AN EXISTENCE RESULT FOR SIMPLE
INDUCTIVE LIMITS OF INTERVAL ALGEBRAS

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Given a $C^*$-algebra $A$ with unit 1 we define the Elliott triple of $A$ to be

$$(K_0(A), T(A), r_A)$$

where $K_0(A)$ has the usual ordering and $[1]$ as order-unit, $T(A)$ is the tracial state space of $A$, and with $S$ the state space functor, $r_A : T(A) \to S(K_0(A))$ is given by $r_A(\tau)([p] - [q]) = \tau(p - q)$ for all $\tau \in T(A)$ and projections $p, q \in M_\infty(A)$ where $\tau$ is extended to $M_\infty(A)$ by $(a_{ij}) \mapsto \sum_i \tau(a_{ii})$. We identify two such triples $(G_i, \Delta_i, f_i)$, $i = 1, 2$ if there are isomorphisms $\phi_0 : G_2 \to G_1$, $\phi_T : \Delta_1 \to \Delta_2$ such that the diagram

$$
\begin{array}{ccc}
\Delta_1 & \xrightarrow{\phi_T} & \Delta_2 \\
\downarrow f_1 & & \downarrow f_2 \\
S(G_1) & \xrightarrow{S(\phi_0)} & S(G_2)
\end{array}
$$

commutes. Elliott [3] proved that this triple is a complete invariant for the simple unital $C^*$-algebras which arise as inductive limits of finite direct sums of matrix algebras over $C([0, 1]) - AI$ algebras for short. The project of determining the range of the Elliott triple when applied to simple unital AI algebras was initiated by Thomsen [6]. When $A$ is a simple unital AI algebra $K_0(A)$ is a simple dimension group, $S(K_0(A))$ is a metrizable Choquet simplex, $T(A)$ is another metrizable Choquet simplex and the map $r_A$ is an affine continuous surjection. Furthermore, we know from [6] that the map $r_A$ preserves extreme points i.e. $r_A(\partial_e T(A)) = \partial_e S(K_0(A))$.

In this paper we show that, whenever $G$ is a simple noncyclic dimension group, such that the extreme boundary of the state space is compact and totally disconnected, $\Delta$ is a metrizable Choquet simplex and $f : \Delta \to S(G)$ an affine continuous map with $f(\partial_e \Delta) = \partial_e S(G)$, then $(G, \Delta, f)$ is the Elliott triple of some simple unital AI algebra.

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Notation 1. For $K$ a compact convex subset of a linear topological space, we denote by Aff$(K)$ the complete order-unit space of affine continuous real-valued functions on $K$ with pointwise ordering and $1$ as the order-unit. For $\lambda: K \to L$ a continuous affine map between compact convex sets, let Aff$(\lambda):$ Aff$(L) \to$ Aff$(K)$ be the positive order-unit-preserving homomorphism given by Aff$(\lambda)(h) = h \circ \lambda$ for all $h \in$ Aff$(L)$. It is well-known that for $K$ a compact convex subset of a locally convex Hausdorff space, the state space of Aff$(K)$ is naturally isomorphic to $K$ via evaluation.

For convenience we write $s^* \circ \phi$ for a homomorphism of ordered groups, and write "homomorphism" instead of "positive order-unit-preserving homomorphism" when dealing with homomorphisms of order-unit spaces.

Definitions 2. Let $\Delta$ be a Choquet simplex. A partition of unity $v_1, \ldots, v_k$ in Aff$(\Delta)$ is said to be extreme if there are closed non-empty faces $\Delta_1, \ldots, \Delta_k$ in $\Delta$ with $\Delta = \hull \{\Delta_1, \ldots, \Delta_k\}$ such that $v_i|_{\Delta_j} \equiv \delta_{ij}$ for $1 \leq i, j \leq k$. A partition of unity $v_1, \ldots, v_k$ in Aff$(\Delta)$ is peaked if $\|v_j\| = 1, j = 1, \ldots, k$. Note that in this case $\span\{v_1, \ldots, v_k\} \cong l^\infty_k$ with $v_1, \ldots, v_k$ as the standard basis.

For $v, 1 - v$ an extreme partition of unity in Aff$(\Delta)$ with corresponding faces $E, E^c$ we let Aff$(\Delta)_v = \{f \in$ Aff$(\Delta): f|_{E^c} \equiv 0\}$, which is an order-unit space with order-unit $v$, and define a homomorphism $\pi_v: \text{Aff}(\Delta) \to \text{Aff}(\Delta)_v$ by $\pi_v(f)|_E = f|_E$ and $\pi_v(f)|_{E^c} \equiv 0$ for $f \in \text{Aff}(\Delta)$.

Lemma 3. Suppose that $\Delta$ is a Choquet simplex and $(l^\infty_n, v_i)$ an inductive system with $\lim l^\infty_n = \text{Aff}(\Delta)$. Let $v_1, \ldots, v_k$ be a peaked partition of unity in Aff$(\Delta)$ and let $\varepsilon > 0$. There is an $i \in \mathbb{N}$ and a homomorphism $\rho: \text{span}\{v_1, \ldots, v_k\} \to l^\infty_n$ such that $\|v_{\infty i} \circ \rho - id\| < \varepsilon$.

Proof. Since $\bigcup_{i=1}^{\infty} v_{\infty i}((l^\infty_n)^+) \cong l^\infty_n$ is dense in Aff$(\Delta)^+$ there are $x_1, \ldots, x_{k-1} \in (l^\infty_n)^+$ with $\left\|\left(1 - \frac{\varepsilon}{2k}\right)v_j - v_{\infty i_0}(x_j)\right\| < \frac{\varepsilon}{2k^2}, j = 1, \ldots, k - 1$. Now

$$v_{\infty i_0} \left(\sum_{j=1}^{k-1} x_j\right) < \left(1 - \frac{\varepsilon}{2k}\right) \sum_{j=1}^{k-1} v_j + \frac{\varepsilon}{2k} \leq 1$$

and there is an $i \geq i_0$ so that $v_{i_0} \left(\sum_{j=1}^{k-1} x_j\right) < 1$. Let $y_j = v_{i_0}(x_j), j = 1, \ldots, k - 1$ and $y_k = 1 - \sum_{j=1}^{k-1} y_j$. Then $(y_j)_{j=1}^k$ is a partition of unity in $l^\infty_n$. Since

$$1 - \frac{\varepsilon}{k} - \left(1 - \frac{\varepsilon}{2k}\right)v_k < \sum_{j=1}^{k-1} v_{\infty i}(y_j) < 1 - \left(1 - \frac{\varepsilon}{2k}\right)v_k$$

we have that
\[
\left(1 - \frac{\varepsilon}{2k}\right)v_k < v_{\infty i}(y_k) < \frac{\varepsilon}{k} + \left(1 - \frac{\varepsilon}{2k}\right)v_k.
\]

Hence, \(\|v_j - v_{\infty i}(y_j)\| < \frac{\varepsilon}{k}\), \(j = 1, \ldots, k\).

