FREE ARCHIMEDEAN $l$-GROUPS

DAO-RONG TON and KAI-YAO HE

Abstract.

In this paper we discuss the existence and description of the free archimedean $l$-group $\mathcal{F}_\mu([G, P])$ generated by a po-group $[G, P]$, and give some properties of the free abelian $l$-group (the free archimedean $l$-group) $\mathcal{H}_\alpha$ of rank $\alpha$.

We use the standard terminologies and notations of $[1, 5, 9]$. We assume that all groups considered will be abelian. The group operation of an $l$-group is written by additive notation. Let $G$ be an $l$-group and $S \subseteq G$. We denote by $[S]$ the $l$-subgroup of $G$ generated by $S$. The convex $l$-subgroup generated by an element $x \in G$ is denoted by $G(x)$. A po-group is a partially ordered group $[G, P]$ where $P = \{x \in G | x \geq 0\}$ is the positive semigroup of $G$. $P$ is said to be semi-group if $p \in P$ whenever $p \in G$ and $np \in P$ for some positive integer $n$. Let $G$ and $H$ be two po-groups. A map $\varphi$ from $G$ into $H$ is called a po-group homomorphism, if $\varphi$ is a group homomorphism and $x \geq y$ implies $\varphi(x) \geq \varphi(y)$ for any $x, y \in G$. A po-group homomorphism $\varphi$ is called a po-group isomorphism if $\varphi$ is an injection and $\varphi^{-1}$ is also a po-group homomorphism. We use $N$ and $Z$ for the natural numbers and the integers, respectively.

1. Sub-product Radical Class of Archimedean $l$-groups.

A family $\mathcal{U}$ of $l$-groups is called a sub-product radical class, if it is closed under taking 1) $l$-subgroups, 2) joins of convex $l$-subgroups and 3) direct products. All our sub-product radical classes are always assumed to contain along with a given $l$-group all its $l$-isomorphic copies. Let $\mathcal{U}$ be a sub-product radical class and $G$ be an $l$-group. Then the join of all convex $l$-subgroups of $G$ belonging to $\mathcal{U}$ is the unique largest convex $l$-subgroup of $G$ belonging to $\mathcal{U}$. It is denoted by $\mathcal{U}(G)$ and is called a sub-product radical of $G$. $\mathcal{U}(G)$ is a characteristic $l$-ideal of $G$.

An $l$-group $G$ is said to be archimedean if it satisfies one of the following three equivalent conditions:

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1. For any \(0 < a, b \in G\), there exists \(n \in N\) such that \(nb \leq a\).
2. For all \(a, b \in G\), if \(nb \leq a \) for all \(n \in Z\), then \(b = 0\).
3. For all \(a, b \in G\), if \(nb \leq a\) for all \(n \in N\), then \(b \leq 0\).

Let \(G\) be an \(l\)-group. An element \(a \in G\) is archimedean if \(a \geq 0\) and if for all \(0 < b \leq a\), there exists \(n \in N\) such that \(nb \leq a\) [12, 18]. Let \(P(G)\) be the set of all archimedean elements of \(G\). An element \(a \in G\) is said to be generally archimedean if the positive part \(a^+\) and the negative part \(a^-\) are both archimedean. The following lemma is easy to show using [18].

**Lemma 1.1.** Let \(G\) be an \(l\)-group and \(g \in G\). Then the following are equivalent:

1. \(g\) is generally archimedean.
2. \(|g|\) is archimedean.
3. \(G(g)\) is archimedean.
4. \(G(|g|)\) is archimedean.

Let \(\mathcal{A}r\) be the family of all archimedean \(l\)-groups. \(\mathcal{A}r\) is a quasi-torsion class [13], that is, \(\mathcal{A}r\) is closed under taking 1) convex \(l\)-subgroups, 2) joins of convex \(l\)-subgroups and 3) complete \(l\)-homomorphisms. It is clear that \(\mathcal{A}r\) is closed under taking \(l\)-subgroups and direct products. So \(\mathcal{A}r\) is a sub-product radical class. Let \(G\) be an \(l\)-group. Then there exists a unique largest archimedean \(l\)-subgroup of \(G\), the \(\mathcal{A}r\) radical \(\mathcal{A}r(G)\). Clearly, \(G\) is archimedean if and only if \(G = \mathcal{A}r(G)\). In [18] it was proved that the \(l\)-subgroup \(A(G)\) of \(G\) is the unique largest archimedean convex \(l\)-subgroup of \(G\). In [12] J. Jakubik also proved the existence of such \(A(G)\). So we have \(\mathcal{A}r(G) = [P(G)]\). By Theorem 1.3 of [5] \(\mathcal{A}r(G)\) consists of the elements \(g = x - y\) where \(x, y \in P(G)\) and \(x \land y = 0\). In fact, \(x = g^+\) and \(y = g^-\). And so such \(g\) are generally archimedean. Conversely, if \(g \in G\) is a generally archimedean element, then \(g \in \mathcal{A}r(G)\). Thus Lemma 1.1 infers

**Lemma 1.2.** \(\mathcal{A}r(G) = [P(G)]\)

\[\{g \in G \ | \ g \text{ is generally archimedean}\}\]
\[= \{g \in G \ | \ |g| \in P(G)\}\]
\[= \{g \in G \ | \ G(g) \text{ is archimedean}\}\]
\[= \{g \in G \ | \ G(|g|) \text{ is archimedean}\}\].

**Corollary 1.3.** The set of all generally archimedean elements of an \(l\)-group \(G\) is closed under the addition, inverse, met and join.

So we obtain a useful result.

**Proposition 1.4.** Suppose that an \(l\)-group \(G\) has a set of generators which consists of generally archimedean elements. Then \(G\) is archimedean.

In what follows we will give an application of Proposition 1.4.
2. Free Archimedean \( l \)-group Generated by a po-group.

A partial \( l \)-group \( G \) is a set with partial operations corresponding to the usual \( l \)-group operations \( \cdot, +, 1, \lor, \land \) such that whenever the operations are defined for elements of \( G \) then the \( l \)-group laws are satisfied. Suppose \([G, P]\) is a po-group. Then \( G \) has implicit partial operations \( \lor \) and \( \land \) as determined by the partial order. That is,

\[
\begin{align*}
x \lor y &= y \lor x = y & \text{if and only if } x \leq y \\
x \land y &= y \land x = x & \text{if and only if } x \leq y.
\end{align*}
\]

Using these two partial lattice operations together with the full group operations, \( G \) can be considered as a partial \( l \)-group. Thus we have the following definition as a special case of the \( \mathcal{U} \)-free algebra generated by a partial algebra.

