A VARIATIONAL PRINCIPLE FOR THE HAUSDORFF DIMENSION OF FRACTAL SETS

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Abstract.
Let $\mathcal{P}(E)$ denote the set of probability measures on a Borel set $E \subseteq \mathbb{R}^n$, and let $R(\mu), \bar{R}(\mu)$ denote respectively the lower and upper Rényi dimensions associated with a measure $\mu \in \mathcal{P}(E)$. We prove that the Hausdorff dimension $\dim(E)$ satisfies

$$\dim(E) \leq \sup_{\mu \in \mathcal{P}(E)} R(\mu)$$

while, if $E$ is additionally bounded, the packing dimension $\text{Dim}(E)$ satisfies

$$\text{Dim}(E) \geq \sup_{\mu \in \mathcal{P}(E)} \bar{R}(\mu).$$

As a consequence, for any bounded Borel set $E$ satisfying Taylor’s definition of a fractal (i.e. $\dim(E) = \text{Dim}(E)$) we obtain the variational principle

$$\dim(E) = \text{Dim}(E) = \sup_{\mu \in \mathcal{P}(E)} R(\mu) = \sup_{\mu \in \mathcal{P}(E)} \bar{R}(\mu).$$

In addition we provide an example showing that the hypothesis “bounded” cannot be eliminated.

1. Introduction.
In recent papers on fractals attention has shifted from sets to measure, cf. [1, 2, 3, 4, 5, 6, 8, 9, 10, 12]. Thus it seems reasonable to make an attempt at finding a relation between the dimension of a fractal $E$ and parameters connected with measures supported by $E$. Such relations have already been investigated, cf. in particular [14, Theorem 1 p. 62] and Young [18]. Our principal result states that if $E \subseteq \mathbb{R}^n$ is a bounded Borel set satisfying Taylor’s definition of a fractal, i.e. the Hausdorff dimension $\dim(E)$ of $E$ is equal to the packing dimension $\text{Dim}(E)$ of $E$, cf. [15] and [16], then

(1) $\dim(E) = \text{Dim}(E) = \sup_{\mu \in \mathcal{P}(E)} R(\mu) = \sup_{\mu \in \mathcal{P}(E)} \bar{R}(\mu)$

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where $R(\mu)$ and $\tilde{R}(\mu)$ denote, respectively, the lower and upper Rényi dimensions and $\mathcal{P}(E)$ is the family of all Borel probability measures on $E$.

Formula (1) is a variational principle – i.e. it establishes an equality between a number naturally connected with a space or a map (in this case $\dim E$) and the supremum of certain numbers connected to a class of probability measures supported by $E$. It is well-known that variational principles play a major role in ergodic theory (cf. e.g. [17, Chapter 8-9]) since these principles yield a canonical way of choosing measures. Formula (1) yields in a similar way a canonical way of choosing measures – namely measures $\mu \in \mathcal{P}(E)$ such that $R(\mu)$ and $\tilde{R}(\mu)$ are close to $\dim E$ and $\text{Dim}(E)$. It is interesting to note that our variational principle is formulated in terms of the Rényi dimension since generalised Rényi dimensions play an important part in so-called multifractal analysis, cf. e.g. Rand [13] and the references therein.

We begin in section 2 by collecting the relevant facts and setting the notation. Then in section 3 we derive some auxiliary inequalities and prove the variational principle contained in formula (1).

2. Preliminaries.

This section contains a survey of the fractal dimensions which we will consider.