Let \(\rho : \text{span}\{v_1, \ldots, v_k\} \rightarrow l^\infty_{m_i}\) be the homomorphism given by \(\rho(v_j) = y_j\) for \(j = 1, \ldots, k\). Clearly, \(\|v_{\infty i} \circ \rho - \text{id}\| < \varepsilon\).

**Lemma 4.** Let \(\Delta\) be a metrizable Choquet simplex and \(w, 1 - w\) an extreme partition of unity in \(\text{Aff}(\Delta)\). Let \(V\) be a subspace of \(\text{Aff}(\Delta)\) with \(1 \in V\) and \(V \cong l^\infty_m\) for some \(m \in \mathbb{N}\). Let \(F \subseteq \text{Aff}(\Delta)_w\) be a finite subset and let \(\varepsilon > 0\). There is a subspace \(W\) of \(\text{Aff}(\Delta)_w\) with \(w \in W \cong l^\infty_n\) for some \(n \in \mathbb{N}\) such that \(\text{dist}(f, W) < \varepsilon\) for all \(f \in F\) and a homomorphism \(\eta : V \rightarrow W\) with \(\|\eta - \pi_{w|V}\| < \varepsilon\).

**Proof.** There is a \(\delta > 0\) such that if \(x_1, \ldots, x_l \in l^\infty_k\) with \(\sum_{i=1}^l x_i - 1 < \delta\) then there are \(y_1, \ldots, y_l \in l^\infty_k\) with \(\sum_{i=1}^l y_i = 1\) and \(\|x_i - y_i\| < \frac{\varepsilon}{2m}\) for \(1 \leq i \leq l\). It follows from Theorem 2.7.2 of [1] that there is a subspace \(W\) of \(\text{Aff}(\Delta)_w\) with \(w \in W\) and \(W \cong l^\infty_m\) for some \(n \in \mathbb{N}\) such that \(\text{dist}(f, W) < \varepsilon\) for all \(f \in F\) and \(\text{dist}(\pi_{w}(e_i^0), W) < \frac{\delta \wedge \varepsilon}{4m}\) for \(1 \leq i \leq m\) where \(e_1, \ldots, e_m\) is the standard basis for \(V \cong l^\infty_m\). Since \(\text{dist}(\pi_{w}(e_i^0), W^+) \leq 2\text{dist}(\pi_{w}(e_i), W)\) there are \(x_1, \ldots, x_m \in W^+\) with \(\|x_i - \pi_{w}(e_i)\| < \frac{\delta \wedge \varepsilon}{2m}\) for \(1 \leq i \leq m\). So \(\sum_{i=1}^m x_i - w < \delta\) and there are \(y_1, \ldots, y_m \in W^+\) with \(\sum_{i=1}^m y_i = w\) and \(\|x_i - y_i\| < \frac{\varepsilon}{2m}\) for \(1 \leq i \leq m\). Let \(\eta : V \rightarrow W\) be the homomorphism given by \(\eta(e_i) = y_i\) for \(1 \leq i \leq m\). Let \(x \in V\) with \(\|x\| \leq 1\). Then \(x = \sum_{i=1}^m \alpha_i e_i\) for some \(\alpha_1, \ldots, \alpha_m \in [-1, 1]\) and

\[
\|\eta(x) - \pi_{w}(x)\| \leq \sum_{i=1}^m |\alpha_i||y_i - \pi_{w}(e_i)|| \leq \sum_{i=1}^m (\|y_i - x_i\| + \|x_i - \pi_{w}(e_i)\|) < \varepsilon.
\]

**Definition 5.** A tree is a triple \((X, \subseteq, x_0)\) where \((X, \subseteq)\) is a partially ordered set with a maximal element \(x_0\) such that \(s(x) = \{y \in X : y < x, \forall z < x : y \neq z\}\) is finite for every \(x \in X\), \(s(x) \cap s(y) = \emptyset\) when \(x \neq y\) and \(X\) is the union of the level sets \(\mathcal{L}_i\) given by \(\mathcal{L}_1 = \{x_0\}\) and \(\mathcal{L}_{i+1} = \cup_{y \in \mathcal{L}_i}s(y)\) for \(i \in \mathbb{N}\). For \(i \in \mathbb{N}\) we let \(\mathcal{L}_i = \mathcal{L}_i \cup \{y \in \bigcup_{j=1}^i \mathcal{L}_j : \forall x \in X : x \preceq y\}\) - the leaves of \(\bigcup_{j=1}^i \mathcal{L}_j\) - and \(c(x) = \{y \in \mathcal{L}_{i+1} : y \preceq x\}\) for \(x \in \mathcal{L}_i\).

**Remark 6.** Let \(\Delta\) be a metrizable Choquet simplex with compact and totally disconnected extreme boundary. Since every locally compact, totally disconnected topological space has a basis of sets being both open and closed, there is
a basis $Y$ for the compact, totally disconnected metric space $\partial e \Delta$ such that $\partial e \Delta \in Y$ and $(Y, \subseteq, \partial e \Delta)$ is a tree, where $s(y)$ is either empty or consists of mutually disjoint sets with union $Y$ for every $y \in Y$. Now put $X = \{ g \in \text{Aff}(\Delta) : \exists y \in Y : g|_{\partial e \Delta} = 1_y \}$. Then $(X, \subseteq, 1)$ is a tree where each $L_i$ is an extreme partition of unity in Aff($\Delta$) and the span of $X$ is dense in Aff($\Delta$).