**Definition 2.1.** Let \( \mathcal{U} \) be a class of \( l \)-groups and \([G, P]\) be a po-group. The \( l \)-group \( \mathcal{F}_\mathcal{U}([G, P]) \) is called the \( \mathcal{U} \)-free \( l \)-group generated by \([G, P]\) (or \( \mathcal{U} \)-free \( l \)-group over \([G, P]\)) if the following conditions are satisfied:

1. \( \mathcal{F}_\mathcal{U}([G, P]) \in \mathcal{U} \);
2. there exists an injective po-group isomorphism \( \alpha : G \to \mathcal{F}_\mathcal{U}([G, P]) \) such that \( \alpha(G) \) generates \( \mathcal{F}_\mathcal{U}([G, P]) \) as an \( l \)-group;
3. if \( K \in \mathcal{U} \) and \( \beta : G \to K \) is a po-group homomorphism, then there exists an \( l \)-homomorphism \( \gamma : \mathcal{F}_\mathcal{U}([G, P]) \to K \) such that \( \gamma \alpha = \beta \).

The classes of \( l \)-groups which will be referred to are \( \mathcal{A}r \) and the following:

\( \mathcal{L} \), the class of all \( l \)-groups,

\( \mathcal{A} \), the class of all abelian \( l \)-groups.

\( \mathcal{L} \), \( \mathcal{A} \) and \( \mathcal{A}r \) are all sub-product radical classes of \( l \)-groups.

In 1963 and 1965, E. C. Weinberg initially considered the \( \mathcal{A} \)-free \( l \)-group generated by a po-group \([G, P]\). He has given a necessary and sufficient condition for existence and a simple description of \( \mathcal{F}_\mathcal{A}([G, P]) \) as follows:

**Proposition 2.2.** [17, 18]. Let \([G, P]\) be a torsion-free abelian po-group.

1. There exists an \( \mathcal{A} \)-free \( l \)-group \( \mathcal{F}_\mathcal{A}([G, P]) \) generated by \([G, P]\) if and only if there exists a po-group isomorphism of \([G, P]\) into an abelian \( l \)-group, if and only if \( P \) is semi-closed.
2. Let \( \mathcal{P} \) be the set of all total orders \( T \) of \( G \) such that \( P \subseteq T \). Then \( \mathcal{F}_\mathcal{A}([G, P]) \) is
the sublattice of the direct product $\prod_{T \in \mathcal{P}} [G, T]$ which is generated by the long constants $\langle g \rangle$ ($g \in G$).

The elements of $\mathcal{F}_\mathcal{A}([G, P])$ have the form

$$x = \bigvee_{i \in I} \bigwedge_{j \in J} \langle x_{ij} \rangle$$

where $I$ and $J$ are both finite and $x_{ij} \in G$ ($i \in I$, $j \in J$).

In 1970, P. Conrad generalized Weiberg's result.

PROPOSITION 2.3 [6]. Let $[G, P]$ be a torsion-free po-group.

1. There exists an $\mathcal{L}$-free l-group $\mathcal{F}_\mathcal{L}([G, P])$ generated by $[G, P]$ if and only if there exists a po-group isomorphism of $[G, P]$ into an l-group, if and only if $P$ is the intersection of right orders on $G$.

2. Suppose that $P = \bigcap_{\lambda \in \Lambda} P_\lambda$ where $\{P_\lambda | \lambda \in \Lambda \}$ is the set of all right orders of $G$ such that $P_\lambda \supseteq P$. If $G_\lambda$ is $G$ with one such right order, then denote by $A(G_\lambda)$ the l-group of order preserving permutations of $G_\lambda$. Each $x \in G$ corresponds to an element $\rho_x$ of $A(G_\lambda)$ defined by $\rho_x g = g + x$. Then $\mathcal{F}_\mathcal{L}([G, P])$ is the sublattice of the direct product $\prod_{\lambda \in \Lambda} A(G_\lambda)$ which is generated by the long constants $\langle g \rangle$ ($g \in G$).

In this section we will discuss the $\mathcal{A}_r$-free l-group $\mathcal{F}_{\mathcal{A}_r}([G, P])$ generated by a po-group $[G, P]$. Because $\mathcal{A}_r$ is a sub-product radical class of l-groups, by Grätzer's existence theorem on a free algebra generated by a partial algebra (see Theorem 28.2 of [10]) we have

THEOREM 2.4. There exists an $\mathcal{A}_r$-free l-group $\mathcal{F}_{\mathcal{A}_r}([G, P])$ generated by a po-group $[G, P]$ if and only if $[G, P]$ is po-group isomorphic into an archimedean l-group.

Now suppose that a po-group $[G, P]$ is po-group isomorphic into an archimedean l-group $[F', \mathcal{F}']$ with the po-group isomorphism $\delta$. Thus $[G, P]$ must be torsion-free abelian and semi-closed. By Proposition 2.2(1) there exists the $\mathcal{A}_r$-free l-group $\mathcal{F}_{\mathcal{A}_r}([G, P])$ generated by $[G, P]$ with the po-group isomorphism $\alpha$ of $[G, P]$ into $\mathcal{F}_{\mathcal{A}_r}([G, P])$. By definition 2.1 there exists an $l$-homomorphism $\gamma$ from $\mathcal{F}_{\mathcal{A}_r}([G, P])$ into $\mathcal{F}'$ such that $\gamma \alpha = \beta$. Let $D = \{F_\lambda | \lambda \in \Lambda \}$ be the set of all archimedean $l$-homomorphism images of $\mathcal{F}_{\mathcal{A}_r}([G, P])$ with the $l$-homomorphism $\beta_\lambda$. Thus $\gamma \mathcal{F}_{\mathcal{A}_r}([G, P]) \subseteq D$ and $D$ is not empty. For each $\lambda \in \Lambda$, $\gamma_\lambda \alpha$ is a po-group homomorphism of $[G, P]$ into $F_\lambda$. The direct product $\prod_{\lambda \in \Lambda} F_\lambda$ is an archimedean l-group. Let $\pi$ be the natural map of the po-group $G$
onto the subgroup $G'$ of long constants of $\prod_{\lambda \in A} F_\lambda$. That is, $\pi(g) = (\cdots, \gamma_\lambda \alpha(g), \cdots)$ for $g \in G$. Because $\gamma \alpha = \delta$ is a po-group isomorphism, $\pi$ is a po-group isomorphism of $G$ onto $G'$. Let $F$ be the sublattice of $\prod_{\lambda \in A} F_\lambda$ generated by $G'$. For each $g \in G$, let $g' = \pi(g)$ denote the long constant of $G'$. Since $\prod_{\lambda \in A} F_\lambda$ is a distributive lattice, the sublattice generated by all $g'$ is

\[
F = \left\{ \bigvee_{i \in I} \bigwedge_{j \in J} g'_{ij} \mid g_{ij} \in G, I \text{ and } J \text{ finite} \right\}.
\]

