then $v_1v_2 \in R(\Delta_A)$. If $u_1v_1 = \Delta_A \Delta_B^{-1}$ and $u_2v_2 = \Delta_B$, then by (6.4) and (6.6)

$$\Delta_A = u_1v_1u_2v_2$$
$$= u_1v_1u_2u_2^B$$
$$= (u_2^B)^A u_1v_1v_2$$

so that $v_1v_2 \in R(\Delta_A)$.

To check surjectivity, suppose that $v \in R(\Delta_A)$, say $uv = \Delta_A$. By (6.6), $\Delta_A = vu^A$. By (6.10), we can write $v = v_1v_2$ and $u^A = u_1u_2$ with $v_2u_1 = \Delta_B$. By (6.4),
\[ u_1^B v_2 = \Delta_B \text{ so } v_2 \in R(\Delta_B). \text{ Since } \Delta_A = v_1 A_B u_2 = u_2^A v_1 A_B, \text{ we get } u_2^A v_1 = \Delta_A A_B^{-1} \text{ so } v_1 \in R(\Delta_A A_B^{-1}). \]

Finally, to check injectivity, note that if \( uv = \Delta_A \) and \( v = v_1 v_2 \) with \( v_1 \in R(\Delta_A A_B^{-1}) \) and \( v_2 \in R(\Delta_B) \), then the proof of (6.10) applied to \( \Delta_A = vu^A \) shows that \( v_2 \) is the greatest common right factor of \( \Delta_B \) and \( v \). Thus \( v_2 \) is uniquely determined by \( v \). By cancellation in \( S \), \( v_1 \) is therefore also uniquely determined by \( v \).

\section{Proof of Theorem A}

Let \( M \) be a fixed Coxeter matrix. In this section, we write \( X, S, G \) and \( W \) for \( X_M, S_M, G_M \) and \( W_M \). Let \( R \) be a commutative ring with 1.

\begin{itemize}
  \item [(7.1)] \textbf{Definition.} If \( w \in S \) and \( \alpha \in R \), then \( D_\alpha(w) \in RS_M \) is defined by
  \[ D_\alpha(w) = \sum_{uv = w} \alpha^{\mid u \mid} v \]
  where the summation ranges over all ordered pairs \((u, v)\) of elements of \( S \) whose product is \( w \).
  
As usual, \( \mid u \mid \) denotes the length of \( u \). The sum is finite, since each element of \( S \) has only finitely many right factors and, by right cancellation, \( u \) is uniquely determined by \( v \) and \( uv \). For all \( \alpha \in R \), we set \( \alpha^0 = 1 \), so that, for example, each \( D_0(w) = w \).

We use the notation \( \mathcal{P} \), \( \Delta_A \) and \( \Delta_A A_B^{-1} \) as in \( \S 6 \). For \( A \in \mathcal{P} \) and \( B \subseteq A \), define \( D_\alpha(A, B) \in RS \) by \( D_\alpha(A, B) = D_\alpha(\Delta_A A_B^{-1}) \). Our first goal is to show that (1.2) is satisfied.

\begin{itemize}
  \item [(7.2)] \textbf{Theorem.} Let \( A \in \mathcal{P} \) and \( B \subseteq A \). Then \( D_\alpha(\Delta_A) = D_\alpha(\Delta_A A_B^{-1}) D_\alpha(\Delta_B) \).
  
\textbf{Proof.}

\[ D_\alpha(\Delta_A) = \sum_{uv = \Delta_A} \alpha^{\mid u \mid} v = \sum_{u_1 v_1 = \Delta_A A_B^{-1}, u_2 v_2 = \Delta_B} \alpha^{\mid u_1 \mid + \mid u_2 \mid} v_1 v_2 \]

\[ = \left( \sum_{u_1 v_1 = \Delta_A A_B^{-1}} \alpha^{\mid u_1 \mid} v_1 \right) \left( \sum_{u_2 v_2 = \Delta_B} \alpha^{\mid u_2 \mid} v_2 \right) \]

\[ = D_\alpha(\Delta_A A_B^{-1}) D_\alpha(\Delta_B) \]

where the second equality uses (6.12).

To show that (1.2b) follows from (7.2), we shall need some cancellation in \( RS \). Recall from \( \S 4 \) that if \( W \in RS \), then \( \mid W \mid \) denotes the maximum \( \mid w \mid \) for \( w \) appearing
with non-zero coefficient in \( w \). Call \( W \in RS \) monic provided \( W = w + W_1 \) with \( w \in S, \ |w| = |W| \) and \(|W_1| < |w|\). Since \( S \) is a right cancellation monoid, if \( U, V, W \in RG \) satisfy \( UW = VW \) and \( W \) is monic, then \( U = V \). Note that each \( D_x(w) \) is monic.

\[(7.3) \text{ Corollary. If } C \subseteq B \subseteq A \in \mathcal{P}, \text{ then } D_x(A_A A_C^{-1}) = D_x(A_{A-B}^{-1}) D_x(A_B A_C^{-1}).\]

**Proof.** By several applications of (7.2),
\[
D_x(A_A A_C^{-1})D_x(A_C) = D_x(A_A)
\]
\[
= D_x(A_A A_B^{-1})D_x(A_B)
\]
\[
= D_x(A_A A_B^{-1})D_x(A_B A_C^{-1})D_x(A_C).
\]

Since \( D_x(A_C) \) is monic, (7.3) follows.

\[(7.4) \text{ Corollary. The functions } D_x(A, B) = D_x(A_A A_B^{-1}), \text{ for } B \subseteq A \in \mathcal{P}, \text{ define a diagram of } (X, \mathcal{P}) \text{ in } RS.\]

**Proof.** Clearly, \( A_A A_A^{-1} = 1 \) and \( D_x(1) = 1 \) which gives (1.2a). (7.3) gives (1.2b).

We let \( \partial_x \) denote the differential associated to the diagram \( D_x(A, B) = D_x(A_A A_B^{-1}) \). (In this section, we omit the dimension subscript from \( \partial_x \).) Interpreting §1 in the current situation, \( C_k \) is the free left \( RS \)-module with basis consisting of all \([A]\) with \( A \) a \( k \)-element subset of \( X \) such that \( A_A \) exists; \( \partial_x \) is given by
\[
\partial_x([A]) = \sum_{x \in A} (-1)^{x(A)} D_x(A_A A_A^{-1} \leftarrow x) [A - x]
\]

\[(7.5) \text{ Theorem. For each } x \in R, \text{ the } RS\text{-complex } (C_*, \partial_x) \text{ is exact in positive dimension.}\]

**Proof.** By (5.1), \( S \) is a right-cancellation monoid. By (4.2), any subset of \( S \) which has a common left multiple has a least common left multiple; it follows that \( X \) is exact (3.1). Thus (3.2) applies: \( (C_*, \partial_0) \) is exact in positive dimension. Clearly, each \( \partial_x \) is dominated by \( \partial_0 \) and the other hypotheses of (4.3) are satisfied. From (4.3), we conclude that each \( (C_*, \partial_x) \) is exact in positive dimension, as required.