**Proposition 7.** Let $G$ be a simple noncyclic dimension group, such that the extreme boundary of the state space is compact and totally disconnected. Let $\Delta$ be a metrizable Choquet simplex and let $f : \Delta \rightarrow S(G)$ be a continuous affine map with $f(\partial e \Delta) = \partial e S(G)$. There is a system $(Z^n_i, s_i)$ with positive order-unit-preserving connecting homomorphisms and inductive limit $\lim (Z^n_i, s_i) = G$, a collection $1 \in X \subseteq \text{AffS}(G)$ with dense span such that $(X, \subseteq, 1)$ is a tree where each $L_i$ is an extreme partition of unity in AffS($G$) and homomorphisms $\rho_i : W_i \rightarrow \text{AffS}(Z^n_i)$, $\delta_i : \text{AffS}(Z^n_i) \rightarrow W_{i+1}$ where $W_i = \text{span}(L_i)$ such that

$$
\| \delta_i \circ \rho_i - \text{id} \| < 2^{-i},
$$

$$
\| \rho_{i+1} \circ \delta_i - s_i^\# \| < 2^{-i} n_i^{-1},
$$

$$
\| s_i^\# - \delta_i \| < 2^{-i}
$$

for all $i \in \mathbb{N}$ and moreover there are Markov operators $\theta_h : C_R([0, 1]) \rightarrow C_R([0, 1])$ of the form $\theta_h = N_i^{-1}(\theta_h^1 + \ldots + \theta_h^N)$ where $\theta_h^1, \ldots, \theta_h^N$ are restrictions of unital $\ast$-endomorphisms of $C([0, 1])$ for $h \in L_i, i \geq 2$ such that

$$
\text{Aff}(\Delta) = \lim (W_i \otimes C_R([0, 1]), \theta_i),
$$

$$
\theta_{\omega_i}(v \otimes 1) = \text{Aff}(f)(v), v \in W_i, i \in \mathbb{N}
$$

where $\theta_i : W_i \otimes C_R([0, 1]) \rightarrow W_{i+1} \otimes C_R([0, 1])$ is the homomorphism given by

$$
\theta_i \left( \sum_{g \in L_i} g \otimes x_g \right) = \sum_{g \in L_i \text{vec}(g)} h \otimes \theta_h(x_g)
$$

and

$$
\text{mult}(s_i) \geq 2^i n_i \not\equiv L_i + 1.
$$

**Proof.** Let $(Z^n_i, s_i)$ be a system with positive order-unit-preserving connecting homomorphisms and inductive limit $\lim (Z^n_i, s_i) = G$. Since $G$ is simple and noncyclic we may assume that mult $(s_i) \rightarrow \infty$ as $i \rightarrow \infty$ where mult denotes the smallest entry of a given matrix. We may therefore also assume that mult $(s_i) \geq 1$ for all $i \in \mathbb{N}$.

By the above remark there is a collection $1 \in X \subseteq \text{AffS}(G)$ with dense span such that $(X, \subseteq, 1)$ is a tree where each $L$ is an extreme partition of unity in AffS($G$).
Note that $\text{Aff}(f)$ takes extreme partitions of unity in $\text{AffS}(G)$ to extreme partitions of unity in $\text{Aff}(\alpha)$ because $f(\partial_x \alpha) = \partial_x S(G)$.

For convenience we write $\text{Aff}(\alpha)_g$ and $\pi_g$ for $\text{Aff}(\alpha)_{\text{Aff}(f)(g)}$ and $\pi_{\text{Aff}(f)(g)}$ respectively for $g \in X$.

For $m \in \mathbb{N}$ a partition of unity $\xi_1, \ldots, \xi_m$ in $C_0([0,1])$ is chosen such that

$$\xi_i \left( i - 1 \right) m = 1, 1 \leq i \leq m,$$

and we let $i_m : l_0^\infty \to C_0([0,1])$, $\kappa_m : C_0([0,1]) \to l_0^\infty$ be the homomorphisms given by $i_m(\varepsilon_i) = \xi_i, 1 \leq i \leq m$ and $\kappa_m(x) = \sum_{i=1}^m x \left( i - 1 \right) m \varepsilon_i$

for $x \in C_0([0,1])$. Note that $\kappa_m \circ i_m = \text{id}$.

Let $(d_i)_{i=1}^\infty$ and $(a_i)_{i=1}^\infty$ be dense sequences in $C_0([0,1])$ and $\text{Aff}(\alpha)$ respectively. We show that there are increasing sequences $(i_p)_{p=1}^\infty, (j_p)_{p=1}^\infty$ in $\mathbb{N}$ ($j_1 = 1$) and

- subspaces $\text{Aff}(f)(g) \subseteq Z_g \subseteq \text{Aff}(\alpha)_g$ such that $Z_g \cong l_0^\infty$ for some $m_g \in \mathbb{N}$ and $\text{dist} (\pi_g(a_q), Z_g) < 2^{-p}$ for all $1 \leq q \leq p, g \in L^r, p \in \mathbb{N}$,

- homomorphisms $\eta_h : Z_g \to Z_h$ such that $\| \eta_h - \pi_h \circ Z_g \| < 2^{-p}$ for all $g \in L^r, h \in L^{r+1}, h \leq g, p \in \mathbb{N}$,

- Markov operators $\theta_h : C_0([0,1]) \to C_0([0,1])$ which are of the form $\theta_h = N_p^{-1}(\theta_1 + \ldots + \theta_N)$ where $\theta_1, \ldots, \theta_N$ are restrictions of unital $\ast$-endomorphisms of $C_0([0,1])$ and $\| \theta_h(f) - i_h \circ \eta_h \circ \kappa_g(f) \| < 2^{-p}$ for all $f \in F_g, g \in L^r, h \in L^{r+1}, h \leq g, p \in \mathbb{N}$ where $i_g = i_m \circ \phi_g : Z_g \to C_0([0,1]), \kappa_g = \phi_g^{-1} \circ \kappa_m : C_0([0,1]) \to Z_g$ for some isomorphism $\phi_g : Z_g \to l_0^\infty$ and

$$\mathcal{F}_g = \bigcup_{q=1}^p \left( \theta_{i_{g,q}} \left( \{d_1, \ldots, d_p\} \right) \cup i_g \circ \eta_{i_{g,q}} \circ \kappa_{g,q} \left( \{d_1, \ldots, d_p\} \right) \right),$$

where $g_q \in L^r$ is the unique function $g_q \geq g$.