Suppose that $\beta$ is a po-group homomorphism of $[G, P]$ into an archimedean $l$-group $[L, L^+]$. Then there exists an $l$-homomorphism $\gamma'$ of $\mathcal{F}_d([G, P])$ into $[L, L^+]$ such that $\gamma' \alpha = \beta$. So $\gamma' \mathcal{F}_d([G, P]) \in D$. Now we extend $\beta$ to $F$ as follows:

\[
\beta^* \left( \bigvee_{i \in I} \bigwedge_{j \in J} g_{ij} \right) = \bigvee_{i \in I} \bigwedge_{j \in J} \beta(g_{ij}).
\]

To see that $\beta^*$ is well defined, suppose that

\[
\bigvee_{i \in I} \bigwedge_{j \in J} \beta(g_{ij}) \neq \bigvee_{m \in M} \bigwedge_{n \in N} \beta(h_{mn}).
\]

Then we have

\[
\bigvee_{i \in I} \bigwedge_{j \in J} \bigvee_{m \in M} \bigwedge_{f \in N^M} \beta(g_{ij} - h_{mf(m)}) \neq 0
\]

in $[L, L^+]$. Because $\gamma' \mathcal{F}_d([G, P]) \in D$,

\[
\bigvee_{i \in I} \bigwedge_{j \in J} \bigvee_{m \in M} \bigwedge_{f \in N^M} \beta(g'_{ij} - h'_{mf(m)}) \neq 0
\]

in $F$. That is, we have

\[
\bigvee_{i \in I} \bigwedge_{j \in J} g'_{ij} \neq \bigvee_{m \in M} \bigwedge_{n \in N} h'_{mn}
\]

in $F$. Therefore $\beta^*$ is single valued.
That \( \beta^* \) is a lattice homomorphism is an immediate consequence of the fact that \( L \) is a distributive lattice. Now consider \( g = \bigvee_{i \in I} \bigwedge_{j \in J} g_{ij} \) and \( h = \bigvee_{m \in M} \bigwedge_{n \in N} h_{mn} \) in \( F \).

\[
\begin{align*}
\beta^*(g - h) &= \bigvee_{i \in I} \bigwedge_{j \in J} \bigvee_{m \in M} \bigwedge_{f \in N} (g_{ij} - h_{mf(m)}) \\
&= \bigvee_{i \in I} \bigwedge_{j \in J} \bigvee_{m \in M} \bigvee_{f \in N} \beta(g_{ij} - h_{mf(m)}) \\
&= \bigvee_{i \in I} \bigwedge_{j \in J} \beta(g_{ij}) - \bigvee_{m \in M} \bigwedge_{n \in N} \beta(h_{mn}) \\
&= \beta^*(g) - \beta^*(h).
\end{align*}
\]

Hence \( \beta^* \) is an \( l \)-homomorphism of \( F \) into \( L \) and \( \beta^*\pi = \beta \).

The above discussion proves the following theorem.

**Theorem 2.5.** Suppose that a po-group \([G, P]\) is po-group isomorphic into an archimedean \( l \)-group. Then the \( \mathcal{A}_r \)-free \( l \)-group \( \mathcal{F}_{\mathcal{A}_r}([G, P]) \) generated by \([G, P]\) is the sublattice \( F \) of the direct product \( \prod_{\lambda \in \Lambda} F_\lambda \) which is generated by the long constants \( g' (g \in G) \) where \( \{ F_\lambda | \lambda \in \Lambda \} \) are all archimedean \( l \)-homomorphic images of the \( \mathcal{A}_r \)-free \( l \)-group \( \mathcal{F}_{\mathcal{A}_r}([G, P]) \) generated by \([G, P]\).

**Note.** Suppose that a po-group \([G, P]\) is po-group isomorphic into an archimedean \( l \)-group. By Proposition 2.3 there exists an \( \mathcal{L} \)-free \( l \)-group \( \mathcal{F}_{\mathcal{L}}([G, P]) \) generated by \([G, P]\). If we take \( \mathcal{F}_{\mathcal{L}}([G, P]) \) instead of \( \mathcal{F}_{\mathcal{A}_r}([G, P]) \) in the above discussion, we obtain another description of \( \mathcal{F}_{\mathcal{A}_r}([G, P]) \).

Let \( \mathcal{U} \) be a class of algebras and \( X \) be a nonempty set. The algebra \( \mathcal{F}_{\mathcal{U}}(X) \) is called the \( \mathcal{U} \)-free algebra on \( X \) if \( X \) generates \( \mathcal{F}_{\mathcal{U}}(X) \) as an algebra, and whenever \( L \in \mathcal{U} \) and \( \lambda: X \rightarrow L \) is a map, then there exists a homomorphism \( \sigma: \mathcal{F}_{\mathcal{U}}(X) \rightarrow L \) which extends \( \lambda \). By Birkhoff's Theorem ([4]) there exists a \( \mathcal{U} \)-free algebra \( \mathcal{F}_{\mathcal{U}}(X) \) on any nonempty set \( X \) if \( \mathcal{U} \) is closed under subalgebras and direct products. Let \( \mathcal{U} \) be a class of \( l \)-groups and \( X \) be a nonempty set with \( |X| = \alpha \). Then the \( \mathcal{U} \)-free \( l \)-group \( \mathcal{F}_{\mathcal{U}}(X) \) on \( X \) is said to be of rank \( \alpha \). We can construct the \( \mathcal{U} \)-free \( l \)-group \( \mathcal{F}_{\mathcal{U}}(X) \) on \( X \) using the \( \mathcal{U} \)-free \( l \)-group generated by a trivially ordered group. Let \( \mathcal{U} \) be a class of \( l \)-groups which is closed under \( l \)-subgroups and direct products. We denote by \( \mathcal{G}(\mathcal{U}) \) the class of all groups that can be embedded (as subgroups) into the members of \( \mathcal{U} \). It is clear that \( \mathcal{G}(\mathcal{U}) \) is closed under subgroups and direct products.