Note that the module \( C_0/\partial_x(C_1) \) of \( D_x \) is \( R \) with the following \( S \)-action: each \( w \in S \) acts as multiplication by \( (-w)^{|w|} \). (To verify this, note that if \( x \in X \), then \( D_x(x) = x + x \).) In particular, \( (C_*, \partial_{-1}) \) is a resolution of \( R \) as a trivial \( RS \)-module.
We would like to show that for each Artin group $G$, the extension of the complex $(C_\ast, \partial_\ast)$ of (7.5) from $RS$ to $RG$ is exact in positive dimension. Unfortunately, we only know how to do this in the situation in which (2.3) applies.

(7.6) Theorem. The following are equivalent:

a) Any two elements of $S$ have a common left multiple.

b) For each finite subset $A$ of $X$, $\Delta_A$ exists.

c) For each finite subset $A$ of $X$, $W_A$ is finite.

Proof. (If $A \subseteq X$, then $W_A$ denotes the subgroup of $W$ generated by $A$.) If $X$ is finite, then (7.6) follows from [8]. If $X$ is infinite, then (7.6) follows from the finite case and the fact that if $A \subseteq X$, then $S_A$ and $W_A$ have the obvious presentation (for $S_A$, this was noted in §5; for $W_A$, see [6]).

Note that (7.6c) holds if and only if every finitely-generated subgroup of $W$ is finite. If $G$ satisfies (7.6), we say that $G$ is locally of finite type. If, in addition, $X$ is finite (so $W$ is finite), we say that $G$ is of finite type.

(7.7) Corollary. If $G$ is locally of finite type, then the extension of $(C_\ast, \partial_\ast)$ from $RS$ to $RG$ is exact in positive dimension.

Proof. By (5.1) and its mirror-image, $S$ is a two-sided cancellation monoid. Thus, by hypothesis and (7.6a), $S$ satisfies the Ore condition. Therefore, (7.7) follows from (7.5) and (2.3).

(7.8) Corollary. If $G$ is locally of finite type, then $G$ is torsion-free.

Proof. First note that if $A \subseteq X$, then $G_A$ ( = the subgroup of $G$ generated by $A$) has the “obvious” presentation. (As already noted, $S_A$ has the obvious presentation and therefore is an Artin monoid. By hypothesis, $S$ satisfies (7.6b) so that $S_A$ satisfies (7.6b). Thus $S_A$ satisfies (7.6a) and therefore satisfies the Ore condition, so that (2.2) applies. Since the natural homomorphism from $S_A$ to $G$ is injective, it follows easily from uniqueness in (2.2) that $G_A$ is the Artin group corresponding to the Artin monoid $S_A$.) Also, since $S_A$ inherits condition (7.6b), $G_A$ is locally of finite type.

Clearly, each cyclic subgroup of $G$ is contained in $G_A$ for some finite $A \subseteq X$. Thus it suffices to prove (7.8) under the stronger assumption that $G$ is of finite type. Under this assumption, we apply (7.7) with $R = Z$ and $\alpha = -1$; the extension of $(C_\ast, \partial_{-1})$ from $ZS$ to $ZG$ is then a finite free resolution of $Z$ as a trivial $ZG$-module, so that $G$ is torsion-free.

§8. Proof of Theorem B.

Let $M$ be a fixed Coxeter matrix we use notation $(X, S, G, W, D_\ast, \mathcal{P}, \Delta_A, \ast, \mathcal{D}, \ast, \mathcal{D})$ as in §7. We show that if $G$ is of finite type, then $G$ is a duality group. We first show that (3.5) applies to $(C_\ast, \partial_0)$. 
(8.1) Lemma. $X$ satisfies the duality condition (3.3).

Proof. By (5.1), $S$ is a right cancellation monoid. By (5.2), $X$ is exact. Thus we need to show: if $A \in \mathcal{P}$ and $x, y \in A$ satisfy $x \neq y$, then $\Delta_A \Delta_A^{-1}_{x-y}$ is the least common right multiple of $\Delta_A \Delta_A^{-1}_{x-x}$ and $\Delta_A \Delta_A^{-1}_{y-x}$ in $S$. From the fact that

$$
\Delta_A \Delta_A^{-1}_{x-x} = (\Delta_A \Delta_A^{-1}_{x-y})(\Delta_A \Delta_A^{-1}_{x-x-y})
= (\Delta_A \Delta_A^{-1}_{x-y})(\Delta_A \Delta_A^{-1}_{y-x-y}),
$$

we conclude that $\Delta_A \Delta_A^{-1}_{x-x}$ and $\Delta_A \Delta_A^{-1}_{y-x}$ have a common right multiple and therefore, by the mirror image of (5.2), they have a (unique) least common right multiple. Let $(\Delta_A \Delta_A^{-1}_{x-y})u = (\Delta_A \Delta_A^{-1}_{x-y})v$ be the least common right multiple of $\Delta_A \Delta_A^{-1}_{x-x}$ and $\Delta_A \Delta_A^{-1}_{x-y}$. Using the expressions above for $\Delta_A \Delta_A^{-1}_{x-y}$ as a right multiple of $\Delta_A \Delta_A^{-1}_{x-x}$ and $\Delta_A \Delta_A^{-1}_{x-y}$ and using left cancellation in $S$, we conclude that there exists $z \in S$ such that $\Delta_A \Delta_A^{-1}_{z-x} = uz$ and $\Delta_A \Delta_A^{-1}_{z-x} = v$. To prove (8.1), it suffices to show that $z = 1$.

If $z \neq 1$, then some $a \in A$ is a right factor of both $\Delta_A \Delta_A^{-1}_{x-x-y}$ and $\Delta_A \Delta_A^{-1}_{y-x-y}$. We show that this contradicts (6.7). Since $\Delta_A \Delta_A^{-1}_{x-x} = (\Delta_A \Delta_A^{-1}_{x-y})$ $\Delta_A \Delta_A^{-1}_{x-y}$ and every element of $A - x - y$ is a left factor of $\Delta_A \Delta_A^{-1}_{x-y}$, it follows from (6.7) that the only element of $A$ which could be a right factor of $\Delta_A \Delta_A^{-1}_{x-y}$ is $y$ itself. (In fact, by (6.8), $y$ is a right factor of $\Delta_A \Delta_A^{-1}_{x-y}$.) Similarly, the only factor of $A$ which could be a right factor of $\Delta_A \Delta_A^{-1}_{y-x-y}$ is $x$ itself. Since $x \neq y$, this contradicts $z \neq 1$. Thus $z = 1$, as required.

From (8.1), we easily obtain the dual of (7.5) when $G$ has finite type. As in §4, we let $(C^*, \partial^2)$ denote the RS-dual complex of $(C_\bullet, \partial_\bullet)$. (As in §7, we suppress the dimension superscript on $\partial^2$.)

(8.2) Theorem. If $A_X$ exists, then for each $z \in R$, the complex $(C^*, \partial^2)$ is exact in positive codimension.