Let $(X, d)$ be a separable metric space, $E \subseteq X$ and $s \geq 0$. Then the $s$-dimensional Hausdorff measure $\mathcal{H}^s(E)$ of $E$ is defined by

$$
\mathcal{H}^s(E) = \sup_{\delta > 0} \inf \left\{ \sum_{i=1}^{\infty} (\text{diam } E_i)^s \mid E \subseteq \bigcup_{i=1}^{\infty} E_i, \text{diam } E_i < \delta \text{ for all } i \in \mathbb{N} \right\}.
$$

The Hausdorff dimension $\dim E$ of $E$ is defined by

$$
\dim E = \inf \{ s \geq 0 \mid \mathcal{H}^s(E) < \infty \} = \sup \{ s \geq 0 \mid \mathcal{H}^s(E) > 0 \}.
$$

The $s$-dimensional packing measure $\mathcal{P}^s(E)$ of $E$ is defined in two stages. First put

$$
\mathcal{P}^s_0(E) = \inf_{\delta > 0} \sup \left\{ \sum_{i=1}^{\infty} (\text{diam } B_i)^s \mid B_i \cap B_j = \emptyset \text{ for } i \neq j \right. \\
\text{and } B_i \text{ is a closed ball of radius at most } \delta \\
\text{with center in } E \text{ for all } i \in \mathbb{N} \right\}.
$$

Then

$$
\mathcal{P}^s(E) = \inf \left\{ \sum_{i=1}^{\infty} \mathcal{P}^s_0(E_i) \mid E \subseteq \bigcup_{i=1}^{\infty} E_i \right\}.
$$

The packing dimension $\text{Dim } E$ of $E$ is defined by
Dim $E = \inf \{ s \geq 0 \mid \mathcal{P}^s(E) < \infty \} = \sup \{ s \geq 0 \mid \mathcal{P}^s(E) > 0 \}$.

It is a well-known fact that $\dim E \leq \Dim E$ for all $E \subseteq \mathbb{R}^n$, cf. [14].

Two other useful dimensions of a bounded set $E$ are the upper and lower box dimensions. For each $\delta > 0$ let $N_\delta(E)$ be the least number of sets of diameter at most $\delta$ that cover $E$. Then the upper and lower box dimensions of $E$ are defined by

$$\bar{C}(E) = \limsup_{\delta \to 0} \log \frac{N_\delta(E)}{-\log \delta}$$

and

$$\underline{C}(E) = \liminf_{\delta \to 0} \log \frac{N_\delta(E)}{-\log \delta}$$

respectively.

Let us introduce the Rényi dimension. Fix $\mu \in \mathcal{P}(X)$ and write

$$h_r(\mu) = \inf \left\{ -\sum_{i=1}^{\infty} \mu(E_i) \log \mu(E_i) \mid \{E_i\} \text{ is a countable Borel partition of } X \text{ and diam } E_i \leq r \right\}$$

for $r > 0$. Then the upper and lower Rényi dimensions of $\mu$ are defined by

$$\bar{R}(\mu) = \limsup_{r \to 0} -\frac{h_r(\mu)}{\log r}$$

and

$$\underline{R}(\mu) = \liminf_{r \to 0} -\frac{h_r(\mu)}{\log r}$$

respectively, (cf. [18]).


We want to prove that

$$\dim(E) \leq \sup_{\mu \in \mathcal{P}(E)} \underline{R}(\mu)$$

for a Borel subset $E$ of $\mathbb{R}^n$, and

$$\Dim(E) \geq \sup_{\mu \in \mathcal{P}(E)} \bar{R}(\mu)$$

for a bounded Borel subset $E$ of $\mathbb{R}^n$. Both proofs are based on the following result:
THEOREM 1. Let $E \subseteq \mathbb{R}^n$ be a Borel set. Then the following assertions hold:

i) $$\dim(E) = \sup_{\mu \in \mathcal{B}(E)} \left( \inf_{x \in E} \inf_{r \downarrow 0} \frac{\log \mu(B(x, r))}{\log r} \right).$$

ii) If $$E \subseteq \left\{ x \mid \limsup_{r \downarrow 0} \frac{\log \mu(B(x, r))}{\log r} \geq \alpha \right\} \text{ and } \mu(E) > 0,$$

then $$\text{Dim}(E) \geq \alpha.$$ 

PROOF. i) Follows easily from [14, Theorem 1]. ii) Follows from [14, Theorem 1], however see also Theorem 3.2 of [5].