**Proposition 2.6.** Let \( \mathcal{U} \) be a class of \( l \)-groups which is closed under \( l \)-subgroups and direct products and \( X \) be a nonempty set. The \( \mathcal{U} \)-free \( l \)-group \( \mathcal{F}_{\mathcal{U}}(X) \) on \( X \) is the \( \mathcal{U} \)-free \( l \)-group generated by the \( \mathcal{G}(\mathcal{U}) \)-free group on \( X \) with trivial order.
PROOF. By Birkhoff's Theorem there exists the $\mathcal{H}(\mathcal{U})$-free group $\mathcal{F}_{\mathcal{G}(\mathcal{U})}(X)$ on $X$. $\mathcal{F}_{\mathcal{G}(\mathcal{U})}(X) \in \mathcal{H}(\mathcal{U})$ means $\mathcal{F}_{\mathcal{G}(\mathcal{U})}(X)$ can be embedded (as a subgroup) into a member of $\mathcal{U}$. By Theorem 28.2 of [10] there exists a $\mathcal{U}$-free $l$-group $\mathcal{F}_{\mathcal{G}(\mathcal{U})}([\mathcal{F}_{\mathcal{G}(\mathcal{U})}(X), \{0\}])$ generated by the trivially ordered $\mathcal{G}(\mathcal{U})$-free group $\mathcal{F}_{\mathcal{G}(\mathcal{U})}(X)$. Now any map from $X$ into an $l$-group $L \in \mathcal{U}$ can be extended to a group homomorphism of $\mathcal{F}_{\mathcal{G}(\mathcal{U})}(X)$ into $L$ and hence to an $l$-homomorphism of $\mathcal{F}_{\mathcal{G}(\mathcal{U})}(X)$ into $L$ and hence to an $l$-homomorphism of $\mathcal{F}_{\mathcal{G}(\mathcal{U})}([\mathcal{F}_{\mathcal{G}(\mathcal{U})}(X), \{0\}])$ into $L$.

Theorem 2.7 of [14] is a special case of the above Proposition 2.6. The following theorem is a consequence of Proposition 2.6.

**THEOREM 2.7.** Let $X$ be a nonempty set. The $\mathcal{A}$-free $l$-group $\mathcal{F}_{\mathcal{A},r}(X)$ on $X$ is the $\mathcal{A}$-free $l$-group generated by the $\mathcal{A}(\mathcal{A})$-free group $\mathcal{F}_{\mathcal{A}(\mathcal{A})}(X)$ with trivial order.

$$X \rightarrow \mathcal{F}_{\mathcal{A}(\mathcal{A})}(X) \rightarrow \mathcal{F}_{\mathcal{A},r}(X).$$

**PROPOSITION 2.8.** Suppose that $\mathcal{F}_{\mathcal{A},r}([G, P_1])$ and $\mathcal{F}_{\mathcal{A},r}([G, P_2])$ are the $\mathcal{A}$-free $l$-groups generated by po-group $[G, P_1]$ and $[G, P_2]$, respectively. If $P_1 \subseteq P_2$. Then $\mathcal{F}_{\mathcal{A},r}([G, P_2])$ is an $l$-homomorphic image of $\mathcal{F}_{\mathcal{A},r}([G, P_1])$.

**PROOF.** $[G, P_2]$ can be embedded into $\mathcal{F}_{\mathcal{A},r}([G, P_2])$ as a po-group and $G$ generates $\mathcal{F}_{\mathcal{A},r}([G, P_2])$. So $[G, P_1]$ is also embedded into $\mathcal{F}_{\mathcal{A},r}([G, P_2])$ as a po-group. Hence there exists an $l$-homomorphism $\phi$ from $\mathcal{F}_{\mathcal{A},r}([G, P_1])$ into $\mathcal{F}_{\mathcal{A},r}([G, P_2])$. But $[G, P_1]$ can be embedded into $\mathcal{F}_{\mathcal{A},r}([G, P_1])$ as a po-group and $G$ generates $\mathcal{F}_{\mathcal{A},r}([G, P_1])$. Therefore $\phi$ is onto $\mathcal{F}_{\mathcal{A},r}([G, P_2])$.

3. The Relation between $\mathcal{F}_{\mathcal{A}}([G, P])$ and $\mathcal{F}_{\mathcal{A},r}([G, P])$.

In [6] P. Conrad has given the relation between the $\mathcal{L}$-free $l$-group $\mathcal{F}_{\mathcal{L}}(X)$ and the $\mathcal{A}$-free $l$-group $\mathcal{F}_{\mathcal{A}}(X)$ on a nonempty set $X$. Let $Y$ be the $l$-ideal generated by the commutator subgroup $[\mathcal{F}_{\mathcal{L}}(X), \mathcal{F}_{\mathcal{L}}(X)]$. Then $\mathcal{F}_{\mathcal{L}}(X) \cong \mathcal{F}_{\mathcal{A}}(X)/Y$.

In this section we will give the relation between $\mathcal{F}_{\mathcal{A}}([G, P])$ and $\mathcal{F}_{\mathcal{A},r}([G, P])$. Clearly, if $\mathcal{F}_{\mathcal{A}}([G, P])$ is archimedean then $\mathcal{F}_{\mathcal{A},r}([G, P]) \cong \mathcal{F}_{\mathcal{A},r}([G, P])$. We will give a necessary and sufficient condition in which $\mathcal{F}_{\mathcal{A}}([G, P])$ is archimedean. First we need some concepts. Let $[G, P]$ be a torsion free abelian po-group and $S$ be a nonempty subset of $G$. $S$ is said to be positively independent if for any finite subset $\{x_1, \cdots, x_k\}$ of $S$ and non-negative integers $\{\lambda_1, \ldots, \lambda_k\}$, $\lambda_i x_i \in -P$ only if $\lambda_i = 0$ ($i = 1, \ldots, k$). There exists a total order $P_1$ of $G$ such that $P_1 \supseteq P \cup S$ if and only if $S$ is positively independent. Let $x' = \bigvee_{i \in I} \bigwedge_{j \in J} x_{ij} \in \mathcal{F}_{\mathcal{A}}([G, P])$. Then $x' \notin 0$ if and only if for some $i \in I$ the set $\{x_{ij} \mid j \in J\}$ is positively independent [3].