Proof. By (5.1) and its mirror-image, $S$ is a two-sided cancellation monoid. By hypothesis, $X$ has a common left multiple. Thus, by (5.2), $X$ is finite. By (8.1), $X$ satisfies the duality condition. Thus (3.5) applies: $(C^*, \partial^0)$ is exact in positive codimension. As in the proof of (7.5), each $\partial^0$ is dominated by $\partial^0$. Clearly, the other hypotheses of (4.4) are satisfied. From (4.4a), we conclude that each $(C^*, \partial^2)$ is exact in positive codimension.

Note that "$A_X$ exists" is equivalent to "$G$ is of finite type."

(8.3) Theorem. Let $G$ be an Artin group of finite type. Then $G$ is a duality group.

Proof. We use the characterization of duality groups given in Theorem 9.2 of [3]. Applying (7.7) with $a = -1$, it follows that $G$ is of type FP (in fact, of type FL) so that $G$ satisfies condition 9.2i of [3].
To prove that $G$ satisfies condition 9.2ii of [3], we let $n - 1$ denote the cardinality of $X$. By (5.1) and its mirror-image, $S$ is a two-sided cancellation monoid. By the mirror-image of (7.6), $S$ satisfies the (right) Ore condition. Thus the mirror-image of (2.3) holds as well. Combining this observation with (8.2), we conclude that the extension of $(C^*, \partial^a)$ to $RG$ (as a right $RG$-complex) is exact in positive codimension. Noting that each $C_k$ is a finitely-generated free left $RS$-module, it follows easily that the $RG$-dual of the extension of $(C^*, \partial^a)$ to $RG$ is isomorphic to the extension of $(C^*, \partial^a)$ to $RG$. From all this, it follows easily that $\theta^k(G, RG) = 0$, if $k \neq n - 1$, which is condition 9.2ii of [3].

Finally, we verify that $G$ satisfies condition 9.2iii of [3]: $H^{n-1}(G, RG)$ is a flat $R$-module. In fact, we will establish a slightly more general result. To state this result, note that, via the inclusion $RS \subseteq RG$, the diagram $D_\alpha(A, B) = D_\alpha(A \Delta B^{-1})$ in $RS$ defines a diagram in $RG$. Clearly, the extension to $RG$ of the $RS$-complex associated to $D_\alpha$ in $RS$ is the $RG$-complex associated to $D_\alpha$ in $RG$.

(8.4) Lemma. If $G$ is an Artin group of finite type and $\alpha$ is a unit of $R$, then, as $R$-modulus, the co-module of $D_\alpha$ in $RS$ and the co-module of $D_\alpha$ in $RG$ are isomorphic.

Recall from (4.4b) that the co-module of $D_\alpha$ in $RS$ is a free $R$-module. Note that the co-module of $D_{-1}$ in $RG$ is $H^{n-1}(G, RG)$. From (8.4), we conclude that condition 9.2iii of [3] is satisfied. From Theorem 9.2 of [3], we conclude that $G$ is a duality group, which completes the proof of (8.3).

The remainder of this section will be devoted to a proof of (8.4). The proof will involve some new ideas. First, a monoid $S$ is said to have the finite factorization property provided for each $w \in S$ there are only finitely many ordered pairs $(u, v) \in S \times S$ such that $uv = w$. Clearly, an Artin monoid has the finite factorization property.

Let $S$ be a monoid with the finite factorization property and let $A$ be an associative ring (with 1). If $f, g: S \to A$ are arbitrary functions, then the convolution $f \ast g$ of $f$ and $g$ is the function from $S$ to $A$ defined as follows:

$$f \ast g(w) = \sum_{uv = w} f(u)g(v)$$

where, as in (7.1), the summation ranges over all pairs of elements of $S$ whose product is $w$. We leave the proof of the following to the reader.

(8.5) Lemma. Convolution is associative. The function $\delta: S \to A$ defined by $\delta(1) = 1$ and $\delta(w) = 0$ if $w \neq 1$ is a two-sided identity for convolution.

Let $S$ be a monoid with the finite factorization property and let $R$ be a commutative ring (with 1). We study convolution with $A = RS$. Define $I: S \to RS$ by
$I(w) = w$. If $S$ admits an $N$-valued length homomorphism $w \mapsto |w|$ and $x \in R$, define $e_x : S \to RS$ by $e_x(w) = x^{|w|}$. (The definition (7.1) of $D_x$ can now be written as $D_x = e_x * I$.)

Finally, we assume that $S$ is an Artin monoid. In this situation, we define $\mu : S \to RS$ by

$$\mu(w) = \begin{cases} (-1)^{|A|} A_A & \text{if } w = A_A \text{ for some } A \subseteq X \\ 0 & \text{otherwise} \end{cases}$$

where $|A|$ denotes the cardinality of $A$. In particular, $\mu(1) = 1$.

(8.6) Theorem. Let $S$ be an Artin monoid. Then $\mu * I = I * \mu = \delta$.

Proof. Clearly $I * \mu(1) = I(1)\mu(1) = 1 = \delta(1)$, since if $uv = 1$ in $S$, then $u = v = 1$.

If $w \neq 1$, let $A = \{ a \in X \mid \text{for some } w' \in S, w = w'a \}$. Since $w \neq 1$, $A \neq 0$. By (5.2), $A$ is finite, since $w$ is a common left multiple of $A$. In particular, $A_A$ exists and there exists $w_1 \in S$ such that $w = w_1 A_A$. Also, if $B \subseteq X$ is finite and $A_B$ exists, then there exists $w' \in S$ such that $w = w' A_B$ if and only if $B \subseteq A$, in which case $w' = w_1 (A_A A_B^{-1})$. Thus

$$I * \mu(w) = \sum_{uv = w} u \mu(v) = \sum_{B \subseteq X : u A_B = w} u (-1)^{|B|} A_B = \sum_{B \subseteq A} (-1)^{|B|} w = (1 - 1)^{|A|} w = 0.$$ 

Thus we have shown that $I * \mu = \delta$. (The proof that $\mu * I = \delta$ follows from the left-right symmetry of the defining relations of $S$.)

(8.7) Corollary. Let $S$ be an Artin monoid and suppose that $\Delta_X$ exists. Then, writing $N = |\Delta_X|$, $\Delta_X - (-1)^{|X|} x^N$ belongs to the right ideal in $RS$ generated by $\{ D_x (\Delta_X A_X^{-1} x^{-1} x) \mid x \in X \}$.