- homomorphisms $\rho_p : W_p \to \text{AffS}(Z^{n_p})$ and $\delta_p : \text{AffS}(Z^{n_p}) \to W_{p+1}$ where $W_p = \text{span}(L^r)$ such that

$$\| s_{r,i_p} \circ \rho_p - \text{id} \| < 2^{-p-1},$$

$$\| s_{r,i_p} - \delta_p \| < 2^{-p-1} n_{i_p}^{-1},$$

$$\| \rho_{p+1} \circ \delta_p - s_{r,i_{p+1}} \| < 2^{-p} n_{i_p}^{-1}$$

for all $p \in \mathbb{N}$ and such that

- mult $(s_{r,i_{p+1}}, i_{p+1}) \geq 2^p N_p \neq L^{r+1}$ for all $p \in \mathbb{N}$.

By Lemma 3, there is an $i_1 \in \mathbb{N}$ and a homomorphism $\rho_1 : W_1 \to \text{AffS}(Z^{n_1})$ such that $\| s_{r,i_1} \circ \rho_1 - \text{id} \| < 2^{-2}$. Since the span of $X$ is dense in $\text{AffS}(G)$ there is a $j_2 > 1$ and a homomorphism $\delta_1 : \text{AffS}(Z^{n_1}) \to W_2$ such that $\| s_{r,i_1} - \delta_1 \| < 2^{-2} n_{i_1}^{-1}$. It follows from Theorem 2.7.2 of [1] that there is a subspace $1 \subseteq Z_1 \subseteq \text{Aff}(\alpha)$ such that $Z_1 \cong l_0^\infty \subseteq \mathbb{N}$ and $\text{dist} (a_1, Z_1) < 2^{-1}$.

Suppose that $(i_p)_{p=1}^\infty, (j_p)_{p=1}^\infty$ are increasing sequences in $\mathbb{N}$ ($j_1 = 1$) and that $Z_g,
\( g \in L^p, 1 \leq p \leq P, \eta_h, \theta_h, h \in L^p, 2 \leq p \leq P, \delta_p, \rho_p, 1 \leq p \leq P \) and \( \text{mult}(s_{i_p, i_p}), 1 \leq p \leq P - 1 \) satisfies the above conditions. It follows from Lemma 4 that for every \( g \in L^p, h \in L^{p+1}, h \leq g \) there is a subspace \( \text{Aff}(f)(h) \in Z_h \subseteq \text{Aff}(\Delta)_h \) such that \( Z_h \cong l^\infty_m \) for some \( m \in \mathbb{N} \) and a homomorphism \( \eta_h : Z_g \to Z_h \) such that

\[
\text{dist}(\pi_h(a_q), Z_h) < 2^{-(P+1)}, 1 \leq q \leq P + 1,
\]

\[\|\eta_h - \pi_h|_{Z_g}\| < 2^{-p}.
\]

It follows from the Krein-Milman theorem for Markov operators of [5] that there is a Markov operator \( \theta_h : C([0, 1]) \) of the form \( \theta_h = N_p^1 (\theta_1^h + \ldots + \theta_{n_p}^h) \) where \( \theta_1^h, \ldots, \theta_{n_p}^h \) are restrictions of unital *-endomorphisms of \( C([0, 1]) \) such that

\[
\|\theta_h(f) - i_k \circ \eta_h \circ \kappa_g(f)\| < 2^{-p}
\]

for all \( f \in F_g \). We may assume that \( N_p = N_p \) for all \( h \in L^{p+1} \). There is a \( k_1 > i_p \) such that \( \text{mult}(s_{k_1, i_p}) \geq 2^p N_p \# L^{p+1} \). It follows from Lemma 3 that there is a \( k_2 > k_1 \) and a homomorphism \( \rho : W_{p+1} \to \text{Aff}(Z_{p+1}) \) such that \( \|s_{k_2, i_p} \circ \rho - \text{id}_{W_{p+1}}\| < 2^{-p-2} n_{i_p}^{-1} \). Since

\[
\|s_{k_2, i_p} \circ \rho \circ \delta_p - s_{i_p, i_p} \circ \rho \circ \delta_p - \delta_p\| \leq \|s_{k_2, i_p} \circ \rho \circ \delta_p - \delta_p\| + \|\delta_p - s_{i_p, i_p} \circ \rho\| < 2^{-p} n_{i_p}^{-1}
\]

and the unit ball in \( \text{Aff}(Z_{p+1}) \) is compact, it follows that there is an \( i_{p+1} \geq k_2 \) such that

\[
\|s_{i_{p+1}, i_p} \circ \rho \circ \delta_p - s_{i_{p+1}, i_p} \circ \rho\| < 2^{-p} n_{i_p}^{-1}.
\]

Note that mult \( (s_{i_{p+1}, i_p}) \geq 2^p N_p \# L^{p+1} \) and put \( \rho_{p+1} = s_{i_{p+1}, i_p} \circ \rho \). Since the span of \( X \) is dense in \( \text{Aff}(G) \) there is an \( j_{p+2} > j_{p+1} \) and a homomorphism \( \delta_{p+1} : \text{Aff}(Z_{p+1}) \to W_{p+2} \) such that \( \|s_{i_{p+1}, i_p} \circ \delta_{p+1}\| < 2^{-(P+1)-1} n_{i_{p+1}}^{-1} \). The result follows from Zorn's lemma.