A po-group $[G, P]$ is said to be strong uniformly archimedean if, given $u \in G$ and a positively independent subset $\{v_1, \ldots, v_k\}$ of $G$, there exists $n \in N$ such that if
\( \lambda_1, \ldots, \lambda_k \) are non-negative integers and \( \sum_{i=1}^{k} \lambda_i \geq mn \) with \( m \in \mathbb{N} \), then
\[
\sum_{i=1}^{k} \lambda_i v_i \geq mu.
\]

**Theorem 3.1.** The \( \mathcal{A} \)-free \( l \)-group \( \mathcal{F}_\mathcal{A}([G, P]) \) generated by a po-group \([G, P]\) is archimedean if and only if \([G, P]\) is strong uniformly archimedean.

**Proof.** Necessity. Suppose that \( u \in G \) and \( \{v_1, \ldots, v_k\} \) is a positively independent subset of \( G \). Then, \( \langle v_1 \rangle^+ \land \cdots \land \langle v_k \rangle^+ \neq 0 \) in \( \mathcal{F}_\mathcal{A}([G, P]) \). Since \( \mathcal{F}_\mathcal{A}([G, P]) \) is archimedean, there exists \( n \in \mathbb{N} \) such that
\[
n(\langle v_1 \rangle \land \cdots \land \langle v_k \rangle)^+ = n(\langle v_1 \rangle^+ \land \cdots \land \langle v_k \rangle^+) \leq \langle u \rangle^+.
\]
It follows that if \( \lambda \geq n \), \( \lambda(\langle v_1 \rangle \land \cdots \land \langle v_k \rangle) \leq \langle u \rangle \). Now if \( \lambda_1, \ldots, \lambda_k \) are non-negative integers and \( \sum_{i=1}^{k} \lambda_i \geq mn \) with \( m \in \mathbb{N} \), then we have
\[
\sum_{i=1}^{k} \lambda_i v_i \geq m(\langle v_1 \rangle \land \cdots \land \langle v_k \rangle) \leq m\langle u \rangle,
\]
because \( P \) is semi-closed and \( mn(\langle v_1 \rangle \land \cdots \land \langle v_k \rangle) \leq m\langle u \rangle \) would imply \( n(\langle v_1 \rangle \land \cdots \land \langle v_k \rangle) \leq \langle u \rangle \), a contradiction. Hence we have \( \sum_{i=1}^{k} \lambda_i v_i \leq mu \) in \([G, P]\).

Sufficiency. It follows from Proposition 1.4 that it suffices to show that \( \langle g \rangle \) is generally archimedean in \( \mathcal{F}_\mathcal{A}([G, P]) \) for each \( g \in G \). And because \( G \) is a group and \( g^- = (-g) \lor 0 \), it suffices to show that \( g^+ \) is archimedean in \( \mathcal{F}_\mathcal{A}([G, P]) \) for each \( g \in G \). Let \( g \in G \) and \( 0 < x = \lor_{i \in I} \langle x_{ij} \rangle \in \mathcal{F}_\mathcal{A}([G, P]) \) where \( x_{ij} \in G \). We must show there exists \( n \in \mathbb{N} \) such that \( nx \leq g^+ \). Since \( x > 0 \), the set \( \{x_{ij} | j \in J\} \) is positively independent for some \( i \). It suffices to show that there exists \( n \in \mathbb{N} \) such that \( n(\lor_{j \in J} \langle x_{ij} \rangle) \leq \langle g \rangle \lor 0 \). And so it suffices to show that if \( \{v_1, \ldots, v_k\} \) is a positively independent subset of \( G \) and \( g \in G \), then there exists \( n \in \mathbb{N} \) and a total order \( T \) of \( G \) such that \( T \supseteq P, v_i \in T \) and \( nv_i - g \in T (i = 1, \ldots, k) \). Then, lifting the identity map of \([G, P]\) onto \([G, T]\) to an \( l \)-homomorphism of \( \mathcal{F}_\mathcal{A}([G, P]) \) onto \([G, T]\) we would have \( \lor_{i=1}^{k} [(nv_i - g) \land nv_i] \leq 0 \), and so \( n(\lor_{i=1}^{k} v_i) \leq g \lor 0 \).

It therefore suffices to show that there exists \( n \in \mathbb{N} \) so that the set
\[
\{v_i | i = 1, \ldots, k\} \cup \{nv_i - g | i = 1, \ldots, k\}
\]
is positively independent. Because \([G, P]\) is strong uniformly archimedean, there
exists \( n \in N \) such that if \( \lambda_1, \ldots, \lambda_k \) are non-negative integers and \( \sum_{i=1}^{k} \lambda_i \geq mn \) with \( m \in N \), then \( \sum_{i=1}^{k} \lambda_i v_i \leq mg \). Suppose that \( \mu_1, \ldots, \mu_k \) and \( v_1, \ldots, v_k \) are all non-negative integers and
\[
\sum_{i=1}^{k} \mu_i v_i + \sum_{i=1}^{k} v_i (nv_i - g) \leq -P.
\]
Then \( \sum_{i=1}^{k} (\mu_i + nv_i) v_i \leq \left( \sum_{i=1}^{k} v_i \right) g \) which contradicts the choice of \( n \) unless all \( v_i \) are zero and then contradicts positive independence of the \( v_i \) unless all \( \mu_i \) are zero. Thus \( \{v_i \mid i = 1, \ldots, k\} \cup \{nv_i - g \mid i = 1, \ldots, k\} \) is positively independent.

**Corollary 3.2.** Suppose that a po-group \([G, P]\) is po-group isomorphic into an archimedean l-group. Then \( \mathcal{F}_a([G, P]) \cong \mathcal{F}_a([G, P]) \) if and only if \([G, P]\) is strong uniformly archimedean.