Proof. Since $D_x = e_x * I$, it follows from (8.5) and (8.6) that $e_x = D_x * \mu$. Applying this equality to $\Delta_X$ (and recalling that $N = |\Delta_X|$), we conclude that
\[ x^N = e_\alpha(A_X) = D_\alpha \ast \mu(A_X) = \sum_{uv = A_X} D_\alpha(u)\mu(v) = \sum_{A \subseteq X} (-1)^{|A|} D_\alpha(A_X A_A^{-1}) A_A = (-1)^{|X|} A_X + \sum_{A \subseteq X} (-1)^{|A|} D_\alpha(A_X A_A^{-1}) A_A. \]

Note that if \( A \) is a proper subset of \( X \), then \( D_\alpha(A_X A_A^{-1}) \) belongs to the right ideal in RS generated by \( \{ D_\alpha(A_X A_A^{-1}) | x \in X \} \); choosing \( x \notin A \), \( D_\alpha(A_X A_A^{-1}) = D_\alpha(A_X A_{X-x}^{-1}) D_\alpha(A_{X-x} A_{A-x}^{-1}) \) by (7.2); (8.7) follows easily.

At last, we complete the proof of (8.3) by providing the

**Proof of (8.4).** Write \( H^{n-1}(S, RS) \) and \( H^{n-1}(G, RG) \) for the co-modules of \( D_\alpha \) in RS and RG, respectively. Since \( S \) is a two-sided cancellation monoid and, by the mirror-image of (7.6), satisfies the right Ore condition, it follows from the mirror-image of (2.3) that the functor \((-) \mapsto (-) \otimes_{RS} RG\) is exact. It follows that \( H^{n-1}(G, RG) \) and \( H^{n-1}(S, RS) \otimes_{RS} RG \) are isomorphic RG-modules.

Now suppose that \( \alpha \) is a unit of \( R \). It follows from (8.7) that \( A_X \) acts invertibly on \( H^{n-1}(S, RS) \). Since each \( x \in X \) is both a left factor and a right factor of \( A_X \), each \( x \in X \) (and therefore all of \( S \)) acts invertibly on \( H^{n-1}(S, RS) \). It follows that \( H^{n-1}(S, RS) \) and \( H^{n-1}(S, RS) \otimes_{RS} RG \) are isomorphic R-modules; (8.4) follows easily.

**Part III. Examples.**

In part III, we apply the results of Part II to certain specific Artin groups of finite type. In §9, we study the three irreducible Artin groups (of finite type) whose generating set \( X_M \) has cardinality 3; in accordance with [6, p. 193], these three groups are denoted \( A_3 \), \( B_3 \) and \( H_3 \). In §10, we briefly treat the braid groups \( B^{(n)} \). (Already, a notational anomaly has arisen: the Artin group denoted \( A_3 \) in §9 is denoted \( B^{(4)} \) in §10.)

In §9, we explicitly compute the ordinary integral homology \( H_\ast(G) \) where \( G \) is one of \( A_3 \), \( B_3 \), \( H_3 \) or their commutator subgroups \( A'_3 \), \( B'_3 \), \( H'_3 \), respectively. Some of our results are not new. \( H_\ast(A_3) \) was described in [30] for example. A description of \( H_\ast(A'_3) \) follows easily from the fact that \( A'_3 \) is a semidirect product of two free groups of rank 2 (see [18]). \( H_\ast(B_3) \) was described in [23]. Our description of \( H_\ast(B'_3) \) contradicts [24], where it is claimed that \( B'_3 \) is a free group of rank 4. Apparently, our descriptions of \( H_2(H_3) \) and \( H_2(H'_3) \) are new.
In §10, we briefly describe the homological algebra of Artin's braid group $B^{(n)}$. In particular, we explicitly describe each $D_{A}(A_{A}A_{X}^{-1}x)$ as in §7. We also describe a finite complex of finitely-generated free abelian groups whose homology is the ordinary integral homology of $B^{(n)}$ (the ordinary integral homology of $B^{(n)}$ is well-known; see [30] or [13]). In addition, we describe a finite complex of finitely-generated free modules over the Laurent-polynomial ring $\mathbb{Z}4[t, t^{-1}]$ whose homology is the ordinary integral homology of the commutator subgroup of $B^{(n)}$.

§9. $A_{3}$, $B_{3}$ and $H_{3}$

In this section, we will carry out some explicit computations for Artin groups whose Coxeter groups are of type $A_{3}$, $B_{3}$ and $H_{3}$ in the notation of [6, p. 193]. It follows from [6] that these Coxeter groups are finite, so that (7.7) applies to the corresponding Artin groups. Before turning to these examples, we establish some notation.

Throughout this section, $G$ will be one of the three Artin groups indicated above, $\alpha$ will be $-1$, $\delta$ will denote $\delta_{-1}$ and $R$ will be $\mathbb{Z}$. In each case, $X$ will be $\{a, b, c\}$ with the total ordering $a < b < c$. If $A \subseteq X$, then $[A]$ and $\Delta_{A}$ will be written by listing the elements of $A$; for example, if $A = \{a, c\}$, then $[A]$ and $\Delta_{A}$ will be written $[ac]$ and $\Delta_{ac}$, respectively.

If $G$ is a group, then $H_{\ast}(G)$ will denote the ordinary integral homology of $G$. For each of the three Artin groups $G$ that we consider, $H_{1}(G)$ will turn out to be a free abelian group of rank 1 or 2. In each case, we write the group-ring $\mathbb{Z}(G/G')$ as a Laurent polynomial ring. Recall Shapiro's lemma (see, for example, [9, p. 73]); if $G$ is a group and $N$ is a normal subgroup of $G$, then $H_{\ast}(N) = H_{\ast}(G, \mathbb{Z}(G/N))$. We will use Shapiro's lemma to compute $H_{\ast}(G')$ where $G'$ is the commutator subgroup of $G$.

Finally, given a monoid $S$ and a generating set $X$ of $S$, we associate to each $w \in S$ a graph $\Gamma(w)$ defined as follows. A vertex of $\Gamma(w)$ is an ordered pair $(u, v) \in S \times S$ which satisfies $uv = w$. (Note the relationship between the vertices of $\Gamma(w)$ and the definition (7.1) of $D_{A}(w)$.) An edge of $\Gamma(w)$ is an ordered triple $(u, a, v) \in S \times X \times S$ which satisfies $uav = w$; the edge $(u, a, v)$ will be directed from the vertex $(u, av)$ to the vertex $(ua, v)$ and will be labelled $a$. We will use the graphs $\Gamma(w)$ to avoid explicitly describing the complex $(C_{\ast}, \partial_{\ast})$ in two of the three examples below.

In each example, we will give the Coxeter matrix $M$, give the corresponding presentation of the Artin group $G$, draw, for each $x \in X$, $\Gamma(A_{X}A_{X^{-1}x})(as an aid in computing $\partial[abc])$, compute $H_{\ast}(G)$ and, at the very least, compute $H_{\ast}(G')$.