We begin with three small technical lemmas

LEMMA 2. Let $\mu$ be a Borel probability measure on $\mathbb{R}^n$. Let $E$ be a Borel set, $t \geq 0$ and $\delta \in ]0, 1[$. Suppose $$\log \mu(B(x, r)) \leq t \log r$$

for all $x \in E$ and $r \in ]0, \delta[$. Then $$R(\mu) \geq \mu(E)t.$$ 

PROOF. Let $r \in ]0, \delta[$ and $(E_i)_i$ be a partition of $\mathbb{R}^n$ such that $\text{diam}(E_i) \leq r$. Let $I = \{ i \mid E_i \cap E \neq \emptyset \}$. If $i \in I$ then we can choose a point $x_i \in E_i \cap E$ such that $E_i \subseteq B(x_i, r)$, whence

(4) $$\log \mu(E_i) \leq \log \mu(B(x_i, r)) \leq t \log r \text{ for } i \in I.$$ 

By (4) we have

$$- \sum_{i} \mu(E_i) \log \mu(E_i) \geq - \sum_{i \in I} \mu(E_i) \log \mu(E_i) \geq - \sum_{i \in I} \mu(E_i) t \log r$$

$$= - \mu \left( \bigcup_{i \in I} E_i \right) t \log r \geq - \mu(E) t \log r.$$ 

Since the partition $(E_i)_i$ was arbitrary this inequality implies that $$h_r(\mu) \geq - \mu(E) t \log r \text{ for } r \in ]0, \delta[,$$

whence $$R(\mu) = \liminf_{r \to 0} \frac{h_r(\mu)}{\log r} \geq t \mu(E).$$
Lemma 3. Let $F \subseteq \mathbb{R}^n$ be a bounded Borel set and $r > 0$. Then there exists a finite collection $F_1, \ldots, F_m$ of disjoint Borel sets with $\text{diam}(F_i) \leq r$ such that $F \subseteq \bigcup_i F_i$ and such that for each $i$, there exists an $x_i \in F$ satisfying

$$B(x_i, \frac{1}{4}r) \subseteq F_i.$$ 

Proof. Construct a sequence of balls $B(x_1, \frac{1}{2}r), B(x_2, \frac{1}{4}r), \ldots$ such that $x_i \in F$ and $d(x_i, x_j) > \frac{1}{2}r$ for $i \neq j$. Because $F$ is totally bounded this process must terminate at some finite stage, giving balls $B(x_1, \frac{1}{2}r), \ldots, B(x_m, \frac{1}{4}r)$ such that any $x \in F$ must satisfy $\min_i d(x, x_i) \leq \frac{1}{2}r$ (consequently $F \subseteq \bigcup_{i=1}^m B(x_i, \frac{1}{2}r)$). Note that the smaller balls $B(x_1, \frac{1}{4}r), \ldots, B(x_m, \frac{1}{4}r)$ are disjoint. Set

$$F_1 = B(x_1, \frac{1}{2}r) \setminus \bigcup_{j=2}^m B(x_j, \frac{1}{4}r)$$

$$F_i = B(x_i, \frac{1}{2}r) \setminus \left( \bigcup_{j=1}^{i-1} F_j \cup \bigcup_{j=i+1}^m B(x_j, \frac{1}{4}r) \right) \text{ for } i = 2, \ldots, m - 1$$

$$F_m = B(x_m, \frac{1}{2}r) \setminus \bigcup_{j=1}^{m-1} F_i.$$ 

It is clear that the $F_i$'s are disjoint, and since $B(x_1, \frac{1}{4}r), \ldots, B(x_m, \frac{1}{4}r)$ are disjoint we can conclude that $B(x_i, \frac{1}{4}r) \subseteq F_i$ and $F \subseteq \bigcup_i F_i$.