Let \( G \) be a group. A nonempty subset \( S \) of \( G \) is said to be indepedent if for any finite subset \( \{x_1, \ldots, x_k\} \) of \( S \) and non-negative integers \( \{\lambda_1, \ldots, \lambda_k\}, \sum_{i=1}^{k} \lambda_i x_i = 0 \) only if \( \lambda_i = 0 \) (i = 1, \ldots, k). Clearly, \( S \) is independent in \( G \) if and only if \( S \) is positively independent in the po-group \([G, \{0\}]\) with the trivial order. Let \( G \) be a torsion-free and abelian group. Weinberg has proved that the \( \mathcal{A} \)-free l-group \( \mathcal{F}_a([G, \{0\}]) \) is archimedean (Corollary 3.4 of [17]). From this we get

**Corollary 3.3.** Suppose that \( G \) is a torsion-free and abelian group. Given \( u \in G \) and an independent subset \( \{v_1, \ldots, v_k\} \) of \( G \), then there exists \( n \in N \) such that if \( \lambda_1, \ldots, \lambda_k \) are non-negative integers and \( \sum_{i=1}^{k} \lambda_i \geq mn \) with \( m \in N \), \( \sum_{i=1}^{k} \lambda_i v_i \neq mu \).

4. Some properties of an archimedean l-group.

In order to discuss properties of \( \mathcal{F}_a([G, P]) \) we need to know some properties of an Archimedean l-group. First we introduce some concepts. Let \( \{G_\alpha \mid \alpha \in A\} \) be a system of l-groups. For \( g \in \prod_{\alpha \in A} G_\alpha \), we denote by \( g_\alpha \) the \( \alpha \) component of \( g \). An l-group \( G \) is said to be an ideal subdirect sum of l-groups \( G_\alpha \), in symbol \( G \leq_\* \prod_{\alpha \in A} G_\alpha \), if \( G \) is a subdirect sum of \( G_\alpha \) and \( G \) is an l-ideal of \( \prod_{\alpha \in A} G_\alpha \). An l-group \( G \) is said to be a completely subdirect sum, if \( G \) is an l-subgroup of \( \prod_{\alpha \in A} G_\alpha \) and \( \sum_{\alpha \in A} G_\alpha \leq G \). We use the symbol \( \leq_\* \) to denote subdirect sum. Let \( G \) be an l-group. We denote by \( vG \) the least cardinal \( \alpha \) such that \( |A| \leq \alpha \) for each bounded disjoint
subset $A$ of $G$. $G$ is said to be $v$-homogeneous if $vH = vG$ for any convex $l$-subgroup $H \neq 0$ of $G$. A $v$-homogeneous $l$-group $G$ is said to be $v$-homogeneous of $\alpha$ type if $vG = \alpha$. By Theorem 3.7 of [11] it is easy to verify the following lemma. The proof is left to the reader.

**Lemma 4.1.** Any complete $l$-group is $l$-isomorphic to an ideal subdirect sum of complete $v$-homogeneous $l$-groups.

By using 4.3 of [11] it is easy to verify that if an $l$-group $G$ is $v$-homogeneous and non-totally ordered, then $vG \geq \aleph_0$. It is well known that any non-zero complete totally ordered group is $l$-isomorphic to a real group $R$ or an integer group $Z$. So from Lemma 4.1 we obtain the structure theorem of a complete $l$-group.

**Theorem 4.2.** Any complete $l$-group $G$ is $l$-isomorphic to an ideal subdirect sum of real groups, integer groups and complete $v$-homogeneous $l$-groups of $\aleph_i$ type ($i \geq 0$).

**Theorem 4.3.** Let $G$ be an archimedean $v$-homogenous $l$-group of $\aleph_i$ type. Then $G$ has the following properties:

1. $G$ has no basic element.
2. $G$ has no basis.
3. The radical $R(G) = G$.
4. $G$ is not completely distributive.
5. The distributive radical $D(G) = G$.

Moreover, every non-trivial convex $l$-subgroup of $G$ enjoys these same five properties.

**Proof.** By Theorems 5.4 and 5.10 of [5] we need only to show (1). For any $0 < g \leq G$, $vG(g) = \aleph_i > 1$. So $G(g)$ is not totally ordered, and $[0, g]$ is also not totally ordered by 4.3 of [11].

An $l$-group $G$ is said to be continuous, if for any $0 < x \leq G$ we have $x = x_1 + x_2$ and $x_1 \land x_2 = 0$ where $x_1 \neq 0$, $x_2 \neq 0$.

**Lemma 4.4** (Lemma 2.4 of [20]). A complete $l$-group $G$ is continuous if and only if $G$ has no basic element.

An $l$-group $G$ is said to be projectable if each of its principal polars is a cardinal summand. The following lema is clear.

**Lemma 4.5.** Let $G$ be a projectable (in particular, complete) and non-totally ordered $l$-group. Then $G$ is directly decomposable.

An $l$-group $G$ is said to be ideal subdirectly irreducible if $G$ cannot be expressed as an ideal subdirect sum of $l$-groups.
LEMMA 4.6 (Lemma 2.6 of [20]). A complete l-group $G$ is directly indecomposable if and only if $G$ is ideal subdirectly irreducible.

LEMMA 4.7 (Lemma 2.7 of [20]). An archimedean l-group $G$ is subdirectly irreducible if and only if the Dedekind completion $G^\wedge$ of $G$ is ideal subdirectly irreducible.

Now from Lemma 4.4, Lemma 4.5 and Lemma 4.6 we have

THEOREM 4.8. Let $G$ be a complete v-homogeneous l-groups of $\aleph_i$ type. Then

1. $G$ is continuous.
2. $G$ is directly decomposable.
3. $G$ is not ideal subdirectly irreducible.

Moreover, every nontrivial convex l-subgroup of $G$ enjoys these same three properties.

From Lemma 4.7 and Theorem 4.8 we obtain

COROLLARY 4.9. An archimedean v-homogeneous l-group of $\aleph_i$ type is not subdirectly irreducible.

A subset $D$ in a lattice $L$ is called a $d$-set if there exists $x \in L$ such that $d_1 \land d_2 = x$ for any pair of distinct elements of $D$ and $d > x$ for each $d \in D$. We denote by $w[a, b]$ the least cardinal $\alpha$ such that $|D| \leq \alpha$ for each $d$-set $D$ of $[a, b]$.

LEMMA 4.10. Let $G$ be a v-homogeneous l-group of $\aleph_i$ type and be a dense l-subgroup of an l-group $G'$. Then $G'$ is also a v-homogeneous l-group of $\aleph_i$ type.