(9.1) Example. $M = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$.
The corresponding Artin group, denoted $A_3$, and Artin monoid have the following presentation: generators $a$, $b$, $c$ and relations $aba = bab$, $ac = ca$, $bcb = cbc$. The Artin group $A_3$ is the braid group on 4 strands and, somewhat inconveniently, is conventionally denoted $B_4$. Here is $(C_\ast, \partial_\ast)$:

\[
\begin{align*}
\partial([\emptyset]) &= 0 \\
\partial([a]) &= (a - 1)[\emptyset] \\
\partial([b]) &= (b - 1)[\emptyset] \\
\partial([c]) &= (c - 1)[\emptyset] \\
\partial([ab]) &= (ba - a + 1)[b] - (ab - b + 1)[a] \\
\partial([ac]) &= (a - 1)[c] - (c - 1)[a] \\
\partial([bc]) &= (cb - b + 1)[c] - (bc - c + 1)[b] \\
\partial([abc]) &= (cba - ba + a - 1)[bc] - (bacb - acb + ab + cb - b + 1)[ac] \\
&\quad + (abc - bc + c - 1)[ab]
\end{align*}
\]

The formula for $\partial([bc])$, for example, follows from the fact that $\Delta_{bc} \Delta_c^{-1} = cb$ and $\Delta_{bc} \Delta_b^{-1} = bc$. The formula for $\partial([abc])$ follows from the fact that $\Delta_{abc} = abcba$, $\Delta_{abc} \Delta_{bc}^{-1} = cba$, $\Delta_{abc} \Delta_{ac}^{-1} = bacb$ and $\Delta_{abc} \Delta_{ab}^{-1} = abc$. In Figure 1, we list the corresponding graphs (edges are directed from left to right).

\[\Gamma(\Delta_{abc} \Delta_{bc}^{-1}):\]

\[\begin{array}{ccc}
\text{c} & \text{b} & \text{a} \\
\bullet & \bullet & \bullet
\end{array}\]

\[\Gamma(\Delta_{abc} \Delta_{ab}^{-1}):\]

\[\begin{array}{ccc}
\text{a} & \text{b} & \text{c} \\
\bullet & \bullet & \bullet
\end{array}\]

\[\Gamma(\Delta_{abc} \Delta_{ac}^{-1}):\]

\[\begin{array}{ccc}
\text{b} & \text{a} & \text{c} \\
\bullet & \bullet & \bullet
\end{array}\]

Figure 1.
The integral homology of the group $A_3$ may be computed by substituting $a = b = c = 1$ into the complex $(C_\ast, \partial_\ast)$ and computing the homology of the resulting complex over $\mathbb{Z}$. This yields

\[ \partial([\emptyset]) = 0 \]
\[ \partial([a]) = \partial([b]) = \partial([c]) = 0 \]
\[ \partial([ab]) = [b] - [a] \]
\[ \partial([ac]) = 0 \]
\[ \partial([bc]) = [c] - [b] \]
\[ \partial([abc]) = -2[ac] \]

It follows easily that $H_0(A_3) = H_1(A_3) = \mathbb{Z}$, $H_2(A_3) = \mathbb{Z}_2$ and $H_k(A_3) = 0$ if $k \neq 0, 1, 2$.

Finally, we compute the integral homology of the commutator subgroup $A'_3$ of $A_3$. From above, $A_3/A'_3 = H_1(A_3)$ is an infinite cyclic group. We identify the group ring $\mathbb{Z}(A_3/A'_3)$ with the Laurent-polynomial ring $\mathbb{Z}[t, t^{-1}]$ where $t$ denotes the common image of $a, b, c$ in $A_3/A'_3$. Using Shapiro's lemma (described above), $H_\ast(A'_3)$ may be computed by substituting $t$ for $a, b, c$ in $(C_\ast, \partial_\ast)$ and computing the homology of the resulting complex over $\mathbb{Z}[t, t^{-1}]$. In describing the resulting complex, we will use the following abbreviation: $\phi_n (n > 0)$ will stand for the $n$th cyclotomic polynomial, with the convention that $\phi_1 = t - 1$. (Aside from $\phi_1$, the only $\phi_n$'s that appear here are $\phi_4 = t^2 + 1$ and $\phi_6 = t^2 - t + 1$.) This yields

\[ \partial([\emptyset]) = 0 \]
\[ \partial([a]) = \partial([b]) = \partial([c]) = \phi_1[\emptyset] \]
\[ \partial([ab]) = \phi_6([b] - [a]) \]
\[ \partial([ac]) = \phi_1([c] - [a]) \]
\[ \partial([bc]) = \phi_6([c] - [b]) \]
\[ \partial([abc]) = \phi_4(\phi_1[bc] - \phi_6[ac] + \phi_1[ab]) \]

It follows easily that $H_0(A'_3) = \mathbb{Z}$, $H_1(A'_3) = H_2(A'_3) = \mathbb{Z} \oplus \mathbb{Z}$ and $H_k(A'_3) = 0$ if $k \neq 0, 1, 2$. We remark that the $\mathbb{Z}[t, t^{-1}]$-module structure of $H_\ast(A'_3)$, induced by conjugation in $A_3$, can be deduced from the $\mathbb{Z}[t, t^{-1}]$-complex above.

In examples (9.2) and (9.3) below, we will not explicitly describe the complex $(C_\ast, \partial_\ast)$. Except for the formulas for $\partial([bc])$ and $\partial([abc])$, $(C_\ast, \partial_\ast)$ in (9.2) and (9.3) will agree with (9.1). The formula for $\partial([bc])$ will be easy enough to describe. The formula for $\partial([abc])$ would take up more space than our apology for not printing it. We will print the graphs $\Gamma(A_{abc}A^{-1}_{bc}), \Gamma(A_{abc}A^{-1}_{ac})$ and $\Gamma(A_{abc}A^{-1}_{ab})$, from these
graphs, it is easy enough to describe $\partial([abc])$. For this reason, we suggest the following exercise: in (9.1), understand $\partial([abc])$ in terms of the graphs $\Gamma(A_{abc}A_{bc}^{-1})$, $\Gamma(A_{abc}A_{ca}^{-1})$ and $\Gamma(A_{abc}A_{ab}^{-1})$.

(9.2) EXAMPLE. $M = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 4 \\ 2 & 4 & 1 \end{pmatrix}$

The corresponding Artin group, denoted $B_3$, and Artin monoid have the following presentation: generators $a$, $b$, $c$ and relations $aba = bab$, $ac = ca$, $bc = cb$. As noted above, we will omit the explicit description of $(C_*, \partial_*)$. It is easy enough to check that

$$\partial([bc]) = (bc - cb + b - 1)[c] - (bc - bc + c - 1)[b]$$

and that $\partial([abc])$ can be read off from the graphs in Figure 2.

![Figure 2](image)

The integral homology of the group $B_3$ may be computed by substituting $a = b = c = 1$ into the complex $(C_*, \partial_*)$ that results from above. This may be easily seen to yield the following complex over $\mathbb{Z}$

$$\partial([\emptyset]) = 0$$
$$\partial([a]) = \partial([b]) = \partial([c]) = 0$$
$$\partial([ab]) = [b] - [a]$$
$$\partial([ac]) = \partial([bc]) = 0$$
$$\partial([abc]) = 0$$
It follows easily that $H_0(B_3) = \mathbb{Z}$, $H_1(B_3) = \mathbb{Z} \oplus \mathbb{Z}$, $H_2(B_3) = \mathbb{Z} \oplus \mathbb{Z}$, $H_3(B_3) = \mathbb{Z}$ and all other $H_k(B_3) = 0$.

