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SERIES REPRESENTATIONS OF LINEAR FUNCTIONALS

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F. John [1] has given a series representation of the form $\int f(x)da(x) = \sum a_n f(x_n)$ for Stieltjes integrals. The purpose of this paper is to prove the following generalisation:

THEOREM. *If X is a compact Hausdorff space with a countable base then for any continuous linear functional Λ on $C(X)$ there exist sequences $a_n \in \mathrm{R}$, $x_n \in X$ such that*

$$\Lambda(f) = \sum a_n f(x_n) \quad \textit{for any } f \in C(X)$$

PROOF. By the Riesz representation theorem this is equivalent to proving

$$\int f d\mu = \sum a_n f(x_n) \tag{1}$$

where μ is a bounded regular Borel measure on X, which we may assume to be non-atomic. It is not difficult to see that there exist closed subsets $E_{k,n}$, $1 \leqq k \leqq n$ of X such that

$$|\mu|(E_{k,n} \cap E_{j,n}) = 0 \quad \text{if} \quad k \neq j, \tag{2}$$

$$\bigcup_{k \leqq n} E_{k,n} = X, \;\; \lim_{k \to \infty} \max_{k \leqq n} |\mu|(E_{k,n}) = 0 \quad \text{and} \quad E_{k,n} \subseteq E_{j,n-1} \quad \text{for some } j,$$

and

$$\int f d\mu = \lim_{n \to \infty} \sum_{k \leqq n} \mu(E_{k,n}) f(y_{k,n}). \tag{3}$$

where $y_{k,n}$ is any sequence such that $y_{k,n} \in E_{k,n}$. One can for example choose a sequence $E_{k,n}$ such that (2) holds and (3) holds for every f_n, where f_n is a dense sequence in $C(X)$, from which it follows that (3) holds for every $f \in C(X)$.

We may suppose the $E_{k,n}$ ordered so that the $E_{k,n}$ contained in $E_{1,n-1}$ come first, then those contained in $E_{2,n-1}$ etc. We now construct the sequence a_n in blocks as follows: the first block is $\mu(E_{1,1})$, for $n > 1$ the nth block consists of

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$\mu(E_{k,n})$ for those $E_{k,n}$ that are contained in $E_{1,n-1}$ followed by $-\mu(E_{1,n-1})$ and then $\mu(E_{k,n})$ for those $E_{k,n}$ that are contained in $E_{2,n-1}$ followed by $-\mu(E_{2,n-1})$ etc. Choose x_n such that $x_n \in E_{j,k}$ where $a_n = + - \mu(E_{j,k})$, and the same x is chosen whenever $E_{j,k}$ occurs.

Then (1) holds. For if we sum to the end of the n first blocks we get

$$\sum_{k \leqq n^2} a_k f(x_k) = \sum_{k \leqq n} \mu(E_{k,n}) f(y_{k,n}) \to \int f d\mu$$

From this it follows that if (1) did not hold we would have for some sequences $n(k) \to \infty$, $c(k)$, $d(k)$ and $\varepsilon > 0$

$$\left| \sum_{j \leqq c(k)} \mu(E_{j,n(k)+1}) f(y_{j,n(k)+1}) - \sum_{j \leqq d(k)} \mu(E_{j,n(k)}) f(y_{j,n(k)}) \right| \geqq \varepsilon \quad \text{for all } k,$$

where $c(k)$, $d(k)$ are such that the expression inside the absolute value sign is the sum of the first $c(k) + d(k)$ terms of the $(n(k) + 1)$th block.

Let $M = \sup |f(x)|$ and $\delta = \varepsilon/(8M + 7)$. Take r so large that $\max_{k \leqq n} |\mu| (E_{k,n}) \leqq \delta$ if $n \geqq r$. Let A be the set of all elements that are contained in $\bigcup_{j \leqq d(k)} E_{j,n(k)}$ for infinitely many k. Since μ is regular there is an open set B containing A and a closed set C contained in A such that $|\mu| (B - C) < \delta$, where C can be chosen to be $\bigcup_{j \leqq m} E_{j,n(k)}$ for some m, k. By Urysohn's lemma there is a continuous g such that $|g| \leqq |f|$, $g = f$ on C and $g = 0$ on the complement of B.

Choose $h \geqq r$ such that $|\mu| (A - \cup E_{j,n(h)})$ and $|\mu| (\cup E_{j,n(h)} - A)$, where the union is taken for all $j \leqq d(h)$, are both less than δ and

$$\left| \int g d\mu - \sum \mu(E_{j,r}) g(y_{j,r}) \right| < \delta \quad \text{for all } r \geqq n(h)$$

We then have

$$\left| \sum_{c(h)} \mu(E_{j,n(h)+1}) f(y_{j,n(h)+1}) - \sum_{d(h)} \mu(E_{j,n(h)}) f(y_{j,n(h)}) \right| \tag{5}$$

$$\leqq \left| \sum_{c(h)} \mu(E_{j,n(h)+1}) f(y_{j,n(h)+1}) - \sum_{c(h)} \mu(E_{j,n(h)+1}) g(y_{j,n(h)+1}) \right|$$

$$+ \left| \sum_{c(h)} \mu(E_{j,n(h)+1}) g(y_{j,n(h)+1}) - \sum_{d(h)} \mu(E_{j,n(h)}) g(y_{j,n(h)}) \right|$$

$$+ \left| \sum_{d(h)} \mu(E_{j,n(h)}) f(y_{j,n(h)}) - \sum_{d(h)} \mu(E_{j,n(h)}) g(y_{j,n(h)}) \right|$$

It follows from the way the $E_{k,n}$ were ordered and the choice of h that $|\mu|\,(S) < 3\delta$ where S is the union of those $E_{j,n(h)+1}$ with $j \leqq c(h)$ that are not contained in C. Hence $\left|\sum_{c(h)} \mu(E_{j,n(h)+1})f(y_{j,n(h)+1}) - \sum_{c(h)} \mu(E_{j,n(h)+1})g(y_{j,n(h)+1})\right| =$

$$\left|\sum_{E_{j,n(h)+1} \subseteq S} \mu(E_{j,n(h)+1})f(y_{j,n(h)+1}) - \sum_{E_{j,n(h)+1} \subseteq S} \mu(E_{j,n(h)+1})g(y_{j,n(h)+1})\right| \leqq 6M\delta.$$

A similar estimation of the other terms of (5) gives $|\sum \mu(E_{j,n(h)+1})f(y_{j,n(h)+1}) - \sum \mu(E_{j,n(h)})f(y_{j,n(h)})| < \delta(8M + 7) = \varepsilon$, a contradiction which proves the theorem.

If we let $X = [1,2]$, $\mu =$ Lebesgue measure $E_{k,n} = [(k-1)/2^n, k/2^n]$, $2^n + 1 \leqq k \leqq 2^{n+1}$, $x_{k,n} = (k-1)/2^n$ we obtain after a change of variables F. John's formula

$$\sum (-1)^{n+1}/n f(\{\log n/\log 2\}) = \log 2 \int f(x)\,dx,$$

for all bounded Riemann integrable f, where $\{x\}$ is the fractional part of x.

(It is clear that the above proof applies also to bounded Riemann integrable functions.)

REFERENCES

1. Fritz John, *A representation of stieltjes integrals by conditionally convergent series*, American Journal of Mathematics 59 (1937), 379–384.

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