PROOF. Suppose that $H$ is an arbitrary convex l-subgroup of $G$. Let $H = H' \cap G$. Then $H$ is dense in $H'$ and $H$ is a convex l-subgroup of $G$. We will prove that $vH' = vH$. It is clear that $vH' \geq vH$. Let $\{x_\alpha \in H' + | \alpha \in A\}$ be a disjoint of $H'$ with an upper bound $x'$. Then there exists $x_\alpha \in H$ such that $0 < x_\alpha \leq x'_\alpha$ for each $\alpha \in A$ and there exists $x \in H$ such that $0 < x \leq x'$. Hence $\{x_\alpha \land x | \alpha \in A\}$ is a disjoint subset of $H$ with an upper bound $x$. Hence $|A| \leq \aleph_i$ and $vH' \leq vH$. Therefore

$vH' = vH = vG = \aleph_i$

and so $G'$ is a v-homogeneous l-group of $\aleph_i$ type.

From Theorem 2.6 and Theorem 5.2 of [8] and the above Lemma 4.10 we get

THEOREM 4.11. Let $G$ be an archimedean v-homogeneous l-group of $\aleph_i$ type. Then the Dedekind completion $G^\wedge$ of $G$ and the lateral completion $G^L$ of $G$ are also v-homogeneous l-groups of $\aleph_i$ type.
Lemma 4.12. Let $G$ be a $v$-homogeneous $l$-group of $\aleph_i$ type and $\{x_\alpha | \alpha \in A\}$ be a disjoint subset in $G$. Then $|A| \leq \aleph_i$.

Proof. Let $G^L$ be the lateral completion of $G$. By Theorem 4.11 $G^L$ is also $v$-homogeneous of $\aleph_i$ type. Let $x$ be the least upper bound of a disjoint subset $\{x_\alpha | \alpha \in A\}$ of $G$ in $G^L$. So $\{x_\alpha | \alpha \in A\}$ is a bounded disjoint subset in $G^L$. Therefore $|A| \leq \aleph_i$.

Theorem 4.13. Let $G$ be a $v$-homogeneous $l$-group of $\aleph_i$ type. Then the divisible hull $G^d$ of $G$ is also a $v$-homogeneous $l$-group of $\aleph_i$ type.

Proof. Let $P$ be any nontrivial convex $l$-subgroup of $G^d$. For any $0 \neq x \in P$ there exists $n \in N$ such that $0 \neq nx \in P \cap G$. So $P \cap G$ is also a nontrivial convex $l$-subgroup of $G$. It is clear that $vP \geq v(P \cap G) = \aleph_i$. On the other hand, $P = \left\{ \frac{1}{n}\sum_{g \in G \cap P} g \in N \right\}$. So if $\{c_j | j \in J\}$ is a bounded disjoint subset in $P$, let $c_j = \frac{1}{n_j} g_j (j \in J, g_j \in G \cap P, n_j \in N)$. By the Bernau representation of an archimedean $l$-group [2] we see that $c_j \wedge c_j' = 0$ if and only if $g_j \wedge g_j' = 0 (j \neq j')$. So $\{g_j | j \in J\}$ is a disjoint subset in $G \cap P$. By Lemma 4.12, $|J| \leq \aleph_i$. Hence $vP \leq \aleph_i$. Therefore $vP = \aleph_i$.

Now we turn to an archimedean $l$-group. In [19] we proved the following result.

Lemma 4.14. An $l$-group $G$ is archimedean if and only if $G$ is $l$-isomorphic to a subdirect sum of subgroups of reals and archimedean $v$-homogeneous $l$-groups of $\aleph_i$ type.

Suppose that $G$ is a subdirect sum of subgroups of reals and $v$-homogeneous $l$-groups of $\aleph_i$ type, $G \subseteq \prod_{\delta \in \Delta} T_\delta$. Let $A_1 = \{\delta \in \Delta | T_\delta$ is a subgroup of reals\}. If $\sum_{\delta \in A_1} T_\delta \subseteq G$, $G$ is said to be a semicomplete subdirect sum of subgroups of reals and $v$-homogeneous $l$-groups of $\aleph_i$ type, in symbols $\sum_{\delta \in A_1 \subseteq \Delta} T_\delta \subseteq G \subseteq \prod_{\delta \in \Delta} T_\delta$.

Theorem 4.15. (Theorem 4.7 of [19]). An $l$-group $G$ is archimedean if and only if $G$ is $l$-isomorphic to a semicomplete subdirect sum of subgroups of reals and archimedean $v$-homogeneous $l$-groups of $\aleph_i$ type.

5. Properties of $A_\lambda$.

We denote by $A_\lambda$ the $A$-free $l$-group $\mathcal{F}_x(X)$ of rank $\alpha$. By Proposition 2.6 $A_\lambda$ is the $A$-free $l$-group $\mathcal{F}_x([G, \{0\}])$ generated by $\mathcal{A}(A)$-free group $G$ with trivial order. It follows from Corollary 3.4 of [17] that $A_\lambda$ is archimedean. Hence $A_\lambda \simeq A r_\alpha$. We
have already known some properties of $\mathcal{A}_\alpha$. For example, $\mathcal{A}_\alpha$ is a subdirect sum of integers (Theorem 2.5 of [3]); $\mathcal{A}_\alpha(\alpha > 1)$ has a countably infinite disjoint subset but no uncountable disjoint subset (Theorem 6.2 of [16]); every infinite chain in $\mathcal{A}_\alpha$ must be countable (Theorem 5.1 of [15]); the word problem for $\mathcal{A}_\alpha$ is solvable (Theorem 2.11 of [14]); $\mathcal{A}_\alpha(\alpha > 1)$ has no singular elements (Theorem 2.8 of [3]). In this section we will give further properties of $\mathcal{A}_\alpha$ using the structure theorem of an archimedean $l$-group.