Finally, we compute the integral homology of $B_3$. Since $B_3/B_3' = \mathbb{Z} \oplus \mathbb{Z}$, we identify $\mathbb{Z}(B_3/B_3')$ with the Laurent-polynomial ring $\mathbb{Z}[s, s^{-1}, t, t^{-1}]$, where, in $B_3/B_3'$, $t$ denotes the image of $a$ and $b$ and $s$ denotes the image of $c$. Proceeding as in (9.1), we substitute $t$ for $a$ and $b$ and $s$ for $c$ in $(C_*, \partial_*)$. It is not difficult to see that this yields

$$\partial([\emptyset]) = 0$$
$$\partial([a]) = \partial([b]) = (t - 1)[\emptyset]$$
$$\partial([c]) = (s - 1)[\emptyset]$$
$$\partial([ab]) = (t^2 - t + 1)[b] - [a]$$
$$\partial([ac]) = (t - 1)[c] - (s - 1)[a]$$
$$\partial([bc]) = (st + 1)[c] - (s - 1)[b]$$
$$\partial([abc]) = (st^2 - 1)(t^2 - t + 1)[bc] - (st + 1)(t^2 - t + 1)[ac] + (st + 1)(s - 1)[ab]$$

To describe $H_*(B_3)$, we let $\Lambda$ denote $\mathbb{Z}[s, s^{-1}, t, t^{-1}]$ and if $p_1, p_2, \ldots \in \Lambda$, we let $(p_1, p_2, \ldots)$ denote the ideal in $\Lambda$ generated by $p_1, p_2, \ldots$. Then

$$H_0(B_3') = \Lambda / (s - 1, t - 1)$$
$$H_1(B_3') = \Lambda / (t^2 - t + 1, (st + 1)(s - 1))$$
$$H_2(B_3') = \Lambda / (st^2 - 1)$$

and all other $H_k(B_3') = 0$. Note that, as abelian groups, $H_1(B_3')$ and $H_2(B_3')$ are free of rank 4 and countably infinite rank, respectively. (It follows that $B_3'$ is not finitely-related. In fact, $B_3'$ is finitely-generated; it is not hard to show that $a^{-1}b$, $ba^{-1}$, $ca^{-1}bc^{-1}$ and $cba^{-1}c^{-1}$ generated $B_3'$.)

(9.3) Example. $M = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 5 \\ 2 & 5 & 1 \end{pmatrix}$

The corresponding Artin group, denoted $H_3$, and Artin monoid have the following presentation: generators $a$, $b$, $c$ and relations $aba = bab$, $ac = ca$, $bcbcb = cbcbc$. We will omit explicit description of $(C_*, \partial_*)$. Clearly,

$$\partial([bc]) = (cbcb - bcb + cb - b + 1)[c] - (bcbc - cbc + bc - c + 1)[b]$$

It is also easy to describe $\partial([abc])$ using the graphs in Figure 3.
\[ \Gamma(D_{abc} D_{bc}^{-1}) : \]

\[ \Gamma(D_{abc} D_{ac}^{-1}) : \]

\[ \Gamma(D_{abc} D_{ab}^{-1}) : \]

Figure 3.
To compute \( H_*(H_3) \), we substitute \( a = b = c = 1 \) in \((C_*, \partial_*)\). This yields
\[
\partial([\emptyset]) = 0 \\
\partial([a]) = \partial([b]) = \partial([c]) = 0 \\
\partial([ab]) = [b] - [a] \\
\partial([ac]) = 0 \\
\partial([bc]) = [c] - [b] \\
\partial([abc]) = 0
\]

It follows \( H_k(H_3) = \mathbb{Z} \) if \( 0 \leq k \leq 3 \) and \( H_k(H_3) = 0 \), otherwise. To compute \( H_*(H'_3) \), since \( H_1(H_3) = \mathbb{Z} \), we identify \( \mathbb{Z}(H_3/H'_3) \) with \( \mathbb{Z}[t, t^{-1}] \) and proceed as in (9.1). In addition to \( \phi_1 \) and \( \phi_5 \), which appeared in (9.1), we will also see \( \phi_3 = t^2 + t + 1, \phi_5 = t^4 + t^3 + t^2 + t + 1 \) and \( \phi_{10} = t^4 - t^3 + t^2 - t + 1 \).

This yields
\[
\partial([\emptyset]) = 0 \\
\partial([a]) = \partial([b]) = \partial([c]) = \phi_1[\emptyset] \\
\partial([ab]) = \phi_5([b] - [a]) \\
\partial([ac]) = \phi_1([c] - [a]) \\
\partial([bc]) = \phi_{10}([c] - [b]) \\
\partial([abc]) = \phi_1 \phi_3 \phi_5 \phi_1 \phi_6 [bc] - \phi_6 \phi_{10} [ac] + \phi_1 \phi_{10} [ab]
\]

It follows that \( H_0(H'_3) = \mathbb{Z}, H_2(H'_3) \) is free abelian of rank 7 and each other \( H_k(H'_3) = 0 \). In particular, \( H'_3 \) is a perfect group and any presentation of \( H'_3 \) requires at least 7 more relations than generators. We remark that \( H'_3 \) is finitely-presented. This can be seen as follows: \( \Lambda_{abc} \) is central in \( H_3 \) and the subgroup \( \langle \Lambda_{abc} \rangle \) generated by \( \Lambda_{abc} \) satisfies \( \langle \Lambda_{abc} \rangle \cap H'_3 = \{1\} \). It follows that \( H'_3 \) is isomorphic to a subgroup of finite index in the finitely-presented group \( H_3/\langle \Lambda_{abc} \rangle \) and is therefore finitely-presented.

§10. \( B^{(n)} \)

In this section, we apply the results of §7 to Artin’s braid group \( B^{(n)} \), which for \( n > 0 \), is presented as follows: \( B^{(n)} \) has generators \( X^{(n)} = \{s_1, \ldots, s_{n-1}\} \) and relations as given in the introduction. For notational convenience, we let \( S^{(n)} \) denote the Artin monoid associated to \( B^{(n)} \). Note that \( B^{(1)} \) is the trivial group, \( B^{(2)} \) is an infinite cyclic group, \( B^{(3)} \) is the fundamental group of the complement of the trefoil knot and that \( B^{(4)} \) is the group denoted \( A_3 \) in §9. The Coxeter group associated to \( B^{(n)} \) is the symmetric group on \( n \) letters so that \( B^{(n)} \) is an Artin group.
of finite type. In particular, (7.7) applies to $B^{(n)}$. Our goal in this section is to give an explicit description of $\partial_x$ in (7.7) for $B^{(n)}$.

We begin with some terminology and notation. First, two subsets $A$ and $B$ of $X^{(n)}$ are said to be separated provided if $s_i \in A$ and $s_j \in B$, then $|i - j| \geq 2$.