For \( p \in \mathbb{N} \) let \( Z_p \) be the subspace of \( \text{Aff}(\Delta) \) spanned by \( (Z_g)_{g \in L^p} \). Note that \( Z_p \) is isomorphic to the order-unit-space direct sum of \( (Z_g)_{g \in L^p} \). Moreover dist \( (a_q, Z_p) < 2^{-p} \) for all \( 1 \leq q \leq p \). Let \( \eta_p : Z_p \to Z_{p+1} \) be the homomorphism given by

\[
\eta_p \left( \sum_{g \in L^p} v_g \right) = \sum_{g \in L^p, h \leq g} \eta_h(v_g)
\]

for \( v_g \in Z_g, g \in L^p \). Then \( \eta_p \) is a homomorphism with \( \eta_p(\text{Aff}(f)(g)) = \text{Aff}(f)(g) \) for all \( g \in W_p \). Let \( v \in Z_p \) with \( \|v\| \leq 1 \). Then \( v \) is of the form \( v = \sum_{g \in L^p} v_g \) where \( v_g \in Z_g, \|v_g\| \leq 1 \) and

\[
\|\eta_p(v) - v\| = \left\| \sum_{g \in L^p, h \leq g} (\eta_h(v_g) - \pi_h(v_g)) \right\| = \max_{g \in L^p, h \leq g} \|\eta_h(v_g) - \pi_h(v_g)\| < 2^{-p}.
\]

Therefore the sequence \( (\eta_q)_{q=1}^\infty - P \) in \( \text{Aff}(\Delta) \) is Cauchy for every \( v \in Z_p \) let \( \alpha_p(v) \) denote the limit. Then \( \alpha_p : Z_p \to \text{Aff}(\Delta) \) is a homomorphism with \( \alpha_{p+1} \circ \eta_p = \alpha_p \) and \( \alpha_p(\text{Aff}(f)(g)) = \text{Aff}(f)(g) \) for all \( g \in W_p \). Thus there is a homomorphism \( \alpha : (Z_p, \eta_p) \to \text{Aff}(\Delta) \) with \( \alpha \circ \eta_p = \alpha_p \). By \( \|\eta_p(v) - v\| < 2^{-p} \|v\| \) the
homomorphism $\alpha$ is seen to be isometric. Using that $(a_i)_{i=1}^\infty$ is dense in $\text{Aff}(A)$, $\text{dist}(a_q, Z_p) < 2^{-p}$ for all $1 \leq q \leq p$ and $1 \in Z_p$ for all $p \in \mathbb{N}$ and the fact that $\alpha$ is isometric, one shows that $\alpha$ is an isomorphism.

Define $I_p: Z_p \to W_p \otimes C_R([0, 1])$ and $K_p: W_p \otimes C(R([0, 1]) \to Z_p$ by

$$I_p \left( \sum_{g \in L_p} v_g \right) = \sum_{g \in L_p} g \otimes I_q(v_g),$$

$$K_p \left( \sum_{g \in L_p} g \otimes x_g \right) = \sum_{g \in L_p} K_q(x_g)$$

for $v_g \in Z_g$, $x_g \in C_R([0, 1])$, $g \in L_p$. The maps $I_p, K_p$ are homomorphisms and $K_p \circ I_p = \text{id}$. Letting $\omega_p = I_p + 1 \circ \eta_{p} \circ K_p$ and $\beta_p = \eta_{\infty, p} \circ K_p$ we have $\beta_{p+1} \circ \omega_p = \beta_p$ and so there is a homomorphism $\beta: \lim_{\longleftarrow} (W_p \otimes C(R([0, 1]), \omega_p)) \to \lim_{\longleftarrow} (Z_p, \eta_p)$ with $\beta \circ \omega_p = \beta_p$, $p \in \mathbb{N}$. It is easy to see that this is an isomorphism. In addition $\alpha \circ \beta_p(g \otimes 1) = \alpha \circ \eta_{\infty, p} \circ K_p(g \otimes 1) = \alpha \circ \eta_{\infty, p}(\text{Aff}(f)(g)) = \alpha_p(\text{Aff}(f)(g)) = \text{Aff}(f)(g)$ for all $g \in W_p$.

Let $\theta_p: W_p \otimes C_R([0, 1]) \to W_{p+1} \otimes C_R([0, 1])$ be the homomorphism given by

$$\theta_p \left( \sum_{g \in L_p} g \otimes x_g \right) = \sum_{g \in L_{p+1}, h \leq g} h \otimes \theta_h(x_g).$$

The set $\left\{ \sum_{g \in L_p} g \otimes x_g : x_g \in \{d_i : i \in \mathbb{N}\} \right\}$ is dense in $W_p \otimes C_R([0, 1])$ and

$$\bigcup_{q=1}^p \left\{ \sum_{h \in L_q} h \otimes x_h : x_h \in \{d_1, \ldots, d_p\} \right\} \cup \omega_{pq} \left\{ \sum_{h \in L_q} h \otimes x_h : x_h \in \{d_1, \ldots, d_p\} \right\}$$

is equal to $\{ \sum_{g \in L_p} g \otimes x_g : x_g \in F_g \}$. For $z = \sum_{g \in L_p} g \otimes x_g$ where $x_g \in F_g$ we have

$$\|\theta_p(z) - \omega_p(z)\| = \sum_{g \in L_{p+1}, h \leq g} \|\theta_h(x_g) - \omega_h \circ \eta_{\infty}(x_g)\| \leq 2^{-p}$$

and it follows from (a slight modification of) Lemma 3.4 of [5] that there is an isomorphism $\gamma: \lim(W_p \otimes C_R([0, 1]), \theta_p) \to \lim(W_p \otimes C_R([0, 1]), \omega_p)$ such that $\gamma \circ \theta_{\infty, p} = \gamma_p$ where $\gamma_p: W_p \otimes C_R([0, 1]) \to \lim(W_p \otimes C_R([0, 1]), \omega_p)$ is the homomorphism given by $\gamma_p(x) = \lim_{\omega_{\infty, q} \circ \theta_{pq}} x$ for $x \in W_p \otimes C_R([0, 1])$. Note that $\gamma_p(g \otimes 1) = \omega_{\infty, p}(g \otimes 1)$ for all $g \in W_p$.

To sum up, there is an isomorphism $\alpha \circ \beta \circ \gamma: \lim(W_p \otimes C_R([0, 1]), \theta_p) \to \text{Aff}(A)$ with $\alpha \circ \beta \circ \gamma \circ \theta_{\infty, i}(g \otimes 1) = \text{Aff}(f)(g)$ for all $g \in W_i$ and $i \in \mathbb{N}$.