**Theorem 5.1.** $\mathcal{A}_\alpha(\alpha > 1)$ is an archimedean $v$-homogeneous $l$-group of $\aleph_0$ type.

**Proof.** Since $\mathcal{A}_\alpha$ is archimedean, by Theorem 4.15, without loss of generality, we have

$$\sum_{\delta \in A_1} T_\delta \subseteq \mathcal{A}_\alpha \subseteq \bigstar \prod_{\delta \in A} T_\delta,$$

where each $T_\delta$ ($\delta \in A_1$) is a subgroup of reals and each $T_\delta$ ($\delta \in A \setminus A_1$) is an archimedean $v$-homogeneous $l$-group of $\aleph_1$ type. By Theorem 3.5 of [3] (or Theorem 1 of [18]), $\mathcal{A}_\alpha(\alpha > 1)$ has no nontrivial direct summands. Hence $A_1 = \emptyset$ and $\mathcal{A}_\alpha(\alpha > 1)$ is a subdirect sum of archimedean $v$-homogeneous $l$-groups of $\aleph_1$ type. Let

$$(1) \quad \sum_{\delta \in \Delta'} T_\delta \subseteq \mathcal{A}_\alpha' \subseteq \bigstar \prod_{\delta \in \Delta'} T_\delta,$$

where $A' = A \setminus A_1$ and each $T_\delta$ ($\delta \in A'$) is an archimedean $v$-homogeneous $l$-groups of $\aleph_1$ type. For any $0 < x_0 \in \mathcal{A}_\alpha$. We denote by $\mathcal{A}_\alpha(x)$ the convex $l$-subgroup in $\mathcal{A}_\alpha$ generated by $x_0$ and $\mathcal{A}_\alpha'(x)$ the convex $l$-subgroup in $\mathcal{A}_\alpha'$ generated by $x_0$. By Theorem 2 of [18] we have

$$(2) \quad v \mathcal{A}_\alpha(x) \leq v \mathcal{A}_\alpha \leq \aleph_0.$$ 

On the other hand, $\mathcal{A}_\alpha(x)$ is dense in $\mathcal{A}_\alpha'(x)$. If $\{x_\alpha | \alpha \in A\}$ is a disjoint subset with an upper bound $x_0$ in $\mathcal{A}_\alpha(x)$. Then there exists $x_0' \in \mathcal{A}_\alpha'(x)$ such that $0 < x_0' < x_0$. Put $x_\alpha' = x_\alpha \wedge x_0'$. Then $\{x_\alpha' | \alpha \in A\}$ is a disjoint subset with an upper bound $x_0'$ in $\mathcal{A}_\alpha(x)$. Hence $v \mathcal{A}_\alpha'(x) \leq v \mathcal{A}_\alpha(x)$. And it is clear that $v \mathcal{A}_\alpha(x) \leq v \mathcal{A}_\alpha'(x)$. Thus,

$$(3) \quad v \mathcal{A}_\alpha(x) = v \mathcal{A}_\alpha'(x).$$

For any $\delta_0 \in A'$, put $\bar{x}_{\delta_0} = (\ldots, 0, \ldots, x_{\delta_0}, \ldots, 0)$. Then $\bar{x}_{\delta_0} \leq x$. Since

$$v T_{\delta_0}(x_{\delta_0}) = v T_{\delta_0} \geq \aleph_0$$

where $T_{\delta_0}(x_{\delta_0})$ is the convex $l$-subgroup of $T_{\delta_0}$ generated by $x_{\delta_0}$, there exists a disjoint subset $\{x^\beta | \beta \in B\}$ in $T_{\delta_0}(x_{\delta_0})$ such that $x^\beta \leq x_{\delta_0}$ and $|B| \geq \aleph_0$. Then $\bar{x}^\beta = (\ldots, 0, \ldots, x^\beta, \ldots, 0) \in \mathcal{A}_\alpha'$ by (1) and $\{\bar{x}^\beta | \beta \in B\}$ is a disjoint subset with an upper bound $\bar{x}_{\delta_0}$ in $\mathcal{A}_\alpha'(\bar{x}_{\delta_0})$. Hence
(4) \( v_{\mathcal{A}_a}(x) = v_{\mathcal{A}'_a}(\bar{x}_{\delta_0}) \geq \mathbb{N}_0. \)

Combining (2), (3) and (4) we get \( v_{\mathcal{A}_a}(x) = \mathbb{N}_0 \) for any \( 0 < x \in \mathcal{A}_a. \) Now for any nontrivial convex \( l \)-subgroup \( K \) in \( \mathcal{A}_a. \) Let \( 0 < x \in K. \) Then

\[ \mathbb{N}_0 = v_{\mathcal{A}_a}(x) \leq vK \leq v_{\mathcal{A}_a} \leq \mathbb{N}_0. \]

Therefore \( vK = \mathbb{N}_0 \) and \( \mathcal{A}_a \) is a \( v \)-homogeneous \( l \)-group of \( \mathbb{N}_0 \) type.

By Theorem 4.3 and Theorem 5.1 we obtain

**Theorem 5.2.** \( \mathcal{A}_a(\alpha > 1) \) has the following properties:

1. \( \mathcal{A}_a \) has no basic element.
2. \( \mathcal{A}_a \) has no basis.
3. The radical \( R(\mathcal{A}_a) = \mathcal{A}_a. \)
4. \( \mathcal{A}_a \) is not completely distributive.
5. The distributive radical \( D(\mathcal{A}_a) = \mathcal{A}_a. \)

Moreover, every nontrivial convex \( l \)-subgroup of \( \mathcal{A}_a \) enjoys these same five properties.

By Theorem 3.6 of [7] and the above Theorem 4.11, Theorem 4.13 and Theorem 5.1 we have

**Theorem 5.3.** (1) The Dedekind completion \( \mathcal{A}_a^* \) of \( \mathcal{A}_a \) is a \( v \)-homogeneous \( l \)-group of \( \mathbb{N}_0 \) type.

2. The lateral completion \( \mathcal{A}_a^L \) of \( \mathcal{A}_a \) is a \( v \)-homogeneous \( l \)-group of \( \mathbb{N}_0 \) type.

3. The divisible hull \( \mathcal{A}_a^d \) of \( \mathcal{A}_a \) is a \( v \)-homogeneous \( l \)-group of \( \mathbb{N}_0 \) type.

4. The essential closure \( \mathcal{A}_a^e \) of \( \mathcal{A}_a \) is a \( v \)-homogeneous \( l \)-group of \( \mathbb{N}_0 \) type.

From Theorem 4.8 we get

**Theorem 5.4.** The Dedekind completion \( \mathcal{A}_a^* \) of \( \mathcal{A}_a \) has the following properties:

1. \( \mathcal{A}_a^* \) is continuous.
2. \( \mathcal{A}_a^* \) is directly decomposable.
3. \( \mathcal{A}_a^* \) is not ideal subdirectly irreducible.
4. \( \mathcal{A}_a^* \) has a closed \( l \)-ideal.

Moreover, each nontrivial convex \( l \)-subgroup of \( \mathcal{A}_a^* \) enjoys these same four properties.

**References**