(10.1) **Lemma.** Let $A, B \subseteq X^{(n)}$ be separated.

a) If $u \in S_A$ and $v \in S_B$, then $uv = vu$.

b) $\Delta_{A \cup B} = \Delta_A \Delta_B$.

**Proof.** (10.1a) follows from the fact that every element of $A$ commutes with every element of $B$. For (10.1b), note first that since, by (10.1a), $\Delta_A \Delta_B = \Delta_B \Delta_A$, $\Delta_A \Delta_B$ is a common left multiple of $A \cup B$, so there exists $z \in S^{(n)}$ such that $\Delta_A \Delta_B = z \Delta_{A \cup B}$. Since $B \subseteq A \cup B$, $\Delta_{A \cup B}$ is a common left multiple of $B$, so there exists $z_1 \in S^{(n)}$ such that $\Delta_A \Delta_B = z_1 \Delta_B$. Since $A \cap B = \emptyset$, by (6.9), every $a \in A$ is a right factor of $z_1$ so that there exists $z_2 \in S^{(n)}$ such that $z_1 = z_2 \Delta_A$. It follows that $\Delta_A \Delta_B = z_2 \Delta_A \Delta_B$ in $S^{(n)}$ so that $z = z_2 = 1$. In particular $\Delta_A \Delta_B = \Delta_{A \cup B}$, as required.

Next, we introduce "open interval" notation on $X^{(n)}$: if $0 \leq i < j \leq n$, then $X(i, j) = \{s_k | i < k < j\}$. Note that each $X(i, i + 1)$ is empty and that $X^{(n)} = X(0, n)$. In particular, we let $S(i, j)$ denote the submonoid of $S^{(n)}$ generated by $X(i, j)$ and let $A(i, j)$ denote the least common left multiple of $X(i, j)$. If $A \subseteq X^{(n)}$, then $X(i, j)$ will be called a subinterval of $A$ provided: if $i < k < j$, then $s_k \in A$. If, in addition, $s_i, s_j \notin A$, then $X(i, j)$ is called a full subinterval of $A$.

(10.2) **Corollary.** Let $A \subseteq X^{(n)}$, let $s_k \in A$ and let $X(i, j)$ be the full subinterval of $A$ which contains $s_k$. Then $\Delta_A \Delta_A^{-1} = \Delta(i, j)(\Delta(i, k)\Delta(k, j))^{-1}$.

**Proof.** Write $A = B \cup X(i, j)$ with $B$ and $X(i, j)$ disjoint. Clearly, $B$ and $X(i, j)$ are separated. By (10.1), $\Delta_A = \Delta_B \Delta(i, j)$ and $\Delta_A^{-1} = \Delta_B \Delta(i, k)\Delta(k, j)$. (10.2) follows easily.

(10.1) and (10.2) reduce the computation of $\Delta_A$ and $\Delta_A \Delta_A^{-1}$ to certain special cases. We begin by studying $A(0, n)$ and $A(0, n)(A(0, i)\Delta(i, n))^{-1}$.

First, if $j > i$, we define the "descending product" $\Pi(j, i)$ inductively as follows: $\Pi(i + 1, i) = 1$ and if $j > i$, then $\Pi(j + 1, i) = s_j \Pi(j, i)$. We record the following:

(10.3) **Lemma.** Let $j > i$.

a) If $k < i$ or $k > j$, then $s_k \Pi(j, i) = \Pi(j, i)s_k$.

b) If $i < k < j - 1$, then $s_k \Pi(j, i) = \Pi(j, i)s_k + 1$.

c) If $i < k < j$, then $\Pi(j, i) = \Pi(j, k - 1)\Pi(k, i)$.

**Proof.** Left to the reader.

Finally, record the well-known.
10.4) THEOREM. \( \Delta(0, 1) = 1 \) and if \( n > 1 \), then \( \Delta(0, n) = \Delta(0, n - 1) \Pi(n, 0) \).

PROOF. See [21] or [4].

Note that \( \Pi(n, 0) = \Delta(0, n - 1) \Delta(0, n) \). It is also easy to check that \( \Pi(n, 0) = \Delta(0, n) \Delta(1, n) \). It also follows from (10.4) that if \( i < j - 1 \), then \( \Delta(i, j) = \Delta(i, j - 1) \Pi(j, i) \); the function \( s_k \rightarrow s_{k+1} \) from \( X(0, j - i) \) to \( X(i, j) \) extends to an isomorphism from \( S(0, j - i) \) onto \( S(i, j) \) and (10.4) applies directly to \( \Delta(0, j - i) \).

Next, we study \( D_\Delta(\Delta_A) \) and \( D_\Delta(\Delta_A^{-1} \Delta_A) \) with \( D_\Delta \) as defined in (7.1). For simplicity, we write \( D \) for \( D_1 \) so that if \( w \in S^{(n)} \), then

\[
D(w) = \sum_{uv = w} v
\]

For \( w \in S^{(n)} \), we also define

\[
D'(w) = \sum_{uv = w} u
\]

(10.5) LEMMA. Let \( A \subseteq X^{(n)} \).

a) \( D(\Delta_A) = D'(\Delta_A) \).

b) If \( B \subseteq A \), then \( D'(\Delta_A) = D'(\Delta_B)D'(\Delta_B^{-1} \Delta_A) \).

PROOF. (6.4) identifies the left factors of \( \Delta_A \) with the right factors of \( \Delta_A \), proving (10.5a). (10.5b) is the mirror-image of (7.2).

(10.6) LEMMA. \( D'(\Pi(n, 0)) = \sum_{j=1}^{n} \Pi(n, n - j) \). In particular, if \( 0 < i < n \), then

\[
D'(\Pi(n, 0)) = D'(\Pi(n, i)) + \Pi(n, i - 1)D'(\Pi(i, 0)).
\]

PROOF. The formula for \( D'(\Pi(n, 0)) \) follows from the fact that no defining relation of \( S^{(n)} \) applies to \( \Pi(n, 0) \). The second part of (10.6) follows from the formula for \( D'(\Pi(n, 0)) \).

For convenience, if \( 0 < i < n \), we let \( \delta^{(n)}_{i} \) denote \( \Delta(0, n)(\Delta(0, i) \Delta(i, n))^{-1} \). The formulae for \( D(\delta^{(n)}_{1}) \) and \( D(\delta^{(n)}_{n-1}) \) can be deduced from (10.6). The general case follows from:

(10.7) THEOREM. Let \( 1 < i < n - 1 \). Then \( D(\delta^{(n)}_{i}) = D(\delta^{(n-1)}_{i}) + D(\delta^{(n-1)}_{i-1}) \Pi(n, i - 1) \).