Lemma 4. Let $E \subseteq \mathbb{R}^n$ be a bounded Borel set and $\mu \in \mathcal{B}(E)$. Let $F \subseteq E$ be a Borel set, $t \geq 0$ and $\delta \in ]0, 1[$. Assume

$$\log \mu(B(x, r)) \geq t \log r$$

for all $x \in F$ and $0 < r < \delta$. Then

$$\tilde{R}(\mu) \leq t + \mu(E \setminus F) \tilde{C}(E \setminus F).$$

Proof. Let $r \in ]0, \delta[$ and choose by Lemma 3 a finite pairwise disjoint covering $(F_1, \ldots, F_m)$ of $F$ with $\text{diam}(F_i) \leq r$ and such that there exists points $x_i \in F$ for all $i$ satisfying

$$B(x_i, \frac{1}{4}r) \subseteq F_i.$$ 

The set $E \setminus F$ can be covered by $N = N_r(E \setminus F)$ closed balls $B_1, \ldots, B_N$ of diameter at most $r$. Define $Q_1, \ldots, Q_N$ by

$$Q_1 = (B_1 \cap (E \setminus F)) \setminus \bigcup_j F_j$$

$$Q_i = (B_i \cap (E \setminus F)) \setminus (\bigcup_j F_j \cup \bigcup_{j=1}^{i-1} Q_j) \text{ for } i = 2, \ldots, N.$$ 

Then $F_1, \ldots, F_m, Q_1, \ldots, Q_N$ are disjoint sets of diameter not exceeding $r$, and

$$E = \bigcup_i (F_i \cap E) \cup \bigcup_i Q_i, \quad \bigcup_i Q_i \subseteq E \setminus F.$$
Hence

\[
h_r(\mu) \leq - \sum_{i=1}^{m} \mu(F_i \cap E) \log \mu(F_i \cap E) - \sum_{i=1}^{N} \mu(Q_i) \log \mu(Q_i)
\]

\[
= - \sum_{i=1}^{m} \mu(F_i) \log \mu(F_i) - \sum_{i=1}^{N} \mu(Q_i) \log \mu(Q_i)
\]

\[
\leq - \sum_{i=1}^{m} \mu(F_i) t \log (\frac{1}{4} r) - \sum_{i=1}^{N} \mu(Q_i) \log \mu(Q_i)
\]

\[
\leq - t \log (\frac{1}{4} r) - \sum_{i=1}^{N} \mu(Q_i) \log \mu(Q_i).
\]

We know that if \( p_1, \ldots, p_k \geq 0 \) and \( \sum_{t=1}^{k} p_t = s \in [0, 1] \) then in fact \( - \sum_{i=1}^{k} p_t \log p_t \leq s \log k - s \log s \leq s \log k + \frac{1}{e} \). Therefore

\[
h_r(\mu) \leq - t \log (\frac{1}{4} r) + \sum_{i=1}^{N} \mu(Q_i) \log N + \frac{1}{e}
\]

\[
\leq - t \log (\frac{1}{4} r) + \mu\left( \bigcup_{i=1}^{N} Q_i \right) \log N_r(E \setminus F) + \frac{1}{e}
\]

\[
\leq - t \log (\frac{1}{4} r) + \mu(E \setminus F) \log N_r(E \setminus F) + \frac{1}{e}
\]

for \( r < \delta \), whence

\[
\bar{R}(\mu) = \lim_{r \downarrow 0} \sup \frac{h_r(\mu)}{-\log r} \leq \lim_{r \downarrow 0} \left( \frac{t \log (\frac{1}{4} r)}{\log r} + \mu(E \setminus F) \frac{\log N_r(E \setminus F)}{-\log r} - \frac{1}{e \log r} \right)
\]

\[
\leq t + \mu(E \setminus F) \bar{C}(E \setminus F).
\]

We are now ready to prove (2) and (3).