**Lemma 8.** Let $M, N \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_m \geq 0$ with $\sum_{j=1}^m \lambda_j > 0$. There are $k_1, \ldots, k_m \in \mathbb{N}_0$ such that
\[ M - N < N \sum_{j=1}^{m} k_j \leq M, \]
\[ \sum_{j=1}^{m} \left| \frac{Nk_j}{M} - \lambda_j \right| < \frac{2Nm}{M} + \left| 1 - \frac{m}{\lambda} \sum_{j=1}^{m} \lambda_j \right|. \]

**Proof.** Put \( \lambda = \sum_{j=1}^{m} \lambda_j \) and \( \mu_j = \frac{M}{\lambda} \sum_{i=1}^{j} \lambda_i \) for \( 1 \leq j \leq m \). There is an \( h_j \in \mathbb{N}_0 \) with \( \mu_j - 1 < h_j \leq \mu_j \) for \( 1 \leq j \leq m \). Put \( k_1 = h_1 \) and \( k_{j+1} = h_{j+1} - h_j \) for \( 1 \leq j \leq m - 1 \). We have

\[ M - N = N(\mu_m - 1) < Nh_m = N \sum_{j=1}^{m} k_j \leq N\mu_m = M, \]
\[ \left| \frac{Nk_1}{M} - \frac{\lambda_1}{\lambda} \right| = \frac{N}{M} \left| k_1 - \frac{M\lambda_1}{N\lambda} \right| = \frac{N}{M} |k_1 - \mu_1| < \frac{N}{M} \]
and
\[ \left| \frac{Nk_j}{M} - \frac{\lambda_j}{\lambda} \right| = \frac{N}{M} \left| k_j - \frac{M\lambda_j}{N\lambda} \right| = \frac{N}{M} |(h_j - h_{j-1}) - (\mu_j - \mu_{j-1})| < 2 \frac{N}{M} \]
for \( 2 \leq j \leq m \). Therefore

\[ \sum_{j=1}^{m} \left| \frac{Nk_j}{M} - \lambda_j \right| < \frac{2Nm}{M} + |1 - \lambda|. \]

**Theorem 9.** Let \( G \) be a simple noncyclic dimension group, such that the extreme boundary of the state space is compact and totally disconnected. Let \( \Delta \) be a metrizable Choquet simplex and let \( f : \Delta \to S(G) \) be a continuous affine map with \( f(\partial_e \Delta) = \partial_e S(G) \). Then \((G, \Delta, f)\) is the Elliott triple of some simple unital AI algebra.

**Proof.** Let \((Z^a, s_i), X, (\rho_i)_{i=1}^\infty, (\delta_i)_{i=1}^\infty, (\theta_h)_{h \in X - \{1\}}\) and mult \( (s_i) \) be as in Proposition 7. Let \((a_{pq}^i, (\lambda_{pq}^i), (\mu_{pq}^i))\) and \((v_{pq}^i)\) be the matrices for \( s_i, \delta_i, s_i^\# \) and \( \rho_i \) respectively and put \( Z_i = \{(p, q); 1 \leq p \leq n_{i+1}, 1 \leq q \leq n_i, \sum_{h \in L_i^+} v_{pq}^{i+1} \lambda_{pq}^i = 0\} \) – the zero entries of the matrix for \( \rho_{i+1} \circ \delta_i \) for \( i \in \mathbb{N} \).

It follows from Lemma 8 that for \((p, q) \in Z_i\) there are \((k_{pq}^h)_{h \in L_i^+}\) in \( \mathbb{N}_0 \) such that

\[ a_{pq}^i - N_i \leq N_i \sum_{h \in L_i^+} k_{pq}^h \leq a_{pq}^i, \]
\[ \sum_{h \in L_i^+} \left| \frac{N_i k_{pq}^h}{a_{pq}^i} - \frac{v_{pq}^{i+1} \lambda_{pq}^i}{\mu_{pq}^i} \right| < \frac{2N_i}{a_{pq}^i} + \left| 1 - \frac{\sum_{h \in L_i^+} v_{pq}^{i+1} \lambda_{pq}^i}{\mu_{pq}^i} \right|. \]

For \( (p, q) \in Z_i \) choose \((k_{pq}^h)_{h \in L_i^+} \subseteq \mathbb{N}_0 \) such that \( a_{pq}^i - N_i \leq N_i \sum_{h \in L_i^+} k_{pq}^h \leq a_{pq}^i \).

Let \( \nu_{pq}^i = a_{pq}^i - N_i \sum_{h \in L_i^+} k_{pq}^h \) for \( 1 \leq p \leq n_{i+1}, 1 \leq q \leq n_i \) and \( i \in \mathbb{N} \). Let \( m_i = \)
\( (m_1^i, \ldots, m_n^i) \) denote the order-unit in \( \mathbb{Z}^n \). Then \( A_i = \bigoplus_{p=1}^{n_i} M_{m_p^i} \otimes C([0,1]) \) is the interval algebra with \( K_0(A_i) = \mathbb{Z}^n \). For all \( h \in L^{i+1}, 1 \leq r \leq N_i \) and \( i \in \mathbb{N} \) there is a continuous function \( \omega_h^r : [0,1] \to [0,1] \) such that \( \theta_h^r(x) = x \circ \omega_h^r \) for all \( x \in C([0,1]) \). Let \( \psi_i: A_i \to A_{i+1} \) be the \(*\)-homomorphism with characteristic functions \( \omega_h^r \) repeated \( k_{pq}^r \) times, \( h \in L^{i+1}, 1 \leq r \leq N_i \) and \( \text{id}_{[0,1]} \) repeated \( r_{pq}^i \) times from the \( q \)th summand of \( A_i \) to the \( p \)th summand of \( A_{i+1} \) for \( 1 \leq p \leq n_{i+1} \), \( 1 \leq q \leq n_i \) and \( i \in \mathbb{N} \). Let \( B = \lim \limits_{\mathbb{N}} (A_i, \psi_i) \) and note that \( K_0(B) = \lim \limits_{\mathbb{N}} (\mathbb{Z}^n, s_i) = G \).