PROOF. First note that

\[
D(\Delta(0, n)) = D(\delta^{(n)}_{i})D(\Delta(0, i))D(\Delta(i, n))
\]

by (7.2) and (10.1). Next note that
\[ D(\Delta(0, n)) = D'(\Delta(0, n)) \]
\[ = D'(\Delta(0, n - 1))D'(\Pi(n, 0)) \]
\[ = D(\Delta(0, n - 1))(D'(\Pi(n, i)) + \Pi(n, i - 1))D'(\Pi(i, 0)) \]

by (10.5a), the mirror-image of (7.2), (10.5a) and (10.6) in succession. Analyzing the pieces individually, we have

\[ D(\Delta(0, n - 1))D'(\Pi(n, i)) \]
\[ = D(\delta^{(n-1)}_i)D(\Delta(0, i))D(\Delta(i, n - 1))D'(\Pi(n, i)) \]
\[ = D(\delta^{(n-1)}_i)D(\Delta(0, i))D(\Delta(i, n)) \]

first using (7.2) and (10.1) as above, and then using (10.5a), the mirror-image of (7.2) and then (10.5a) again. We also have

\[ D(\Delta(0, n - 1))\Pi(n, i - 1)D'(\Pi(i, 0)) \]
\[ = D(\delta^{(n-1)}_i)D(\Delta(0, i - 1))D(\Delta(i - 1, n - 1))\Pi(n, i - 1)D'(\Pi(i, 0)) \]
\[ = D(\delta^{(n-1)}_i)D(\Delta(0, i - 1))\Pi(n, i - 1)D(\Delta(i, n))D'(\Pi(i, 0))D(\Delta(i, n)) \]
\[ = D(\delta^{(n-1)}_i)\Pi(n, i - 1)D(\Delta(0, i))D(\Delta(i, n)) \]

first using (7.2) and (10.1) as above, second using (10.3a) repeatedly and finally using (10.5a), the mirror-image of (7.2) and then (10.5a) again, as above. Comparing the two expressions for \( D(\Delta(0, n)) \) and then, as in the proof of (7.3), cancelling \( D(\Delta(0, i))D(\Delta(i, n)) \), (10.7) follows easily.

We remark that (10.7) includes a proof of the formula \( \delta_i^{(n)} = \delta_{i-1}^{(n-1)}\Pi(n, i - 1) \). Define \( \delta_0^{(n)} = \delta_n^{(n)} = 1 \), so that \( D(\delta_0^{(n)}) = D(\delta_n^{(n)}) = 1 \). It is easy to check that the recursion in (10.7) now holds for \( 0 < i < n \).

\[ (10.8) \textbf{Corollary. Let } 0 < i < n. \text{ Then } D_x(\delta_i^{(n)}) = x^iD_x(\delta^{(n-1)}_i) + D_x(\delta^{(n-1)}_{i-1}) \]
\[ \Pi(n, i - 1). \]

\textbf{Proof.} The case \( x = 1 \) follows from (10.7) and the remark above. For the general case, we assume that \( x \) is an indeterminate over \( \mathbb{Z} \) and work over the Laurent-polynomial ring \( \mathbb{Z}[x, x^{-1}] \) in order to prove (10.7) over the polynomial ring \( \mathbb{Z}[x] \); the general case follows by taking a suitable homomorphism from \( \mathbb{Z}[x] \) to the given ring \( R \).

For the remainder of the proof, we let \( R \) denote \( \mathbb{Z}[x, x^{-1}] \). For \( w \in S^{(n)} \), define \( \phi_x(w) = x^{|w|}w \). Extend \( D \), \( D_x \) and \( \phi_x \) to \( R \)-linear functions from \( RS^{(n)} \) to itself.
Since $\alpha$ is a unit, $\phi_\alpha$ is an $R$-linear (ring) automorphism of $RS^{(n)}$. It is easy to check that $D = \phi_\alpha D_\alpha \phi_\alpha^{-1}$. In particular, if $w \in S^{(n)}$, then $D(w) = \alpha^{-|w|} \phi_\alpha D_\alpha(w)$.

It follows easily from (10.4) that if $n > 0$, then $|A(0, n)| = \frac{1}{2} n(n - 1)$. In turn, it follows that if $0 < i < n$, then $|\delta_i^{(n)}| = i(n - i)$. Noting that $|II(n, i - 1)| = n - i$, (10.7) follows from (10.6) by a simple computation.

By (7.7), we can use (10.2) and (10.8) to explicitly describe a resolution of $R$ as a trivial left $RB^{(n)}$-module ($\alpha = -1$).

For example, the ordinary integral homology of $B^{(n)}$ may be computed from the complex of free abelian groups that arises from (10.2) and (10.8) by letting $R = \mathbb{Z}$, $\alpha = -1$ and substituting $1$ for each $s_i$. The image of $D_{-1}(\delta_i^{(n)})$ after substituting each $s_i = 1$ will be denoted $\binom{n}{i}_{-1}$. Clearly, each $\binom{n}{0}_{-1} = \binom{n}{n}_{-1} = 1$ and if $0 < i < n$, then $\binom{n}{i}_{-1} = (-1)^i \binom{n-1}{i-1} + \binom{n-1}{i}_{-1}$. It is easy to check that if $0 \leq i \leq n$, then

$$\binom{n}{i}_{-1} = \begin{cases} 0 & \text{if } n \text{ is even, } i \text{ is odd,} \\
\left\lfloor \frac{n}{2} \right\rfloor & \text{otherwise} \\
\left\lfloor \frac{i}{2} \right\rfloor & \text{otherwise}
\end{cases}$$

where $[x]$ denotes the greatest integer $\leq x$ and $\binom{n}{i}$ denotes an ordinary binomial coefficient. In particular, this complex agrees with that described by in [30]; see [30] (or [13]) for a complete description of the ordinary integral (co)homology of $B^{(n)}$.

For a second example, the ordinary integral homology of the commutator subgroup of $B^{(n)}$ may be computed, using Shapiro's lemma as in §9, from the complex of $\mathbb{Z}[t, t^{-1}]$-modules that arises from (10.2) and (10.8) by letting $R = \mathbb{Z}$, $\alpha = -1$ and substituting $t$ for each $s_i$. The image of $D_{-1}(\delta_i^{(n)})$ after substituting each $s_i = t$ will be denoted $\binom{n}{i}_t$. Clearly, each $\binom{n}{0}_t = \binom{n}{n}_t = 1$ and if $0 < i < n$, then $\binom{n}{i}_t = (-1)^i \binom{n-1}{i}_t + t^{n-i} \binom{n-1}{i-1}_t$. It is easy to check that if $0 \leq i \leq n$, then $\binom{n}{i}_t = p_n/p_i p_{n-i}$, where, if $n \geq 0$, then

$$p_n = \prod_{i=1}^{n} \frac{t^i - (-1)^i}{t + 1}$$
so that, in particular, \( p_0 = 1 \). It is not difficult to use the complex just described to show that if \( n \geq 5 \), then the second commutator subgroup of \( B^{(n)} \) coincides with the first commutator subgroup (see [22]). Apparently (see [15]), a complete description of the ordinary integral (co)homology of the commutator subgroup of \( B^{(n)} \) is not known.

REFERENCES

13. F. R. Cohen, *The homology of \( \mathbb{E}_{n+1} \)-spaces, \( n \geq 0 \)*, in *The Homology of Iterated Loop Spaces*, Lecture Notes in Math. 533 (1976).