**Proposition 5.** Let \( E \subseteq \mathbb{R}^n \). Then the following assertions hold:

i) If \( E \) is a Borel set then

\[
\dim E \leq \sup_{\mu \in \mathcal{P}(E)} R(\mu).
\]

ii) If \( E \) is a bounded Borel set then
\[ \sup_{\mu \in \mathcal{P}(E)} \tilde{R}(\mu) \leq \dim E. \]

**Proof.** i) Let \( t < \dim E \). Then Theorem 1 part i) implies that there exists a measure \( \mu \in \mathcal{P}(E) \) such that

\[
(5) \quad t < \lim_{r \downarrow 0} \inf \frac{\log \mu(B(x, r))}{\log r} \text{ for all } x \in E.
\]

Now put

\[ E_m = \left\{ x \in E \mid \frac{\log \mu(B(x, r))}{\log r} > t \text{ for } 0 < r < \frac{1}{m} \right\}, \ m \in \mathbb{N}. \]

Let \( \varepsilon > 0 \) and observe that (5) implies that \( E_m \uparrow E \). We can thus choose an integer \( N \in \mathbb{N} \) so \( \mu(E_N) \geq \mu(E) - \varepsilon = 1 - \varepsilon \). An application of Lemma 2 then yields

\[ \sup_{\lambda \in \mathcal{P}(E)} R(\lambda) \geq R(\mu) \geq \mu(E_N)t \geq (1 - \varepsilon)t \]

which proves the first part of the proposition since \( t < \dim E \) and \( \varepsilon > 0 \) were arbitrary.

ii) Let \( \mu \in \mathcal{P}(E) \) and \( t > \dim (E) \). Then Theorem 1 part ii) implies that

\[ \limsup_{r \downarrow 0} \frac{\log \mu(B(x, r))}{\log r} \leq \dim (E) \mu\text{-a.s.} \]

and we can thus choose a subset \( F \) of \( E \) with \( \mu(F) = 1 \) such that \( \limsup_{r \downarrow 0} \frac{\log \mu(B(x, r))}{\log r} < t \) for all \( x \in F \). Now put

\[ F_m = \left\{ x \in F \mid \frac{\log \mu(B(x, r))}{\log r} < t \text{ for } 0 < r < \frac{1}{m} \right\}, \ m \in \mathbb{N}. \]

An application of Lemma 4 then yields

\[ \tilde{R}(\mu) \leq t + \mu(E \setminus F_m) \tilde{C}(E) = t + \mu(F \setminus F_m) \tilde{C}(E). \]

Since \( F_m \uparrow F \) we conclude that \( \tilde{R}(\mu) \leq t \). This completes the proof since both \( \mu \in \mathcal{P}(E) \) and \( t > \dim (E) \) were arbitrary.

**Proposition 5** immediately yields the following variational principle

**Proposition 6.** If \( E \subseteq \mathbb{R}^n \) is a bounded Borel set satisfying \( \dim (E) = \dim (E) \), then

\[ \dim (E) = \dim (E) = \sup_{\mu \in \mathcal{P}(E)} R(\mu) = \sup_{\mu \in \mathcal{P}(E)} \tilde{R}(\mu). \]
It is easily seen that the inequality in Proposition 5) ii) may not hold if the assumption "bounded" is omitted. Indeed put $E = N$ and $q_n = c((n + 1)(\log (n + 1))^2)^{-1}$ for $n \in N$ where $c = 1/\sum_{n=2}^{\infty} 1/n(\log n)^2$, and define $\mu \in \mathcal{P}(E)$ by $\mu = \sum_n q_n \delta_n$ (here $\delta_x$ denotes the Dirac measure concentrated at $x$). If $0 < r < 1$ and $(E_i)_i$ is a countable partition of $E = N$ then $(E_i \cap E)_i = \{\{n\}\}_n \in N$, whence
\[ \frac{h_r(\mu)}{-\log r} = \frac{-\sum_n \mu(\{n\}) \log (\mu(\{n\}))}{-\log r} = \frac{-\sum_n q_n \log q_n}{-\log r} = \infty \]
which implies that $\bar{R}(\mu) = R(\mu) = \infty > 0 = \text{Dim}(E)$.

REFERENCES


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