Let \( i: W_i \hookrightarrow W_{i+1} \) be inclusion and let \( \iota_i: W_{i+1} \otimes C_R([0,1]) \to W_{i+1} \otimes C_R([0,1]) \) be the homomorphism given by

\[
\iota_i \left( \sum_{h \in L^{i+1}} h \otimes x_h \right) = \sum_{h \in L^{i+1}} h \otimes \theta_h(x_h).
\]

Observe that \( \theta_i \circ (i \otimes \text{id}_{C_R(0,1)}) \). Define

\[
\zeta_i = \iota_i \circ (\delta_i \otimes \text{id}_{C_R(0,1)}): \text{AffT}(A_i) \to W_{i+1} \otimes C_R([0,1]),
\]

\[
\eta_i = \rho_i \otimes \text{id}_{C_R(0,1)}: W_i \otimes C_R([0,1]) \to \text{AffT}(A_i)
\]

for \( i \in \mathbb{N} \).

We show that the triangles of the following diagram commutes up to an error which is summable.

\[
\begin{array}{ccc}
W_i \otimes C_R([0,1]) & \xrightarrow{\theta_i} & W_{i+1} \otimes C_R([0,1]) \\
\eta_i \downarrow & & \downarrow \eta_{i+1} \\
\text{AffT}(A_i) & \xrightarrow{\text{AffT}(\psi_i)} & \text{AffT}(A_{i+1})
\end{array}
\]

As for the upper triangle we have

\[
\| \theta_i - \zeta_i \circ \eta_i \| = \| \iota_i \circ (i \otimes \text{id}_{C_R(0,1)}) - \iota_i \circ (\delta_i \otimes \text{id}_{C_R(0,1)}) \circ (\rho_i \otimes \text{id}_{C_R(0,1)}) \|
\leq \| i - \delta_i \circ \rho_i \|
< 2^{-i}.
\]

Let \( x = (x_1, \ldots, x_n) \in \text{AffT}(A_i) \) and \( 1 \leq p \leq n_{i+1} \). The \( p \)th coordinate of \( \text{AffT}(\psi_i)(x) \) is

\[
\sum_{q=1}^{n_i} \frac{m_q^i}{m_{p+1}^i} \left( \sum_{h \in L^{i+1}} k_{pq}^h \sum_{r=1}^{N_i} x_q \circ \omega_h^r + r_{pq}^i x_q \right)
\]

and the \( p \)th coordinate of \( \eta_{i+1} \circ \zeta_i(x) \) is

\[
\sum_{q=1}^{n_i} \sum_{h \in L^{i+1}} \mu_{pq}^i \lambda_h^i \theta_h(x_q).
\]
Assume that \( \|x\| \leq 1 \). Then
\[
\| \text{Aff}^T(\psi_i)(x) - \eta_{i+1} \circ \zeta_i(x) \|
\leq \max_{1 \leq p \leq n_i+1} \left\{ \sum_{q=1}^{n_i} \sum_{h \in L^{i+1}} \left| \frac{m_q^{k_{pq}} N_i}{m^{i+1}_p} - v^{i+1}_{ph} \lambda_{hq} \frac{1}{N_i} \right| \left\| \sum_{r=1}^{N_i} x^q \circ \omega^r_k \right\| + \sum_{q=1}^{n_i} \frac{m_q^{k_{pq}}}{m^{i+1}_p} \| x^q \| \right\}
\leq \max_{1 \leq p \leq n_i+1} \left\{ \sum_{q=1}^{n_i} \sum_{h \in L^{i+1}} \left| \frac{m_q^{k_{pq}} N_i}{m^{i+1}_p} - v^{i+1}_{ph} \lambda_{hq} \right| + \sum_{q=1}^{n_i} \frac{m_q a_{pq}^{2^{-i}}}{m^{i+1}_p} \right\}
\leq \max_{1 \leq p \leq n_i+1} \left\{ \sum_{q=1}^{n_i} \mu_{pq} \sum_{h \in L^{i+1}} \left| \frac{k_{pq} N_i}{a_{pq}^{i+1}} - v^{i+1}_{ph} \lambda_{hq} \frac{1}{\mu_{pq}} \right| + 2^{-i} \right\}
\leq \max_{1 \leq p \leq n_i+1} \left\{ \sum_{1 \leq q \leq n_i \atop (p,q) \in Z_i} \mu_{pq} \sum_{h \in L^{i+1}} \frac{k_{pq} N_i}{a_{pq}^{i+1}} + \sum_{1 \leq q \leq n_i \atop (p,q) \in Z_i} \mu_{pq} \right\}
\times \left( \frac{2N_i \# L^{i+1}}{a_{pq}} + 1 - \sum_{1 \leq q \leq n_i \atop (p,q) \in Z_i} \frac{v^{i+1}_{ph} \lambda_{hq}}{\mu_{pq}} \right) + 2^{-i}
\leq \max_{1 \leq p \leq n_i+1} \left\{ \sum_{1 \leq q \leq n_i \atop (p,q) \in Z_i} \mu_{pq} + 2^{-i} + \sum_{1 \leq q \leq n_i \atop (p,q) \in Z_i} \mu_{pq} - \sum_{1 \leq q \leq n_i \atop (p,q) \in Z_i} \frac{v^{i+1}_{ph} \lambda_{hq}}{\mu_{pq}} \right\} + 2^{-i}
\leq \max_{1 \leq p \leq n_i+1} \left\{ \sum_{1 \leq q \leq n_i \atop (p,q) \in Z_i} \| \rho_{i+1} \circ \delta_i - s_i^\# \| + 2^{-i} + \sum_{1 \leq q \leq n_i \atop (p,q) \in Z_i} \| \rho_{i+1} \circ \delta_i - s_i^\# \| + 2^{-i} \right\}
= n_i \| \rho_{i+1} \circ \delta_i - s_i^\# \| + 3 \cdot 2^{-i}
\leq 4 \cdot 2^{-i}.
\]

Since
\[
\| \theta_{i+1} \circ \zeta_i - \zeta_{i+1} \circ \text{Aff}^T(\psi_i) \|
\leq \| \theta_{i+1} \circ \zeta_i - \zeta_{i+1} \circ \eta_{i+1} \circ \zeta_i \| + \| \zeta_{i+1} \circ \eta_{i+1} \circ \zeta_i - \zeta_{i+1} \circ \text{Aff}^T(\psi_i) \|
\leq \| \theta_{i+1} - \zeta_{i+1} \circ \eta_{i+1} \| + \| \eta_{i+1} \circ \zeta_i - \text{Aff}^T(\psi_i) \|
\leq 5 \cdot 2^{-i}
\]

the sequence \( \left( \theta_{\alpha k+1} \circ \zeta_k \circ \text{Aff}^T(\psi_k) \right)_{k=1}^{\infty} \) is Cauchy for all \( x \in \text{Aff}^T(A_i) \) – let \( \alpha_i(x) \) denote the limit. Then \( \alpha_i: \text{Aff}^T(A_i) \to \text{Aff}(\Delta) \) is a homomorphism with \( \alpha_{i+1} \circ \text{Aff}^T(\psi_i) = \alpha_i \). Thus there is a homomorphism \( \alpha: \text{Aff}^T(B) \to \text{Aff}(\Delta) \) with \( \alpha \circ \text{Aff}^T(\psi_{\alpha i}) = \alpha_i \). Using that \( (\| \text{Aff}^T(\psi_i) - \eta_{i+1} \circ \zeta_i \|)_{i=1}^{\infty} \) and \( (\| \zeta_i \circ \eta_i - \theta_i \|)_{i=1}^{\infty} \) are summable one shows that \( \alpha \) is an isomorphism.

Note that \( T(B) \cong \Delta \) via \( S(\alpha): \Delta \to T(B) \). We now show that \( r_B \circ S(\alpha) = f \). Let \( \kappa_i \) and \( \chi_i \) be the inclusions.
\( \kappa_i : W_i \hookrightarrow W_i \otimes C_R([0, 1]), g \mapsto g \otimes 1, \)
\( \chi_i : \text{AffS}(\mathbb{Z}^n) \hookrightarrow \text{AffS}(\mathbb{Z}^n) \otimes C_R([0, 1]) = \text{AffT}(A_i), g \mapsto g \otimes 1. \)

With these definitions we have that
\( \theta_{\infty i} \circ \kappa_i = \text{Aff}(f)|_{W_i}, \)
\( \zeta_{i} \circ \chi_i = \kappa_{i+1} \circ \delta_i \)

and \( p = \chi_i[p] \) for every projection \( p \in A_i \subset \text{AffT}(A_i). \) Let \( w \in A \) and \( p \in A_i \) be a projection.

\[
\begin{align*}
& r_B \circ S(\alpha)(w)[\psi_{\infty i}(p)] \\
& = S(\alpha)(w)(\psi_{\infty i}(p)) \\
& = w \circ \alpha \circ \psi_{\infty i}(p) \\
& = w \circ \chi_i(p) \\
& = \lim_{k \to \infty} w \circ \theta_{\infty k+1} \circ \varsigma_k \circ \psi_{k}(p) \\
& = \lim_{k \to \infty} w \circ \theta_{\infty k+1} \circ \varsigma_k \circ \chi_k[\psi_{k}(p)] \\
& = \lim_{k \to \infty} w \circ \theta_{\infty k+1} \circ \kappa_{k+1} \circ \delta_k[\psi_{k}(p)] \\
& = f(w) \left( \lim_{k \to \infty} \delta_k[\psi_{k}(p)] \right) \\
& = f(w)[\psi_{\infty i}(p)].
\end{align*}
\]

Hence \((G, A, f)\) is the Elliott triple of \( B. \)

The final step of the proof consists of replacing \( B \) by a simple \( \mathcal{AI} \) algebra with the same Elliott triple. Let \( \phi_i \) be the \(*\)-homomorphism obtained from \( \psi_i \) by replacing two of the characteristic functions in each entry of \( \psi_i \) by \( h_0 \) and \( h_1 \) where \( h_0(t) = \frac{t}{2} \) and \( h_1(t) = \frac{t + 1}{2} \) for \( t \in [0, 1] \) and all \( i \in \mathbb{N} \). It follows from [2] that the C*-algebra \( A = \lim_{\rightarrow} (A_i, \phi_i) \) is simple. Note that \( K_0(\phi_i) = K_0(\psi_i) = s_i. \) Since

\[
\| \text{AffT}(\phi_i) - \text{AffT}(\psi_i) \| < 2 \text{mult } (s_i)^{-1} \leq 2^{1-i}
\]

the sequence \( (\text{AffT}(\psi_{\infty k} \circ \phi_{ki})(x))_{k=i}^\infty \) is Cauchy for every \( x \in \text{AffT}(A_i), i \in \mathbb{N} \) – let \( \gamma_i(x) \) denote the limit. Then \( \gamma_i : \text{AffT}(A_i) \to \text{AffT}(B) \) is a homomorphism with \( \gamma_{i+1} \circ \text{AffT}(\phi_i) = \gamma_i \) for all \( i \in \mathbb{N} \). There is an isomorphism \( \gamma : \text{AffT}(A) \to \text{AffT}(B) \) such that \( \gamma \circ \text{AffT}(\phi_{\infty i}) = \gamma_i \) for all \( i \in \mathbb{N} \). Let \( \tau \in T(B) \) and \( p \in A_i \) be a projection.
Then

\[ r_A \circ S(\gamma)(\tau)[\phi_{\infty i}(p)] = S(\gamma)(\tau)(\phi_{\infty i}(p)) \]

\[ = \gamma \circ \text{AffT}(\phi_{\infty i})(p)(\tau) \]

\[ = \gamma_i(p)(\tau) \]

\[ = \lim_{k \to \infty} \text{AffT}(\psi_{\infty k} \circ \phi_{ki})(p)(\tau) \]

\[ = \lim_{k \to \infty} \tau(\psi_{\infty k} \circ \phi_{ki}(p)) \]

\[ = \tau(\psi_{\infty i}(p)) \]

\[ = r_B(\tau)[\psi_{\infty i}(p)]. \]

We conclude that \( r_A \circ S(\gamma) \circ S(\alpha) = f \).

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